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# Studies on Brutal Contraction and Severe Withdrawal

**Abstract.** In this paper we present an axiomatic characterization for brutal contractions. Then we consider the particular case of the brutal contractions that are based on a bounded ensconcement and also the class of severe withdrawals which are based on bounded epistemic entrenchment relations that are defined by means of bounded enscongements (using the procedure proposed by Mary-Anne Williams). We present axiomatic characterizations for each one of those classes of functions and investigate the interrelation among them.

**Keywords:** Belief Change, Contraction, Withdrawal, Ensconcement, Epistemic entrenchment, Axiomatic characterization.

## 1. Introduction

The central goal underlying the research area of *logic of theory change* (for an overview see [3]) is the study of the changes which can occur in the belief state of a rational agent when he receives new information.

The most well known model of theory change was proposed by Alchourrón, Gärdenfors, and Makinson in [1] and is, nowadays, known as the AGM model. Assuming that the belief state of an agent is modelled by a *belief set* (i.e. a logically closed set of sentences), this framework essentially provides a definition for contractions—i.e. functions that receive a sentence (representing the new information received by the agent), and return a belief set which is a subset of the original one that does not contain the received sentence. In the mentioned paper, the class of *partial meet contractions* was introduced and axiomatically characterized. Subsequently several constructive models have been presented for the class of contraction functions proposed in the AGM framework (such as the *system of spheres-based contractions* [8], *safe/kernel contractions* [2, 9, 14], and the *epistemic entrenchment-based contractions* [6, 7]). Also several adaptations and variations of those constructive models have been presented and studied in the literature as it is

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the case, for example, of *severe withdrawals* (or mild contractions or Rott's contractions) [12, 15] which results of *simplifying* the definition of *epistemic entrenchment-based contractions*.

Although the AGM model has quickly acquired the status of standard model of theory change, several researchers (for an overview see [3]) have pointed out its inadequacy in several contexts and proposed several extensions and generalizations to that framework. One of the most relevant of the proposed extensions of the AGM model of contraction is to use sets of sentences not (necessarily) closed under logical consequence—which are designated *belief bases*—rather than belief sets to represent belief states.

Hence, several of the existing models (of AGM contractions) were generalized to the case when belief states are represented by belief bases instead of belief sets. Among those we emphasize the *ensconcement-based contractions* and the *brutal contractions* (of belief bases) proposed in [17], which can be seen as adaptations to the case of belief bases of the *epistemic entrenchment-based contractions* and of the *severe withdrawals*, respectively. In fact, the definitions of those operations are both based on the concept of *ensconcement*, which is an adaptation of the concept of epistemic entrenchment relation to the case of belief bases. In the mentioned paper Mary-Anne Williams has also presented a method for constructing an epistemic entrenchment from an ensconcement relation.

In the present paper we will study the interrelation among *brutal contractions* (of belief bases) and *severe withdrawals* (of belief sets). More precisely, we will present an axiomatic characterization of the class of *brutal contractions*. After that we devote special attention to the class of *brutal contractions* which are based on bounded ensconcements—the so-called *bounded brutal contractions* and also to the class of the so-called *ensconcement-based severe withdrawals*, which is formed by the severe withdrawals that are based on an epistemic entrenchment relation defined from a bounded ensconcement using Mary-Anne William's method. We shall provide axiomatic characterizations to each one of those classes of functions and study the interrelation among them.

This paper is organized as follows: In Section 2 we provide the notation and background needed for the rest of the paper. In Section 3 we present axiomatic characterizations for the classes of *brutal contractions* and *bounded brutal contractions*. In Section 4 we show how to define a *bounded brutal contraction* from an *ensconcement-based severe withdrawal* and vice-versa. Furthermore we present an axiomatic characterization for the class of *ensconcement-based severe withdrawals*. Finally, in Section 5, we briefly

summarize the main contributions of the paper. In the Appendix we provide proofs for all the original results presented.

## 2. Background

### 2.1. Formal Preliminaries

We will assume a language  $\mathcal{L}$  that is closed under truth-functional operations and a consequence operator  $Cn$  for  $\mathcal{L}$ .  $Cn$  satisfies the standard Tarskian properties, namely inclusion ( $A \subseteq Cn(A)$ ), monotony (if  $A \subseteq B$ , then  $Cn(A) \subseteq Cn(B)$ ), and iteration ( $Cn(A) = Cn(Cn(A))$ ). It is supra-classical and compact, and satisfies deduction (if  $\beta \in Cn(A \cup \{\alpha\})$ , then  $(\alpha \rightarrow \beta) \in Cn(A)$ ).  $A \vdash \alpha$  will be used as an alternative notation for  $\alpha \in Cn(A)$ ,  $\vdash \alpha$  for  $\alpha \in Cn(\emptyset)$  and  $Cn(\alpha)$  for  $Cn(\{\alpha\})$ . Upper-case letters denote subsets of  $\mathcal{L}$ . Lower-case Greek letters denote elements of  $\mathcal{L}$ .  $\perp$  stands for an arbitrary contradiction.

A *well-ranked* preorder on a set  $X$  is a preorder such that every nonempty subset of  $X$  has a minimal element, and similarly an *inversely well-ranked* preorder on a set  $X$  is a preorder such that every nonempty subset of  $X$  has a maximal element. A total preorder on  $X$  is *bounded* if and only if it is both *well-ranked* and *inversely well-ranked*.<sup>1</sup>

### 2.2. Epistemic Entrenchment and Severe Withdrawals

We start by recalling, in the following definition, the concept of *epistemic entrenchment relation*.

DEFINITION 1 ([6, 7]). An ordering of epistemic entrenchment with respect to a belief set  $K$  is a binary relation  $\leq$  on  $\mathcal{L}$  which satisfies the following properties:

- (EE1) For all  $\alpha, \beta, \delta \in \mathcal{L}$ , if  $\alpha \leq \beta$  and  $\beta \leq \delta$  then  $\alpha \leq \delta$ . (Transitivity)
- (EE2) For all  $\alpha, \beta \in \mathcal{L}$ , if  $\alpha \vdash \beta$  then  $\alpha \leq \beta$ . (Dominance)
- (EE3) For all  $\alpha, \beta \in \mathcal{L}$ ,  $\alpha \leq \alpha \wedge \beta$  or  $\beta \leq \alpha \wedge \beta$ . (Conjunctiveness)
- (EE4) When  $K \not\vdash \perp$ ,  $\alpha \notin K$  iff  $\alpha \leq \beta$  for all  $\beta \in \mathcal{L}$ . (Minimality)
- (EE5) If  $\beta \leq \alpha$  for all  $\beta \in \mathcal{L}$ , then  $\vdash \alpha$ . (Maximality)

We shall denote the strict part and the symmetric part of  $\leq$  by  $<$  and  $=_{\leq}$ , respectively.

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<sup>1</sup>In [17] a preorder in these conditions is designated by *finite*, however we think the denomination *bounded* is more adequate.

Now we proceed to the presentation of the definition of the *severe withdrawals* (also known as mild contractions or Rott's contractions) which was introduced by Rott in [12].

DEFINITION 2 ([12]). Let  $K$  be a belief set and  $\leq$  be an epistemic entrenchment relation with respect to  $K$ . The  $\leq$ -based severe withdrawal on  $K$  is the operation  $\div_{\leq}$  defined, for any  $\alpha \in \mathcal{L}$ , by:

$$K \div_{\leq} \alpha = \begin{cases} \{\beta \in K : \alpha < \beta\}, & \text{if } \not\vdash \alpha \\ K, & \text{if } \vdash \alpha. \end{cases} \quad (R_{\leq})$$

An operation  $\div$  on  $K$  is a severe withdrawal if and only if there is an epistemic entrenchment relation  $\leq$  with respect to  $K$  such that, for all sentences  $\alpha \in \mathcal{L}$ ,  $K \div \alpha = K \div_{\leq} \alpha$ .

*Severe withdrawals* were axiomatically characterized independently by Rott and Pagnucco in [15] and by Fermé and Rodriguez in [5].

OBSERVATION 3 ([15]). Let  $K$  be a belief set and  $\div$  be a contraction function on  $K$ . Then  $\div$  is a severe withdrawal if and only if it satisfies the following postulates:

- ( $\div 1$ )  $K \div \alpha = Cn(K \div \alpha)$
- ( $\div 2$ )  $K \div \alpha \subseteq K$
- ( $\div 3$ ) If  $\alpha \notin K$  or  $\vdash \alpha$ , then  $K \subseteq K \div \alpha$
- ( $\div 4$ ) If  $\not\vdash \alpha$ , then  $\alpha \notin K \div \alpha$
- ( $\div 6$ ) If  $Cn(\alpha) = Cn(\beta)$ , then  $K \div \alpha = K \div \beta$
- ( $\div 9$ ) If  $\alpha \notin K \div \beta$ , then  $K \div \beta \subseteq K \div \alpha$

We note also that in [15, Proof of Lemma 1 (i)] it is shown that, in the presence of ( $\div 1$ ) to ( $\div 4$ ), the postulate ( $\div 9$ ) is equivalent to the following two postulates (taken together):

- ( $\div 7a$ ) If  $\not\vdash \alpha$ , then  $K \div \alpha \subseteq K \div (\alpha \wedge \beta)$
- ( $\div 8$ ) If  $\alpha \notin K \div (\alpha \wedge \beta)$ , then  $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$

Hence, in [15], it is also presented an alternative axiomatization of *severe withdrawals* consisting of the postulates ( $\div 1$ ) to ( $\div 4$ ), ( $\div 6$ ), ( $\div 7a$ ) and ( $\div 8$ ).

However, at this point it is worth mentioning that, as attested by the following observation, postulate ( $\div 6$ ) is redundant in both of the above recalled axiomatic characterizations.

OBSERVATION 4. Let  $K$  be a belief set and  $\div$  an operator that satisfies ( $\div 1$ ) to ( $\div 4$ ) and ( $\div 9$ ). Then  $\div$  satisfies ( $\div 6$ ).

### 2.3. Ensconcement and Brutal Contractions

We start by recalling the definition of *ensconcement* and some related concepts, which was originally proposed by Mary-Anne Williams [16–18].

DEFINITION 5 ([17]). An ensconcement is a pair  $(A, \preceq)$  where  $A$  is a belief base and  $\preceq$  is a transitive and connected relation on  $A$  that satisfies the following three conditions:<sup>2</sup>

- ( $\preceq 1$ ) If  $\beta \in A \setminus Cn(\emptyset)$ , then  $\{\alpha \in A : \beta \prec \alpha\} \not\vdash \beta$
- ( $\preceq 2$ ) If  $\not\vdash \alpha$  and  $\vdash \beta$ , then  $\alpha \prec \beta$ , for all  $\alpha, \beta \in A$
- ( $\preceq 3$ ) If  $\vdash \alpha$  and  $\vdash \beta$ , then  $\alpha \preceq \beta$ , for all  $\alpha, \beta \in A$

If  $\preceq$  is well-ranked/inversely well-ranked, then  $(A, \preceq)$  is a well-ranked/inversely well-ranked ensconcement. If  $\preceq$  is both well-ranked and inversely well-ranked then  $(A, \preceq)$  is a bounded ensconcement.

Williams has also introduced the concepts of *cut* and *proper cut*, which we recall in the following definition.

DEFINITION 6 ([17]). Let  $(A, \preceq)$  be an ensconcement.

- For all  $\alpha \in Cn(A)$  the cut of  $\alpha$ , denoted  $cut_{\preceq}(\alpha)$  is the following subset of  $A$ :

$$cut_{\preceq}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \prec \gamma\} \not\vdash \alpha\}.$$

- For all  $\alpha \in \mathcal{L}$  the proper cut of  $\alpha$ , denoted  $cut_{\prec}(\alpha)$  is the subset of  $A$  defined by:

$$cut_{\prec}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \preceq \gamma\} \not\vdash \alpha\}.$$

The following observation states that when  $\alpha$  is an explicit belief, its *proper cut* consists of the set of sentences which are strictly more ensconced than  $\alpha$ .

OBSERVATION 7 ([17]). If  $\alpha \in A$ ,  $cut_{\prec}(\alpha) = \{\beta \in A : \alpha \prec \beta\}$ .

We notice that, when  $\alpha \notin A$  the proper cut  $cut_{\prec}(\alpha)$  can be seen as the set formed by the sentences of  $A$  which may be considered to be *strictly better* than  $\alpha$ .

The following observation exposes some other properties of cuts and proper cuts.

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<sup>2</sup> $\alpha \prec \beta$  means  $\alpha \preceq \beta$  and  $\beta \not\preceq \alpha$ .  $\alpha =_{\preceq} \beta$  means  $\alpha \preceq \beta$  and  $\beta \preceq \alpha$ .

OBSERVATION 8. Let  $(A, \preceq)$  be a bounded *ensconcement* and  $\alpha, \beta \in Cn(A)$ .

- (a) Let  $\nVdash \beta$ . If  $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ , then  $cut_{\preceq}(\alpha) \subseteq cut_{\preceq}(\beta)$ .
- (b) If  $\vdash \beta$  and  $\nVdash \alpha$ , then  $cut_{\preceq}(\beta) \subset cut_{\preceq}(\alpha)$ .

Williams [17] has shown how an *epistemic entrenchment* can be defined from an *ensconcement*:

OBSERVATION 9 ([17]). Let  $(A, \preceq)$  be an *ensconcement* and let  $\leq_{\preceq}$  be the binary relation on  $\mathcal{L}$  defined by:  $\alpha \leq_{\preceq} \beta$  if and only if either

i)  $\alpha \notin Cn(A)$ , or ii)  $\alpha, \beta \in Cn(A)$  and  $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$ . Then:

- (a)  $\leq_{\preceq}$  is an *epistemic entrenchment* related to  $Cn(A)$ .
- (b)  $\preceq$  is *well-ranked* (inversely well-ranked, bounded) if and only if  $\leq_{\preceq}$  is *well-ranked* (inversely well-ranked, bounded).

Finally, we recall the definition of the *brutal contraction* which was introduced in [17] and is essentially based on the above presented notion of proper cut.

DEFINITION 10 ([17]). Let  $(A, \preceq)$  be an *ensconcement*. The  $\preceq$ -based brutal contraction on  $A$  is the operation  $-_{\preceq}$  such that:

$$A -_{\preceq} \alpha = \begin{cases} cut_{\prec}(\alpha), & \text{if } \nVdash \alpha \\ A, & \text{if } \vdash \alpha \end{cases}$$

An operation  $-$  on  $A$  is a brutal contraction if and only if there is an *ensconcement*  $(A, \preceq)$  such that for all sentences  $\alpha$ :  $A - \alpha = A -_{\preceq} \alpha$ .

### 3. Axiomatic Characterization of Brutal Contraction Functions

In this section we present an axiomatic characterization for the class of *brutal contractions*. Furthermore we also provide a representation theorem regarding the subclass of that class of functions formed by the *brutal contractions* that are based on a *bounded ensconcement*.

In the following theorem we present an axiomatic characterization of *brutal contraction functions*.

THEOREM 11. (Axiomatic characterization of brutal contraction functions) *Let  $A$  be a belief base. An operator  $-$  on  $A$  is a brutal contraction if and only if it satisfies the following postulates:*

- (Success) If  $\nVdash \alpha$ , then  $A - \alpha \nVdash \alpha$
- (Inclusion)  $A - \alpha \subseteq A$

**(Vacuity)** If  $A \not\vdash \alpha$ , then  $A \subseteq A - \alpha$

**(Failure)** If  $\vdash \alpha$ , then  $A - \alpha = A$

**(Relative Closure)**  $A \cap Cn(A - \alpha) \subseteq A - \alpha$

**(Strong Inclusion)** If  $A - \beta \not\vdash \alpha$ , then  $A - \beta \subseteq A - \alpha$

**(Uniform Behaviour)** If  $\beta \in A$ ,  $A \vdash \alpha$  and  $A - \alpha = A - \beta$ , then  $\alpha \in Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$

The first five postulates listed above are well known in the literature of belief change. *Strong Inclusion*, is presented in [5, 15] as meaning that if  $\alpha$  is not deducible from the set that results of contracting  $A$  by  $\beta$  then anything given up in removing  $\alpha$  from  $A$  should also be given up when removing  $\beta$  from  $A$ . *Uniform Behaviour* asserts that if a sentence  $\alpha$  that is deducible from  $A$  is such that the result of its contraction from  $A$  coincides with the result of contracting  $A$  by a sentence which is (explicitly) present in  $A$  then  $\alpha$  should be deducible from the union of the set of all the sentences fulfilling that property with the set that results of contracting  $A$  by  $\alpha$ . We note that this postulate is trivial when  $\alpha \in A$ .

At this point it is worth to compare the above representation theorem with the axiomatic characterization for the *severe withdrawals* which results of combining Observations 3 and 4. We note that the postulates of *relative closure*, *inclusion*, *failure* and *vacuity* (together), *success* and *strong inclusion* can be seen as the analogues in the belief bases setting of the postulates  $(\div 1)$ ,  $(\div 2)$ ,  $(\div 3)$ ,  $(\div 4)$  and  $(\div 9)$ . Thus, the main difference among the two axiomatizations is the presence of the postulate of *uniform behaviour* in the characterization of *brutal contractions*. In this regard we recall that *uniform behaviour* holds trivially when  $\alpha \in A$ . Therefore, this postulate can be seen as the property that captures the behaviour of *brutal contractions* by implicit sentences (a kind of contraction that does not occur in the belief sets setting since, in that context all beliefs are explicit).

The following observation lists some other properties that are satisfied by a brutal contraction function.

**OBSERVATION 12.** *Let  $A$  be a belief base and  $-$  be an operator on  $A$  that satisfies success, inclusion, vacuity, failure, relative closure and strong inclusion. Then  $-$  satisfies:*

- (a) If  $\alpha \in A \setminus A - \beta$ , then  $A - \beta \subseteq A - \alpha$ .
- (b) If  $\not\vdash \alpha$ , then  $A - \alpha \subseteq A - (\alpha \wedge \beta)$ . (7a)
- (c) If  $A - (\alpha \wedge \beta) \not\vdash \alpha$ , then  $A - (\alpha \wedge \beta) \subseteq A - \alpha$ . (Conjunctive Inclusion)
- (d) If  $\not\vdash \alpha, \not\vdash \beta$  and  $A - \alpha \vdash \beta$ , then  $A - \beta \subseteq A - \alpha$ .

- (e) If  $\not\vdash \alpha$  and  $\alpha \in A - \beta$ , then  $A - \alpha \subset A - \beta$ .
- (f) If  $A - \alpha \subset A - \beta$ , then  $A - \beta \vdash \alpha$ .
- (g)  $A - \alpha \subseteq A - \beta$  or  $A - \beta \subseteq A - \alpha$ . (*Linearity*)
- (h) If  $\not\vdash \alpha, \not\vdash \beta$  then  $A - \beta \not\vdash \alpha$  or  $A - \alpha \not\vdash \beta$ . (*Expulsiveness*)
- (i) If  $\vdash \alpha \leftrightarrow \beta$ , then  $A - \alpha = A - \beta$ . (*Extensionality*)

A brief comment concerning a couple of arguably undesirable properties of the above list is in order. *Expulsiveness* was first presented in [11, p. 102] as a highly implausible property of belief contraction, since according to it two unrelated sentences influence the result of the contraction by each other. Other one of the above listed postulates that also suffers from this same excessive strength is *linearity*, which was originally presented in [5, 15]. Nevertheless, Rott and Pagnucco [13, 15] argue that the concept of severe withdrawal (a contraction function that satisfies the two mentioned postulates) is still interesting and well-motivated.

### 3.1. Bounded Brutal Contraction Functions

In this subsection we introduce the *bounded brutal contractions* and obtain an axiomatic characterization for that class of functions.

**DEFINITION 13.** Let  $A$  be a belief base. An operation  $-$  is a bounded brutal contraction on  $A$  if and only if it is a brutal contraction based on a bounded ensconcement.

We introduce the following postulates:

- (Upper Bound)** For every non-empty set  $X \subseteq A$  of nontautological formulae, there exists  $\alpha \in X$  such that  $A - \beta \subseteq A - \alpha$  for all  $\beta \in X$
- (Lower Bound)** For every non-empty set  $X \subseteq A$  of nontautological formulae, there exists  $\alpha \in X$  such that  $A - \alpha \subseteq A - \beta$  for all  $\beta \in X$
- (Clustering)** If  $\beta \in A$ , then there exists  $\alpha \in A \cup Cn(\emptyset)$  such that  $A - \alpha = A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\}$

*Upper Bound* (respectively *Lower Bound*) states that every non-empty set of nontautological formulae of  $A$  contains an element which is such that the result of contracting  $A$  by that sentence is a superset (respectively a subset) of any set which results of contracting  $A$  by one of the remaining sentences of that set. *Clustering* asserts that for any sentence  $\beta$  in  $A$  there exists some sentence  $\alpha$  in  $A \cup Cn(\emptyset)$  such that the result of the contraction of  $\alpha$  from  $A$  is the set consisting of the union of the result of contracting  $A$  by  $\beta$  with the set formed by all the sentences of  $A$  which are such that



the result of contracting it from  $A$  coincides with the result of contracting  $A$  by  $\beta$ .

The two following observations present some interrelations among the above proposed postulates and some of the postulates included in the axiomatic characterization that was obtained for the class of *brutal contraction*.

OBSERVATION 14. *Let  $A$  be a belief base and  $-$  be an operator on  $A$  that satisfies success, inclusion, failure, relative closure, strong inclusion and lower bound. Then  $-$  satisfies clustering.*

OBSERVATION 15. *Let  $A$  be a belief base and  $-$  an operator on  $A$  that satisfies failure, success, strong inclusion and clustering. Then  $-$  satisfies uniform behaviour.*

We are now in a position to present an axiomatic characterization for the class of *bounded brutal contractions*.

THEOREM 16. (Axiomatic characterization of bounded brutal contraction functions) *Let  $A$  be a belief base. An operator  $-$  on  $A$  is a bounded brutal contraction on  $A$  if and only if it satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion, lower bound and upper bound.*

The following observation asserts that for any non-tautological sentence  $\alpha$  which is deducible from  $A$  it holds that the result of contracting  $A$  by  $\alpha$  coincides with the result of the contraction of  $A$  by some sentence explicitly included in  $A$ .

OBSERVATION 17. *Let  $A$  be a belief base and  $-$  be an operator on  $A$  that satisfies success, inclusion, failure, relative closure, strong inclusion and lower bound. Then for all  $\alpha \in Cn(A) \setminus Cn(\emptyset)$  there exists  $\beta \in A$  such that  $A - \alpha = A - \beta$ .*

#### 4. Relation Between Bounded Brutal Contraction and Ensconcement-based Severe Withdrawal

In this section we will define and axiomatically characterize a particular kind of *severe withdrawals* which we will show to be the contraction functions that correspond to the *bounded brutal contractions* in the context of belief set contractions.

We start by noticing that, given a *bounded ensconcement*  $(A, \preceq)$ , we can combine Observation 9 and Definition 2 in order to obtain the contraction function on the belief set  $Cn(A)$  that is formally introduced in the following definition.

DEFINITION 18. Let  $(A, \preceq)$  be a bounded ensconcement. An operation  $\div$  on  $Cn(A)$  is an ensconcement-based withdrawal related to  $(A, \preceq)$  if and only if  $Cn(A) \div \alpha = Cn(A) \div_{\leq_{\preceq}} \alpha$ , where  $\leq_{\preceq}$  is the epistemic entrenchment with respect to  $Cn(A)$  presented in Observation 9 and  $\div_{\leq_{\preceq}}$  is the  $\leq_{\preceq}$ -based severe withdrawal on  $Cn(A)$  defined by  $(R_{\preceq})$ .

Comparing the above definition with Definitions 10 and 13 it becomes clear that there is a strong interrelation among the *ensconcement-based severe withdrawals* and the (bounded) *brutal contractions*. That interrelation is explicitly presented in the two following theorems:

THEOREM 19. Let  $(A, \preceq)$  be a bounded ensconcement,  $-$  be the  $\preceq$ -based brutal contraction, and  $\div_{\leq_{\preceq}}$  be the ensconcement-based severe withdrawal related to  $(A, \preceq)$ , then  $A - \alpha = (Cn(A) \div_{\leq_{\preceq}} \alpha) \cap A$ .

THEOREM 20. Let  $(A, \preceq)$  be a bounded ensconcement,  $-$  be the  $\preceq$ -based brutal contraction, and  $\div_{\leq_{\preceq}}$  be the ensconcement-based severe withdrawal related to  $(A, \preceq)$ , then  $Cn(A) \div_{\leq_{\preceq}} \alpha = Cn(A - \alpha)$ .

Given a bounded ensconcement  $(A, \preceq)$ , these two theorems expose how the  $\preceq$ -based brutal contraction on  $A$  can be defined from the *ensconcement-based withdrawal* related to  $(A, \preceq)$  and, vice-versa, how the latter can be defined by means of the former.

#### 4.1. Axiomatic Characterization of Ensconcement-Based Severe Withdrawals

In this subsection we will present an axiomatic characterization for the class of *ensconcement-based severe withdrawals*. To do that we must start by introducing the following postulate:

**(Base-reduction)** If  $Cn(A) \div \alpha \vdash \beta$ , then  $(Cn(A) \div \alpha) \cap A \vdash \beta$

This postulate essentially states that the result of contracting the belief set  $Cn(A)$  by any sentence  $\alpha$  coincides with the logical closure of some subset of  $A$ . Indeed, it is not hard to see that *base-reduction* is equivalent to:  $\forall \alpha \exists A' \subseteq A : Cn(A') = Cn(A) \div \alpha$  (which is quite similar to the postulate *finitude* proposed by Hansson in [10]).

The following observation relates a contraction by an implicit sentence with a contraction by an explicit sentence.

OBSERVATION 21. Let  $\div$  be an operator on  $Cn(A)$  that satisfies  $(\div 1)$ ,  $(\div 2)$ ,  $(\div 4)$ ,  $(\div 9)$ , *base-reduction* and *lower bound*, then for all  $\alpha \in Cn(A) \setminus Cn(\emptyset)$  there exists  $\beta \in A$  such that  $Cn(A) \div \alpha = Cn(A) \div \beta$ .

We are now in a position to present the following axiomatic characterization for the *ensconcement-based severe withdrawals*.

**THEOREM 22.** (Axiomatic characterization of ensconcement-based withdrawals) *Let  $A$  be a belief base. An operator  $\div$  on  $Cn(A)$  satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , base-reduction, upper bound and lower bound if and only if there exists a bounded ensconcement  $(A, \preceq)$  such that  $\div$  is an ensconcement-based withdrawal related to  $(A, \preceq)$ .*

Theorems 19 and 20 expose how a base contraction function can be defined from a belief set contraction function and, vice-versa. Combining those two results with the axiomatic characterizations presented in Theorems 16 and 22 we can obtain the following results which highlight the correspondence among sets of postulates for base contraction and sets of postulates for belief set contraction.

**COROLLARY 23.** *An operator  $-$  on  $A$  satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion, upper bound and lower bound if and only if there exists an operator  $\div$  on  $Cn(A)$  that satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , base-reduction, upper bound and lower bound such that:  $A - \alpha = Cn(A \div \alpha) \cap A$ .*

**COROLLARY 24.** *An operator  $\div$  on  $Cn(A)$  satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , base-reduction, upper bound and lower bound if and only if there exists an operator  $-$  on  $A$  that satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion, upper bound and lower bound such that:  $Cn(A) \div \alpha = Cn(A - \alpha)$ .*

The two following observations consist of a slight refinement of the right to left part of Corollary 24. These results specify more precisely which properties of the belief base contraction are needed in order to assure that the belief set contraction obtained from it as exposed in Theorem 20 satisfies certain postulates.

**OBSERVATION 25.** *Let  $A$  be a belief base and  $-$  be an operator on  $A$  that satisfies success, inclusion, vacuity, failure, relative closure and strong inclusion. If  $\div$  is an operator on  $Cn(A)$  defined by  $Cn(A) \div \alpha = Cn(A - \alpha)$  then  $\div$  satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$  and base-reduction.*

**OBSERVATION 26.** *Let  $A$  be a belief base and  $-$  be an operator on  $A$  that satisfies success, inclusion, failure, relative closure, upper bound, lower bound and strong inclusion. If  $\div$  is an operator on  $Cn(A)$  defined by  $Cn(A) \div \alpha = Cn(A - \alpha)$  then  $\div$  satisfies upper bound and lower bound.*

## 5. Conclusion

We have presented two axiomatic characterizations for *brutal contractions* one for *brutal contractions* based on a general ensconcement relation and another one for the particular case of *brutal contractions* that are based on bounded enscongements. We have also introduced and axiomatically characterized the class of *ensconcement-based severe withdrawals* which is formed by the *severe withdrawals* that are based on epistemic entrenchment relations which are obtained from an ensconcement relation using the construction proposed by Mary-Anne Williams. Some results were presented concerning the interrelation among the classes of *bounded brutal contractions* and of *ensconcement-based severe withdrawals*. Finally we presented some results relating base contraction postulates and belief set contraction postulates by means of explicit definitions of belief set contractions from base contractions and vice-versa.

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## Appendix: Proofs

LEMMA 1 ([4, Lemma 11]).

- (a) If  $\vdash \alpha$ , then  $cut_{\prec}(\alpha) = \emptyset$ .
- (b) If  $\nvdash \alpha$ ,  $cut_{\prec}(\alpha) \nvdash \alpha$ .
- (c) If  $A \nvdash \alpha$ ,  $cut_{\prec}(\alpha) = A$ .
- (d) If  $\beta \vdash \alpha$ , then  $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ .
- (e) If  $\vdash \alpha \leftrightarrow \beta$ , then  $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$ .
- (f) If  $\alpha \preceq \beta$ , then  $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$ .
- (g) If  $\alpha \prec \beta$ , then  $cut_{\prec}(\alpha) \vdash \beta$  and  $cut_{\prec}(\beta) \nvdash \alpha$ .
- (h) If  $\alpha \prec \beta$ , then  $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$ .

- (i) If  $\beta =_{\preceq} \alpha$ , then  $\text{cut}_{\prec}(\alpha \wedge \beta) = \text{cut}_{\prec}(\alpha) = \text{cut}_{\prec}(\beta)$ .
- (j) If  $\text{cut}_{\prec}(\alpha) \vdash \beta$ , then  $\text{cut}_{\prec}(\alpha \wedge \beta) = \text{cut}_{\prec}(\alpha)$ .
- (k) If  $\text{cut}_{\prec}(\alpha) \nvdash \beta$ , then  $\text{cut}_{\prec}(\alpha \wedge \beta) = \text{cut}_{\prec}(\beta)$ .

LEMMA 2. If  $\alpha, \beta \in A$ , then  $\alpha =_{\preceq} \beta$  if and only if  $\text{cut}_{\prec}(\alpha) = \text{cut}_{\prec}(\beta)$ .

PROOF. If  $\vdash \alpha$ , then from  $(\preceq 2)$  it follows that  $\vdash \beta$ . Hence, the proof follows from  $(\preceq 3)$  and Lemma 1 (a). Assume now that  $\nvdash \alpha$  and consequently that  $\nvdash \beta$ . From left to right it follows from Lemma 1 (i). For the other direction: Let  $\alpha, \beta \in A$ , if  $\alpha \prec \beta$ , then by Lemma 1 (g)  $\text{cut}_{\prec}(\alpha) \vdash \beta$  and so  $\text{cut}_{\prec}(\beta) \vdash \beta$  which contradicts Lemma 1 (b). Due to the symmetry of the case we may conclude that  $\beta \not\prec \alpha$ . Since  $\alpha \not\prec \beta, \beta \not\prec \alpha$ , and  $\preceq$  is connected, we can conclude that  $\alpha \preceq \beta$  and  $\beta \preceq \alpha$ . ■

LEMMA 3 ([15, Lemma 2]). Let  $K$  be a belief set. If  $\div$  is a severe withdrawal on  $K$ , then  $\div$  satisfies the following postulates:

**(Linearity)** Either  $K \div \alpha \subseteq K \div \beta$  or  $K \div \beta \subseteq K \div \alpha$ .

**(Expulsiveness)** If  $\nvdash \alpha$  and  $\nvdash \beta$ , then either  $\alpha \notin K \div \beta$  or  $\beta \notin K \div \alpha$ .

LEMMA 4 ([15, Observation 19(ii)]). Let  $K$  be a belief set. If  $\div$  is a severe withdrawal on  $K$ , then  $\div$  can be represented as an entrenchment-based withdrawal where the relation  $\leq$  on which  $\div$  is based is obtained by (Def  $\leq$  from  $\div$ )  $\alpha \leq \beta$  if and only if  $\alpha \notin K \div \beta$  or  $\vdash \beta$  and  $\leq$  satisfies (EE1) to (EE5).

LEMMA 5. Let  $(A, \preceq)$  be an ensconcement and  $\beta \in \text{Cn}(A)$ . If  $\alpha \in \text{cut}_{\preceq}(\beta)$ ,  $\gamma \in A$  and  $\alpha \preceq \gamma$ , then  $\gamma \in \text{cut}_{\preceq}(\beta)$ .

PROOF. Let  $(A, \preceq)$  be an ensconcement  $\alpha \in \text{cut}_{\preceq}(\beta)$ ,  $\gamma \in A$  and  $\alpha \preceq \gamma$ . From  $\alpha \in \text{cut}_{\preceq}(\beta)$  it follows that  $\{\delta \in A : \alpha \prec \delta\} \nvdash \beta$ . Hence, since  $\alpha \preceq \gamma$ , it follows that  $\{\delta \in A : \gamma \prec \delta\} \nvdash \beta$ . Therefore  $\gamma \in \text{cut}_{\preceq}(\beta)$ . ■

LEMMA 6. Let  $(A, \preceq)$  be a bounded ensconcement and  $\text{cut}_{\preceq}(\alpha) \neq \emptyset$ . Then there exists  $\beta \in \text{cut}_{\preceq}(\alpha)$  such that  $\text{cut}_{\preceq}(\beta) = \text{cut}_{\preceq}(\alpha)$ .

PROOF. Let  $(A, \preceq)$  be a bounded ensconcement and  $\text{cut}_{\preceq}(\alpha) \neq \emptyset$ . Then, there exists  $\beta \in \text{cut}_{\preceq}(\alpha)$  such that  $\beta \preceq \gamma$  for all  $\gamma \in \text{cut}_{\preceq}(\alpha)$ . Therefore  $\text{cut}_{\preceq}(\beta) \subseteq \text{cut}_{\preceq}(\alpha)$ . If  $\text{cut}_{\preceq}(\beta) \subset \text{cut}_{\preceq}(\alpha)$ , then there exists  $\psi \in \text{cut}_{\preceq}(\alpha) \setminus \text{cut}_{\preceq}(\beta)$ . Hence  $\psi \prec \beta$  and  $\psi \in \text{cut}_{\preceq}(\alpha)$ . Contradiction. Therefore  $\text{cut}_{\preceq}(\beta) = \text{cut}_{\preceq}(\alpha)$ . ■

LEMMA 7. Let  $(A, \preceq)$  be a bounded ensconcement and  $\alpha \in \text{Cn}(A)$ . Then  $\text{cut}_{\preceq}(\alpha) \vdash \alpha$ .

PROOF. Let  $(A, \preceq)$  be a bounded ensconcement and  $\alpha \in Cn(A)$ . We will prove by cases:

Case 1)  $cut_{\preceq}(\alpha) = A$ . Trivial.

Case 2)  $cut_{\preceq}(\alpha) = \emptyset$ .  $(A, \preceq)$  is a bounded ensconcement, hence there exists  $\beta \in A$  such that  $\gamma \preceq \beta$  for all  $\gamma \in A$ .  $\beta \notin cut_{\preceq}(\alpha)$ . Thus,  $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$ . Hence  $\emptyset \vdash \alpha$ .

Case 3)  $cut_{\preceq}(\alpha) \neq A$  and  $cut_{\preceq}(\alpha) \neq \emptyset$ .  $A \setminus cut_{\preceq}(\alpha) \neq \emptyset$ . From  $cut_{\preceq}(\alpha) \neq \emptyset$  it follows, from Lemma 6, that there exists  $\beta \in cut_{\preceq}(\alpha)$  such that  $cut_{\preceq}(\beta) = cut_{\preceq}(\alpha)$ , where (as seen in the proof of Lemma 6)  $\beta \preceq \gamma$  for all  $\gamma \in cut_{\preceq}(\alpha)$ . Let  $\delta \in A \setminus cut_{\preceq}(\alpha)$  be such that  $\gamma \preceq \delta$  for all  $\gamma \in A \setminus cut_{\preceq}(\alpha)$ . From  $\delta \notin cut_{\preceq}(\alpha)$  it follows that  $X = \{\gamma \in A : \delta \prec \gamma\} \vdash \alpha$ . Let  $\theta \in X$ . Then  $\theta \in A$  and  $\delta \prec \theta$ . Then  $\theta \notin A \setminus cut_{\preceq}(\alpha)$ , from which follows that  $\theta \in cut_{\preceq}(\alpha)$ . Hence  $\beta \preceq \theta$ , from which follows that  $X \subseteq \{\gamma \in A : \beta \preceq \gamma\}$  and, since  $X \vdash \alpha$ , it follows that  $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$ . It remains to prove that  $\{\gamma \in A : \beta \preceq \gamma\} \subseteq cut_{\preceq}(\alpha)$ , that follows trivially from Lemma 5. ■

LEMMA 8. Let  $(A, \preceq)$  be a bounded ensconcement and  $\alpha, \beta \in Cn(A)$ , then:

- (a) If  $\nvdash \alpha$ ,  $\delta \notin cut_{\prec}(\alpha)$  and  $\gamma \preceq \delta$  for all  $\gamma \in A \setminus cut_{\prec}(\alpha)$ , then  $cut_{\preceq}(\alpha) = \{\theta \in A : \delta \preceq \theta\}$ .
- (b) If  $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta)$ , then  $cut_{\preceq}(\alpha) \subset cut_{\preceq}(\beta)$ .
- (c) If  $cut_{\prec}(\alpha) \neq \emptyset$  and  $cut_{\preceq}(\alpha) \subset cut_{\preceq}(\beta)$ , then  $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta)$ .

PROOF.

(a): Let  $\nvdash \alpha$ ,  $\delta \notin cut_{\prec}(\alpha)$  and  $\gamma \preceq \delta$  for all  $\gamma \in A \setminus cut_{\prec}(\alpha)$ , we will prove that  $cut_{\preceq}(\alpha) = \{\theta \in A : \delta \preceq \theta\}$  by double inclusion.

( $\subseteq$ ) Let  $\beta \in cut_{\preceq}(\alpha)$ . Assume by *reductio* that  $\beta \prec \delta$ . Hence,  $\{\theta \in A : \beta \prec \theta\} \vdash \alpha$ . It follows that  $\{\theta \in A : \delta \preceq \theta\} \nvdash \alpha$ . Therefore  $\delta \in cut_{\prec}(\alpha)$ . Contradiction. Hence,  $\delta \preceq \beta$  and so  $cut_{\preceq}(\alpha) \subseteq \{\theta \in A : \delta \preceq \theta\}$ .

( $\supseteq$ ) We will prove first that  $\delta \in cut_{\preceq}(\alpha)$ . Assume by *reductio* that  $\delta \notin cut_{\preceq}(\alpha)$ . Hence  $\{\theta \in A : \delta \prec \theta\} \vdash \alpha$ . Let  $X = \{\theta \in A : \delta \prec \theta\}$ . It follows, from  $\nvdash \alpha$  that  $X \neq \emptyset$ . Hence, since  $\preceq$  is a bounded ensconcement, there exists a minimal element  $\psi$  of  $X$ . It follows that  $\{\theta \in A : \psi \preceq \theta\} \vdash \alpha$ . Hence  $\psi \notin cut_{\prec}(\alpha)$ , from which follows that  $\psi \preceq \delta$ . Contradiction, since  $\psi \in X$ . Therefore  $\delta \in cut_{\preceq}(\alpha)$ , from which follows that  $\{\theta \in A : \delta \preceq \theta\} \subseteq cut_{\preceq}(\alpha)$ .

(b): Let  $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta)$ . There are only two cases to consider:

Case 1)  $\vdash \alpha$  and  $\nvdash \beta$ . Hence,  $cut_{\preceq}(\alpha) = \emptyset$  and, since  $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta)$ ,  $cut_{\prec}(\beta) \neq \emptyset$ . Therefore  $cut_{\preceq}(\beta) \neq \emptyset$ . Hence  $cut_{\preceq}(\alpha) \subset cut_{\preceq}(\beta)$ .

Case 2)  $\nvdash \alpha$  and  $\nvdash \beta$ .  $A \setminus cut_{\prec}(\alpha) \neq \emptyset$ , since  $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta) \subseteq A$ . Then, since the ensconcement is bounded, there exists  $\delta \in A \setminus cut_{\prec}(\alpha)$  such that  $\gamma \preceq \delta$  for all  $\gamma \in A \setminus cut_{\prec}(\alpha)$ . Then, by (a),  $cut_{\preceq}(\alpha) = \{\theta \in A : \delta \preceq \theta\}$ .

$A \setminus \text{cut}_{\prec}(\beta) \neq \emptyset$  (because  $\beta \in \text{Cn}(A)$ ). Then there exists  $\delta' \in A \setminus \text{cut}_{\prec}(\beta)$  such that  $\gamma \preceq \delta'$  for all  $\gamma \in A \setminus \text{cut}_{\prec}(\beta)$ . Then, by (a),  $\text{cut}_{\preceq}(\beta) = \{\theta \in A : \delta' \preceq \theta\}$ . Let  $\epsilon \in \text{cut}_{\prec}(\beta) \setminus \text{cut}_{\prec}(\alpha)$ . Hence  $\epsilon \in A \setminus \text{cut}_{\prec}(\alpha)$  and  $\epsilon \notin A \setminus \text{cut}_{\prec}(\beta)$ . From which follows that  $\epsilon \preceq \delta$  and  $\delta' \prec \epsilon$ . Hence  $\delta' \prec \delta$ . Hence  $\text{cut}_{\preceq}(\alpha) \subset \text{cut}_{\preceq}(\beta)$ .

**(c):** Let  $\text{cut}_{\prec}(\alpha) \neq \emptyset$  and  $\text{cut}_{\preceq}(\alpha) \subset \text{cut}_{\preceq}(\beta)$  and assume by *reductio* that  $\text{cut}_{\prec}(\alpha) \not\subseteq \text{cut}_{\prec}(\beta)$ . Hence there exists  $\gamma \in A$  such that  $\gamma \in \text{cut}_{\prec}(\alpha)$  and  $\gamma \notin \text{cut}_{\prec}(\beta)$ . From definition of proper cut it follows that  $\{\psi \in A : \gamma \preceq \psi\} \not\vdash \alpha$  and  $\{\psi \in A : \gamma \preceq \psi\} \vdash \beta$ . On the other hand, since  $\text{cut}_{\preceq}(\alpha) \subset \text{cut}_{\preceq}(\beta)$  there exists  $\delta \in A$  such that  $\delta \in \text{cut}_{\preceq}(\beta)$  and  $\delta \notin \text{cut}_{\preceq}(\alpha)$ . Hence, by definition of cut, it follows that  $\{\psi \in A : \delta \prec \psi\} \not\vdash \beta$  and  $\{\psi \in A : \delta \prec \psi\} \vdash \alpha$ . From  $\{\psi \in A : \gamma \preceq \psi\} \not\vdash \alpha$  and  $\{\psi \in A : \delta \prec \psi\} \vdash \alpha$  it follows that  $\delta \prec \gamma$ . Hence, from  $\{\psi \in A : \gamma \preceq \psi\} \vdash \beta$  it follows that  $\{\psi \in A : \delta \prec \psi\} \vdash \beta$ . Contradiction. Hence  $\text{cut}_{\prec}(\alpha) \subseteq \text{cut}_{\prec}(\beta)$ . It remains to show that  $\text{cut}_{\prec}(\alpha) \neq \text{cut}_{\prec}(\beta)$ . From  $\text{cut}_{\prec}(\alpha) \neq \emptyset$  it follows that  $\text{cut}_{\preceq}(\alpha) \neq \emptyset$  and  $\text{cut}_{\preceq}(\beta) \neq \emptyset$ . Let  $\delta$  be a minimal member of  $\text{cut}_{\preceq}(\alpha)$  and  $\delta'$  be a minimal element of  $\text{cut}_{\preceq}(\beta)$ . According to Lemma 5  $\text{cut}_{\preceq}(\alpha) = \{\gamma \in A : \delta \preceq \gamma\}$  and  $\text{cut}_{\preceq}(\beta) = \{\gamma \in A : \delta' \preceq \gamma\}$ . Hence, since  $\text{cut}_{\preceq}(\alpha) \subset \text{cut}_{\preceq}(\beta)$ ,  $\delta' \prec \delta$ . Assume by *reductio* that  $\delta \in \text{cut}_{\prec}(\alpha)$ . Then  $\{\gamma \in A : \delta \preceq \gamma\} \not\vdash \alpha$ . Contradiction (by Lemma 7). Assume by *reductio* that  $\delta \notin \text{cut}_{\prec}(\beta)$ . Hence  $\{\gamma \in A : \delta \preceq \gamma\} \vdash \beta$ . Therefore,  $\{\gamma \in A : \delta' \prec \gamma\} \vdash \beta$ . Contradiction, since  $\delta' \in \text{cut}_{\preceq}(\beta)$ . ■

**PROOF OF OBSERVATION 4.** Let  $K$  be a belief set and  $\div$  an operator that satisfies  $(\div 1)$  to  $(\div 4)$  and  $(\div 9)$ . Assume that  $\text{Cn}(\alpha) = \text{Cn}(\beta)$ . We intent to prove that  $K - \alpha = K - \beta$ . Assume first that  $\vdash \alpha \wedge \beta$ . Hence  $\vdash \alpha$  and  $\vdash \beta$ . Therefore, by  $(\div 2)$  and  $(\div 3)$ , it follows that  $K - \alpha = K - \beta = K$ . Assume now that  $\not\vdash \alpha \wedge \beta$ . From  $\text{Cn}(\alpha) = \text{Cn}(\beta)$  it follows that  $\not\vdash \alpha$  and  $\not\vdash \beta$  furthermore it follows that  $\beta \in \text{Cn}(\alpha)$  and  $\alpha \in \text{Cn}(\beta)$ . From  $(\div 4)$  and  $(\div 1)$  it follows that  $K - \alpha \not\vdash \alpha$  and  $K - \beta \not\vdash \beta$ . Therefore  $\beta \notin K - \alpha$  and  $\alpha \notin K - \beta$ . Hence, by  $(\div 9)$ , it follows that  $K \div \alpha = K \div \beta$ . ■

**PROOF OF OBSERVATION 8.**

**(a)** Let  $\not\vdash \beta$  and  $\text{cut}_{\prec}(\alpha) \subseteq \text{cut}_{\prec}(\beta)$  We will prove by cases:

Case 1)  $\text{cut}_{\prec}(\alpha) \subset \text{cut}_{\prec}(\beta)$ . It follows trivially by Lemma 8 (b).

Case 2)  $\text{cut}_{\prec}(\alpha) = \text{cut}_{\prec}(\beta)$ .

Case 2.1)  $\text{cut}_{\prec}(\alpha) \neq \emptyset$ . Assume by *reductio* that  $\text{cut}_{\preceq}(\alpha) \not\subseteq \text{cut}_{\preceq}(\beta)$ . Hence  $\text{cut}_{\preceq}(\beta) \subset \text{cut}_{\preceq}(\alpha)$ . It follows, from Lemma 8 (c) that  $\text{cut}_{\prec}(\beta) \subset \text{cut}_{\prec}(\alpha)$ . Contradiction.

Case 2.2)  $\text{cut}_{\prec}(\alpha) = \emptyset$ . If  $\text{cut}_{\preceq}(\alpha) = \emptyset$  trivial. Assume now that  $\text{cut}_{\preceq}(\alpha) \neq \emptyset$ . Since  $\preceq$  is inversely well-ranked, there exists  $\gamma \in A$  such that  $\psi \preceq \gamma$  for all  $\psi \in A$ . Let  $\theta \in \text{cut}_{\preceq}(\alpha)$ . Hence  $\theta =_{\preceq} \gamma$ , otherwise  $\gamma \in \text{cut}_{\prec}(\alpha)$ . Therefore

$\{\psi \in A : \theta \prec \psi\} = \emptyset$ . Hence, since  $\nVdash \beta$ ,  $\theta \in \text{cut}_{\preceq}(\beta)$ .

(b) Let  $\vdash \beta$  and  $\nVdash \alpha$ . By definition of cut it follows that  $\text{cut}_{\preceq}(\beta) = \emptyset$ . Since  $\preceq$  is inversely well-ranked, there exists  $\gamma \in A$  such that  $\psi \preceq \gamma$  for all  $\psi \in A$ . Hence,  $\{\psi \in A : \gamma \prec \psi\} = \emptyset$ . Therefore, since  $\nVdash \alpha$ , it follows that  $\gamma \in \text{cut}_{\preceq}(\alpha)$ . Hence,  $\text{cut}_{\preceq}(\beta) \subset \text{cut}_{\preceq}(\alpha)$ . ■

PROOF OF THEOREM 11. Throughout this proof we will often use Observation 12. However, this is not an issue because the result that is proven here is not used in the proof of Observation 12 that is presented immediately after this one.

*From Brutal Contraction to Postulates:*

**Success** Let  $\nVdash \alpha$  and assume by *reductio* that  $A - \alpha \vdash \alpha$ . Then it follows from the definition of  $-$  that  $\text{cut}_{\prec}(\alpha) \vdash \alpha$ . Contradiction by Lemma 1 (b).

**Inclusion** and **Failure** follow trivially.

**Vacuity** follows trivially from Lemma 1 (c).

**Relative Closure** If  $\vdash \alpha$ , trivial from *failure*. Let  $\nVdash \alpha$  and assume by *reductio* that  $\beta \in A, A - \alpha \vdash \beta$  and  $\beta \notin A - \alpha$ . It follows from the definition of  $-$  that  $\text{cut}_{\prec}(\alpha) \vdash \beta$  and  $\beta \notin \text{cut}_{\prec}(\alpha)$ . From  $\beta \notin \text{cut}_{\prec}(\alpha)$  it follows that  $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$ . If  $\vdash \beta$ , then it follows from  $(\preceq 2)$  that  $\vdash \alpha$ . Contradiction.

Assume that  $\nVdash \beta$ . Since  $\text{cut}_{\prec}(\alpha) \vdash \beta$  it follows that  $\text{cut}_{\prec}(\alpha) \neq \emptyset$ . Let  $\delta \in \text{cut}_{\prec}(\alpha)$ .

$\delta, \beta \in A$  and  $\preceq$  is a connected relation, then  $\delta \preceq \beta$  or  $\beta \prec \delta$ .

If  $\delta \preceq \beta$ , then  $\{\gamma \in A : \beta \preceq \gamma\} \subseteq \{\gamma \in A : \delta \preceq \gamma\}$ . Contradiction, since  $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$  and  $\delta \in \text{cut}_{\prec}(\alpha)$ .

If  $\beta \prec \delta$ , and since  $\delta$  is an arbitrary element of  $\text{cut}_{\prec}(\alpha)$ , then for all  $\gamma \in \text{cut}_{\prec}(\alpha)$ , we have that  $\beta \prec \gamma$ . Contradiction, since  $\text{cut}_{\prec}(\alpha) \vdash \beta$  contradicts  $(\preceq 1)$ .

**Strong Inclusion** Let  $\vdash \beta$ , then by *failure* it follows that  $A - \beta = A$ . Hence  $A \nVdash \alpha$  and, by *vacuity* and *inclusion*, it follows that  $A - \alpha = A$ . Therefore  $A - \beta \subseteq A - \alpha$ .

Assume now that  $\nVdash \beta$ . Let  $A - \beta \nVdash \alpha$ , it follows by the definition of  $-$  that  $\text{cut}_{\prec}(\beta) \nVdash \alpha$ . By Lemma 1 (k) it follows that  $\text{cut}_{\prec}(\alpha \wedge \beta) = \text{cut}_{\prec}(\alpha)$ . By Lemma 1 (d)  $\text{cut}_{\prec}(\beta) \subseteq \text{cut}_{\prec}(\alpha \wedge \beta)$ , then  $\text{cut}_{\prec}(\beta) \subseteq \text{cut}_{\prec}(\alpha)$ . Thus  $A - \beta \subseteq A - \alpha$  (by  $-$  definition).

**Uniform Behaviour** If  $\vdash \beta$  follows trivially from *failure*. Let  $\nVdash \beta$ ,  $A \vdash \alpha, \beta \in A$ ,  $A - \alpha = A - \beta$  and assume by *reductio* that  $\alpha \notin \text{Cn}(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$ .

Since  $\nVdash \beta$  and  $\beta \in A$ , by *success* it follows that  $A - \beta \neq A$ . Hence, for all



$\delta$ , if  $A - \delta = A - \beta$  it follows that  $A - \delta \neq A$  and, by *failure*,  $\not\vdash \delta$ . Then it follows from the hypothesis and the definition of  $-$  that  $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$  and  $cut_{\prec}(\beta) \cup \{\gamma \in A : cut_{\prec}(\gamma) = cut_{\prec}(\beta)\} \not\vdash \alpha$ . From Observation 7 and Lemma 2 it follows that  $\{\gamma \in A : \beta \prec \gamma\} \cup \{\gamma \in A : \gamma =_{\preceq} \beta\} \not\vdash \alpha$ . Hence,  $\{\gamma \in A : \beta \preceq \gamma\} \not\vdash \alpha$ , and so  $\beta \in cut_{\prec}(\alpha) = cut_{\prec}(\beta)$ . Contradiction (Lemma 1 (b)).

*From Postulates to Brutal Contraction:*

Let  $-$  be an operator on  $A$  that satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion* and *uniform behaviour*. In order to prove that  $-$  is a brutal contraction we must prove that there exists a transitive, connected relation  $\preceq$  that satisfies  $(\preceq 1)$  -  $(\preceq 3)$  and such that

$$A - \alpha = \begin{cases} cut_{\prec}(\alpha) & \text{if } \not\vdash \alpha \\ A & \text{otherwise} \end{cases}, \text{ where } cut \text{ is defined in terms of } \preceq.$$

Let  $\preceq$  be defined for  $\alpha, \beta \in A$  as follows:

$$\alpha \preceq \beta \text{ iff either } (A - \beta \subseteq A - \alpha \text{ and } \not\vdash \alpha) \text{ or } \vdash \beta.$$

$(\preceq 1)$  Let  $\gamma \in A \setminus Cn(\emptyset)$ , we must show that  $H = \{\alpha \in A : \gamma \prec \alpha\} \not\vdash \gamma$ . Let  $\alpha \in A$  and  $\gamma \prec \alpha$ , then, according to our construction, there are two possible cases to consider (the other two cases are excluded by *failure*, *success* and *inclusion*):

Case 1)  $A - \alpha \subseteq A - \gamma$ ,  $A - \gamma \not\subseteq A - \alpha$  and  $\not\vdash \gamma$ . Since  $\alpha \in A$  and  $A - \gamma \not\subseteq A - \alpha$  then, by Observation 12 (a), it follows that  $\alpha \in A - \gamma$ .

Case 2)  $\vdash \alpha$  and  $\not\vdash \gamma$ . By *relative closure*, it follows that  $\alpha \in A - \gamma$ .

Hence  $H \subseteq A - \gamma$ . Therefore, by *success*, it follows that  $H \not\vdash \gamma$ .

$(\preceq 2)$  Let  $\alpha, \beta \in A$  such that  $\not\vdash \alpha$  and  $\vdash \beta$ . We need to prove that  $\alpha \preceq \beta$  and  $\beta \not\preceq \alpha$ . Both follow from the definition of  $\preceq$ .

$(\preceq 3)$  Follows trivially from the definition of  $\preceq$ .

**$(\preceq \text{ is connected})$**  By the definition of  $\preceq$ , it follows that  $\alpha \not\preceq \beta$  if and only if  $(A - \beta \not\subseteq A - \alpha \text{ or } \vdash \alpha)$  and  $\not\vdash \beta$ . We will prove by cases:

Case 1)  $\vdash \alpha$  and  $\not\vdash \beta$ . It follows from  $(\preceq 2)$  that  $\beta \preceq \alpha$ .

Case 2)  $A - \beta \not\subseteq A - \alpha$  and  $\not\vdash \beta$ . From *linearity* (Observation 12 (g)), it follows that  $A - \alpha \subseteq A - \beta$ . Hence, by the definition of  $\preceq$ , it follows that  $\beta \preceq \alpha$ .

**$(\preceq \text{ is transitive})$**  Let  $\alpha, \beta, \gamma \in A$  and assume that  $\alpha \preceq \beta$  and  $\beta \preceq \gamma$ . If  $\vdash \gamma$ , then, by  $(\preceq 2)$  and  $(\preceq 3)$ , it follows that  $\alpha \preceq \gamma$ . Assume now that  $\not\vdash \gamma$ . Then, by  $(\preceq 2)$  it follows that  $\not\vdash \alpha$  and  $\not\vdash \beta$ . From  $\alpha \preceq \beta$  and  $\beta \preceq \gamma$  and the definition of  $\preceq$  it follows that  $A - \beta \subseteq A - \alpha$  and  $A - \gamma \subseteq A - \beta$ . Therefore, since  $\subseteq$  is transitive  $A - \gamma \subseteq A - \alpha$ , and it follows from the definition of  $\preceq$  (since  $\not\vdash \alpha$ ) that  $\alpha \preceq \gamma$ .

It remains to prove that:

$$A - \alpha = \begin{cases} cut_{\prec}(\alpha) & \text{if } \not\vdash \alpha \\ A & \text{otherwise} \end{cases}$$

We will prove by cases:

1.  $\vdash \alpha$ . Follows trivially by *failure*.

2.  $\not\vdash \alpha$

2.1.  $A \not\vdash \alpha$ . It follows from *vacuity*, *inclusion* and Lemma 1 (c) that  $cut_{\prec}(\alpha) = A = A - \alpha$ .

2.2.  $A \vdash \alpha$ .

2.2.1.  $\alpha \in A$ . We will prove that  $A - \alpha = cut_{\prec}(\alpha)$  by double inclusion.

( $\supseteq$ ) Let  $\beta \in cut_{\prec}(\alpha)$  and assume by *reductio* that  $\beta \notin A - \alpha$ . From  $\beta \notin A - \alpha$  it follows, by Observation 12 (a), that  $A - \alpha \subseteq A - \beta$ . On the other hand, since  $\beta \in cut_{\prec}(\alpha)$ , it follows from Observation 7 that  $\alpha \prec \beta$ . According to the definition of  $\preceq$  there are four cases to consider:

Case 1)  $A - \beta \subseteq A - \alpha$ ,  $A - \alpha \not\subseteq A - \beta$  and  $\not\vdash \alpha$ . Contradiction, since  $A - \alpha \subseteq A - \beta$ .

Case 2)  $A - \beta \subseteq A - \alpha$ ,  $\not\vdash \alpha$  and  $\vdash \beta$ . Hence, since  $A - \alpha \subseteq A - \beta$ , it follows that  $A - \alpha = A - \beta$ . Contradiction, since by *failure*  $A - \beta = A$  and, by *success* and *inclusion*,  $A - \alpha \subset A$ .

Case 3)  $A - \alpha \not\subseteq A - \beta$ ,  $\not\vdash \alpha$  and  $\vdash \beta$ . Contradiction.

Case 4)  $\vdash \beta$  and  $\not\vdash \alpha$ . Then, by *relative closure*, it follows that  $\beta \in A - \alpha$ . Contradiction.

It follows that  $cut_{\prec}(\alpha) \subseteq A - \alpha$ .

( $\subseteq$ ) Let  $\beta \in A - \alpha$  and assume by *reductio* that  $\beta \notin cut_{\prec}(\alpha)$ . We will prove by cases:

Case 1)  $\vdash \beta$ . Then, by ( $\preceq$  2),  $\alpha \prec \beta$ . Therefore, from Observation 7, it follows that  $\beta \in cut_{\prec}(\alpha)$ . Contradiction.

Case 2)  $\not\vdash \beta$ . Since  $\beta \notin cut_{\prec}(\alpha)$ , by Observation 7, it follows that  $\alpha \not\prec \beta$ . Therefore, since  $\preceq$  is a connected relation, it follows that  $\beta \preceq \alpha$ . According to the definition of  $\preceq$  this means that:

i)  $A - \alpha \subseteq A - \beta$  and  $\not\vdash \beta$  or ii)  $\vdash \alpha$ . In the latter we obtain a contradiction. In the former we also obtain a contradiction, since  $\beta \in A - \alpha$  and by *success*  $\beta \notin A - \beta$ .

Therefore  $cut_{\prec}(\alpha) = A - \alpha$ .

2.2.2.  $\alpha \notin A$ . We will prove that  $A - \alpha = cut_{\prec}(\alpha)$  by double inclusion.

( $\supseteq$ ) Let  $\beta \in cut_{\prec}(\alpha)$  and assume by *reductio* that  $\beta \notin A - \alpha$ . If  $\vdash \beta$  then, by *relative closure*, it follows that  $\beta \in A - \alpha$ . Contradiction. Assume now that  $\not\vdash \beta$ . From  $\beta \notin A - \alpha$  it follows, from Observation 12 (a), that  $A - \alpha \subseteq A - \beta$ .

We will consider two cases:

Case 1)  $A - \alpha \subset A - \beta$ . Hence, by Observation 12 (f),  $A - \beta \vdash \alpha$ . Therefore, since  $\beta \in A$ , from the case 2.2.1. it follows that  $cut_{\prec}(\beta) \vdash \alpha$ . Hence, from Observation 7,  $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$ . Contradiction, since  $\beta \in cut_{\prec}(\alpha)$ .

Case 2)  $A - \alpha = A - \beta$ . From  $\beta \in cut_{\prec}(\alpha)$  it follows that  $\{\gamma \in A : \beta \preceq \gamma\} \not\vdash \alpha$ . Hence, by Observation 7, it follows that  $cut_{\prec}(\beta) \cup \{\gamma \in A : \beta =_{\preceq} \gamma\} \not\vdash \alpha$ . Hence, by Lemma 2, it follows that  $cut_{\prec}(\beta) \cup \{\gamma \in A : cut_{\prec}(\beta) = cut_{\prec}(\gamma)\} \not\vdash \alpha$ . Therefore, from case 2.2.1., it follows that  $(A - \beta) \cup \{\gamma \in A : A - \beta = A - \gamma\} \not\vdash \alpha$ . Contradiction, from *uniform behaviour*. It follows that  $cut_{\prec}(\alpha) \subseteq A - \alpha$ .

( $\subseteq$ ) Let  $\beta \in A - \alpha$  and assume by *reductio* that  $\beta \notin cut_{\prec}(\alpha)$  (note that it follows from *inclusion* that  $\beta \in A$ ). We will consider two cases:

Case 1)  $\vdash \beta$ . Then, by ( $\preceq$  2), it follows that, if  $\gamma \in A$  and  $\beta \preceq \gamma$ , then  $\vdash \gamma$ . Therefore  $\{\gamma \in A : \beta \preceq \gamma\} \subseteq Cn(\emptyset)$ . Hence, since  $\not\vdash \alpha$ , it follows that  $\{\gamma \in A : \beta \preceq \gamma\} \not\vdash \alpha$ . Therefore  $\beta \in cut_{\prec}(\alpha)$ . Contradiction.

Case 2)  $\not\vdash \beta$ . From  $\beta \notin cut_{\prec}(\alpha)$  it follows that  $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$ . Then according to the definition of  $\preceq$ ,  $(\{\gamma \in A : A - \gamma \subseteq A - \beta\} \cup \{\gamma \in A : \vdash \gamma\}) \vdash \alpha$ . Hence, by Observation 12 (e), it follows that  $\{\gamma \in A : A - \gamma \subset A - \alpha\} \vdash \alpha$ . If  $A - \gamma \subset A - \alpha$ , then, by Observation 12 (a), it follows that  $\gamma \in A - \alpha$ . Hence,  $\{\gamma \in A : A - \gamma \subset A - \alpha\} \subseteq A - \alpha$ . Therefore,  $A - \alpha \vdash \alpha$ . Contradiction (by *success*). ■

#### PROOF OF OBSERVATION 12.

(a) Let  $\alpha \in A \setminus A - \beta$ , then it follows by *relative closure* that  $A - \beta \not\vdash \alpha$  and so, by *strong inclusion*,  $A - \beta \subseteq A - \alpha$ .

(b) Let  $\not\vdash \alpha$ . Hence, by *success*  $A - \alpha \not\vdash \alpha$ , and so  $A - \alpha \not\vdash \alpha \wedge \beta$ . Therefore, by *strong inclusion*,  $A - \alpha \subseteq A - (\alpha \wedge \beta)$ .

(c) It follows from replacing  $\beta$  by  $\alpha \wedge \beta$  in *strong inclusion*.

(d) It follows from  $\not\vdash \alpha$  and (b) that  $A - \alpha \subseteq A - (\alpha \wedge \beta)$ . Hence,  $A - (\alpha \wedge \beta) \vdash \beta$ . Therefore, since  $\not\vdash \alpha \wedge \beta$ , due to *success*, it follows that  $A - (\alpha \wedge \beta) \not\vdash \alpha$ . From (c) it follows that  $A - (\alpha \wedge \beta) \subseteq A - \alpha$ , and so  $A - (\alpha \wedge \beta) = A - \alpha$ . On the other hand, since  $\not\vdash \beta$  it follows from (b) that  $A - \beta \subseteq A - (\alpha \wedge \beta) = A - \alpha$ .

(e) Let  $\not\vdash \alpha$  and  $\alpha \in A - \beta$ , then  $A - \alpha \neq A - \beta$ , since from *success*  $\alpha \notin A - \alpha$ . We will prove by cases:

Case 1)  $A - \alpha \not\vdash \beta$ . By *strong inclusion*,  $A - \alpha \subseteq A - \beta$ . Hence  $A - \alpha \subset A - \beta$ .

Case 2)  $\vdash \beta$ . It follows from *failure* that  $A - \beta = A$  and from *inclusion* that  $A - \alpha \subseteq A - \beta$ . Hence  $A - \alpha \subset A - \beta$ .

Case 3)  $A - \alpha \vdash \beta$  and  $\nvdash \beta$ . It follows from  $\nvdash \alpha$  and (d) that  $A - \beta \subseteq A - \alpha$ . Contradiction, since  $\alpha \in A - \beta$  and  $\nvdash \alpha$ .

(f) Follows by *strong inclusion*.

(g) We will prove by cases:

Case 1)  $\vdash \alpha$ , it follows from *failure* that  $A - \alpha = A$  and so (by *inclusion*)  $A - \beta \subseteq A - \alpha$ .

Case 2)  $\vdash \beta$ , due to the symmetry of the case, it follows that  $A - \alpha \subseteq A - \beta$ .

Case 3)  $A - \alpha \nvdash \beta$ , then by *strong inclusion*  $A - \alpha \subseteq A - \beta$ .

Case 4)  $\nvdash \alpha, \nvdash \beta$  and  $A - \alpha \vdash \beta$ . It follows from (d) that  $A - \beta \subseteq A - \alpha$ .

(h) Follows by *success* and (g).

(i) If  $\vdash \alpha \wedge \beta$  it follows trivially from *failure*. Assume now that  $\nvdash \alpha \wedge \beta$ . It follows from  $\vdash \alpha \leftrightarrow \beta$  that  $\nvdash \alpha, \nvdash \beta, \vdash \alpha \rightarrow \beta$  and  $\vdash \beta \rightarrow \alpha$ . Then, due to *success*,  $A - \beta \nvdash \alpha$  and  $A - \alpha \nvdash \beta$ . Hence, by *strong inclusion*,  $A - \alpha = A - \beta$ . ■

PROOF OF OBSERVATION 14. Let  $\beta \in A$  and  $H = A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\}$  we must prove that there exists  $\alpha \in A \cup Cn(\emptyset)$  such that  $A - \alpha = H$ . If  $\vdash \beta$ , then it follows, from *failure*, that  $A - \beta = A$  and that  $H = A$ . Then it is enough to consider  $\alpha = \beta$ . Assume now that  $\nvdash \beta$ . We will prove by cases: Case 1)  $A - \delta \subseteq A - \beta$  for all  $\delta \in A$ . We will consider  $\alpha \in Cn(\emptyset)$ . Then  $H \subseteq A - \alpha = A$ . It remains to prove that  $A - \alpha \subseteq H$ . Let  $\theta \in A - \alpha = A$ . Hence  $A - \theta \subseteq A - \beta$ . We will consider two cases:

Case 1.1)  $A - \theta \subset A - \beta$ . By Observation 12 (f),  $A - \beta \vdash \theta$ . Hence, by *relative closure*,  $\theta \in A - \beta$ .

Case 1.2)  $A - \theta = A - \beta$ . Then  $\theta \in \{\gamma \in A : A - \beta = A - \gamma\}$ . Therefore  $A - \alpha = H$ .

Case 2) There exists  $\delta \in A$  such that  $A - \delta \not\subseteq A - \beta$ . Then, by *linearity*, it follows that  $A - \beta \subset A - \delta$ . Let  $\theta \in \{\gamma \in A : A - \gamma = A - \beta\}$ . Then  $A - \theta = A - \beta$ . Hence  $A - \theta \subset A - \delta$  and, by Observation 12 (f)  $A - \delta \vdash \theta$ . Therefore, by *relative closure*,  $\theta \in A - \delta$ . Hence there exists  $\delta$  such that  $H \subseteq A - \delta$ . Consider the (non-empty) set  $S = \{\psi \in A : H \subseteq A - \psi\}$ . We will consider two cases:

Case 2.1)  $S \subseteq Cn(\emptyset)$ . Then we take  $\alpha \in Cn(\emptyset)$ .  $H \subseteq A - \alpha$ . It remains to prove that  $A - \alpha \subseteq H$ . Let  $\theta \in A - \alpha$ . If  $\vdash \theta$ , then, by *relative closure*,  $\theta \in A - \beta$  and so  $\theta \in H$ .

Consider now that  $\nvdash \theta$  and assume by *reductio* that  $\theta \notin H$ . Then  $\theta \notin A - \beta$  and  $A - \theta \neq A - \beta$ . Hence, by Observation 12 (a),  $A - \beta \subset A - \theta$ . Let  $\psi \in \{\gamma \in A : A - \gamma = A - \beta\}$ . It follows that  $A - \psi \subset A - \theta$ . By Observation

12 (f) and *relative closure* it follows that  $\psi \in A - \theta$ . Therefore  $H \subseteq A - \theta$ . Contradiction since  $\theta \in S \subseteq Cn(\emptyset)$  and  $\nvdash \theta$ .

Case 2.2) There are non-tautological formulae on  $S$ . Consider  $S' = S \setminus Cn(\emptyset)$ . Then by *lower bound* there exists  $\gamma' \in S'$  such that  $A - \gamma' \subseteq A - \gamma$  for all  $\gamma \in S'$ . Hence  $H \subseteq A - \gamma'$ . It remains to prove that  $A - \gamma' \subseteq H$ . Let  $\theta \in A - \gamma'$ . If  $\vdash \theta$ , then by *relative closure*  $\theta \in A - \beta$ . If  $\nvdash \theta$ , assume by *reductio* that  $\theta \notin H$ . Hence  $\theta \notin A - \beta$  and  $A - \theta \neq A - \beta$ . Hence, by *linearity*,  $A - \theta \subset A - \beta$  or  $A - \beta \subset A - \theta$ . Thus we have two cases to consider:

Case 2.2.1)  $\theta \notin A - \beta$  and  $A - \theta \subset A - \beta$ . From  $A - \theta \subset A - \beta$  it follows, from Observation 12 (f) and *relative closure*,  $\theta \in A - \beta$ . Contradiction.

Case 2.2.2)  $\theta \notin A - \beta$  and  $A - \beta \subset A - \theta$ . Hence  $H \subseteq A - \theta$ , since  $A - \beta \subset A - \theta$  and, by Observation 12 (f) and *relative closure*, for all  $\psi \in A$  such that  $A - \psi = A - \beta$ ,  $\psi \in A - \theta$ . Hence  $A - \gamma' \subseteq A - \theta$ . Therefore  $\theta \in A - \theta$ . Contradiction, since by *success*  $A - \theta \nvdash \theta$ . ■

PROOF OF OBSERVATION 15. Let  $\beta \in A$ ,  $A \vdash \alpha$  and  $A - \alpha = A - \beta$ . If  $\vdash \beta$  follows trivially by *failure*. If  $\nvdash \beta$ , then it follows, by *success*  $A - \beta \nvdash \beta$ . From *clustering* it follows that there exists  $\psi \in A \cup Cn(\emptyset)$  such that  $A - \psi = A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\}$ .  $\beta \in A - \psi$ . Hence,  $A - \alpha = A - \beta \subset A - \psi$ . By Observation 12 (f)  $A - \psi \vdash \alpha$ . ■

PROOF OF THEOREM 16.

*From bounded brutal contraction to postulates:*

Let  $-$  be a bounded brutal contraction operator on  $A$ . By Theorem 11  $-$  satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure* and *strong inclusion*. It remains to show that  $-$  satisfies *upper bound* and *lower bound*.

**Upper Bound** Let  $X \subseteq A$  be a non empty set of non-tautological formulae. Since  $\preceq$  is well ranked there exists  $\beta \in X$  such that  $\beta \preceq \alpha$  for all  $\alpha \in X$ . Hence, by Lemma 1 (f), there exists  $\beta \in X$  for all  $\alpha \in X$  such that  $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ . Therefore, by definition of  $-$  there exists  $\beta \in X$  for all  $\alpha \in X$  such that  $A - \alpha \subseteq A - \beta$ .

**Lower Bound** Analogous to *upper bound*.

*From postulates to bounded brutal contraction:*

Let  $-$  be an operator on  $A$  that satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *lower bound*, *upper bound* and *strong inclusion*. From Observation 14 and Observation 15 it follows that  $-$  satisfies *uniform behaviour*. Let  $\preceq$  be defined by:

$$\alpha \preceq \beta \text{ iff } \begin{cases} A - \beta \subseteq A - \alpha \text{ and } \not\vdash \alpha \\ \text{or} \\ \vdash \beta \end{cases}$$

According to the Postulates to Construction part of the proof of Theorem 11  $\preceq$  satisfies  $(\preceq 1)$  -  $(\preceq 3)$  and is such that

$$A - \alpha = \begin{cases} \text{cut}_{\prec}(\alpha) & \text{if } \not\vdash \alpha \\ A & \text{otherwise} \end{cases}$$

It remains to prove that  $\preceq$  is bounded. To do so we must prove that  $\preceq$  is well-ranked and inversely well-ranked.

**( $\preceq$  is well-ranked)** Let  $X \neq \emptyset$  and  $X \subseteq A$ . We will prove by cases:

Case 1) All formulae in  $X$  are tautologies. Let  $\beta$  be one of those formulas. Hence by  $(\preceq 3)$   $\beta \preceq \alpha$  for all  $\alpha \in X$ .

Case 2) All formulae in  $X$  are non-tautological. By *upper bound* there exists  $\beta \in X$  such that  $A - \alpha \subseteq A - \beta$  for all  $\alpha \in X$ . Hence, by definition of  $\preceq$ , there exists  $\beta \in X$  such that  $\beta \preceq \alpha$  for all  $\alpha \in X$ .

Case 3) There are some formulae in  $X$ , that are tautological and others that are not. Consider  $X' = X \setminus \text{Cn}(\emptyset)$ . Hence, by the previous case, there exists  $\beta \in X'$  such that  $\beta \preceq \alpha'$  for all  $\alpha' \in X'$ . Therefore, it follows from  $(\preceq 3)$  that  $\beta \preceq \alpha$  for all  $\alpha \in X$ .

**( $\preceq$  is inversely well-ranked)** Let  $X \neq \emptyset$  and  $X \subseteq A$ . We will prove by cases:

Case 1) There are some  $\beta \in X \cap \text{Cn}(\emptyset)$ . Then, by definition of  $\preceq$ ,  $\alpha \preceq \beta$  for all  $\alpha \in X$ .

Case 2) All formulae in  $X$  are non-tautological. By *lower bound* there exists  $\beta \in X$  such that  $A - \beta \subseteq A - \alpha$  for all  $\alpha \in X$ . Hence, by definition of  $\preceq$ , there exists  $\beta \in X$  such that  $\alpha \preceq \beta$  for all  $\alpha \in X$ . ■

**PROOF OF OBSERVATION 17.** Let  $\alpha \in \text{Cn}(A) \setminus \text{Cn}(\emptyset)$ . By *inclusion* and *success*  $A - \alpha \subset A$ . Hence, by *relative closure*,  $A \setminus A - \alpha$  is a non-empty set of non-tautological formulae. Therefore, from *lower bound*, there exists  $\beta \in A \setminus A - \alpha$  such that  $A - \beta \subseteq A - \delta$  for all  $\delta \in A \setminus A - \alpha$ . It follows from  $\beta \in A \setminus A - \alpha$ , by Observation 12 (a), that  $A - \alpha \subseteq A - \beta$ . Hence  $A - \alpha = A - \beta$  or  $A - \alpha \subset A - \beta$ . The latter leads to a contradiction. If  $A - \alpha \subset A - \beta$  there exists  $\psi \in A - \beta \setminus A - \alpha$ . By *relative closure* it follows that  $\not\vdash \psi$ . Hence, by *linearity* and *success*, it follows that  $A - \psi \subset A - \beta$ . From  $\psi \in A - \beta \setminus A - \alpha$  it follows that  $\psi \in A \setminus A - \alpha$ , and so  $A - \beta \subseteq A - \psi$ . Contradiction. Hence  $A - \alpha = A - \beta$ . ■

**PROOF OF THEOREM 19.** We will prove by cases:

Case 1)  $\vdash \alpha$ . It follows that  $A - \alpha = A$  and  $(\text{Cn}(A) \div_{\preceq} \alpha) \cap A = A$ .

Case 2)  $A \not\vdash \alpha$ . It follows that  $(\text{Cn}(A) \div_{\preceq} \alpha) \cap A = \bar{A}$  and that  $A - \alpha =$

$cut_{\prec}(\alpha)$ . By Lemma 1 (c), it follows that  $cut_{\prec}(\alpha) = A$ .

Case 3)  $A \vdash \alpha$  and  $\nvdash \alpha$ . We will prove that  $A - \alpha = (Cn(A) \div_{\leq} \alpha) \cap A$  by double inclusion.

( $\subseteq$ ) Let  $\beta \in A - \alpha$ . It follows that  $\beta \in A$ . It remains to prove that  $\beta \in Cn(A) \div_{\leq} \alpha$ , i.e. that  $\beta \in \{\psi \in Cn(A) : cut_{\prec}(\psi) \subset cut_{\prec}(\alpha)\}$ .

If  $\vdash \beta$ , it follows trivially by Observation 8 (b). Assume now that  $\nvdash \beta$ . Then  $\beta \in cut_{\prec}(\alpha)$ . Hence  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . It follows, from Lemma 8 (b) that  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ .

( $\supseteq$ ) Let  $\beta \in (Cn(A) \div_{\leq} \alpha) \cap A$ . If  $\vdash \beta$ , then it follows from ( $\leq 2$ ) that  $\{\psi \in A : \beta \preceq \psi\} \subseteq Cn(\emptyset)$ . Therefore, since  $\nvdash \alpha$ , it follows that  $\beta \in cut_{\prec}(\alpha) = A - \alpha$ . Assume now that  $\nvdash \beta$ . From  $\beta \in (Cn(A) \div_{\leq} \alpha) \cap A$  it follows that  $\beta \in A$  and  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . Hence there exists  $\gamma \in A$  such that  $\gamma \in cut_{\prec}(\alpha)$  and  $\gamma \notin cut_{\prec}(\beta)$ . Hence,  $\{\psi \in A : \gamma \prec \psi\} \nvdash \alpha$  and  $\{\psi \in A : \gamma \prec \psi\} \vdash \beta$ . Assume by *reductio* that  $\beta \notin A - \alpha$  i.e. that  $\beta \notin cut_{\prec}(\alpha)$ . Hence,  $\{\psi \in A : \beta \preceq \psi\} \vdash \alpha$ . From  $\{\psi \in A : \beta \preceq \psi\} \vdash \alpha$  and  $\{\psi \in A : \gamma \prec \psi\} \nvdash \alpha$  it follows that  $\beta \preceq \gamma$ . Therefore, since  $\{\psi \in A : \gamma \prec \psi\} \vdash \beta$ , it follows that  $\{\psi \in A : \beta \prec \psi\} \vdash \beta$  which contradicts ( $\leq 1$ ). ■

PROOF OF THEOREM 20. We will prove by cases:

Case 1)  $\vdash \alpha$ . Then  $Cn(A) \div_{\leq} \alpha = Cn(A)$  and  $A - \alpha = A$ . Hence  $Cn(A - \alpha) = Cn(A) = Cn(A) \div_{\leq} \alpha$ .

Case 2)  $A \nvdash \alpha$ . Then  $Cn(A) \div_{\leq} \alpha = Cn(A)$  and, by Lemma 1 (c),  $A - \alpha = cut_{\prec}(\alpha) = A$ . Hence  $Cn(A - \alpha) = Cn(A) = Cn(A) \div_{\leq} \alpha$ .

Case 3)  $A \vdash \alpha$  and  $\nvdash \alpha$ . Hence  $Cn(A) \div_{\leq} \alpha = \{\psi \in Cn(A) : \alpha <_{\prec} \psi\} = \{\psi \in Cn(A) : cut_{\prec}(\psi) \subset cut_{\prec}(\alpha)\}$ . We will prove that  $Cn(A - \alpha) = Cn(A) \div_{\leq} \alpha$  by double inclusion.

( $\subseteq$ ) Let  $\beta \in Cn(A - \alpha)$ . If  $\vdash \beta$ , then  $\beta \in Cn(A)$  and, by Observation 8 (b),  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . Hence  $\beta \in Cn(A) \div_{\leq} \alpha$ . Assume now that  $\nvdash \beta$ . From  $\beta \in Cn(A - \alpha)$  it follows that  $cut_{\prec}(\alpha) \vdash \beta$ . Hence, by Lemma 1 (j),  $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$ . From  $\alpha \wedge \beta \vdash \beta$  by Lemma 1 (d) it follows that  $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha \wedge \beta)$ . Hence  $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$ . From which, together with Lemma 1 (b) and  $cut_{\prec}(\alpha) \vdash \beta$  it follows that  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . Hence, by Lemma 8 (b), it follows that  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . Therefore, since  $\beta \in Cn(A)$ , it follows that  $\beta \in Cn(A) \div_{\leq} \alpha$ .

( $\supseteq$ ) Let  $\beta \in Cn(A) \div_{\leq} \alpha$ . Hence,  $\beta \in Cn(A)$  and  $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$ . Assume by *reductio* that  $\beta \notin Cn(A - \alpha)$ . Therefore  $cut_{\prec}(\alpha) \nvdash \beta$ . By Lemma 1 (k) it follows that  $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$ . From  $\alpha \wedge \beta \vdash \alpha$ , by Lemma 1 (d), it follows that  $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ . From Observation 8 (a) it follows that  $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ . Contradiction. ■

**PROOF OF OBSERVATION 21.** If  $A = \emptyset$  vacuously true. Assume now that  $A \neq \emptyset$ . Let  $\div$  be an operator on  $Cn(A)$  that satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*, and  $\alpha \in Cn(A) \setminus Cn(\emptyset)$ . It follows trivially if  $\alpha \in A$ . Assume now that  $\alpha \in Cn(A) \setminus A$ . It holds that  $A \setminus Cn(A) \div \alpha \neq \emptyset$ , otherwise, from  $(\div 1)$  and  $(\div 2)$ , it would follow that  $Cn(A) \div \alpha = Cn(A)$  which contradicts  $(\div 4)$ . On the other hand, by  $(\div 1)$ ,  $(A \setminus Cn(A) \div \alpha) \cap Cn(\emptyset) = \emptyset$ . Hence, by  $\div$  *lower bound*, there exists  $\beta \in A \setminus Cn(A) \div \alpha$  such that  $Cn(A) \div \beta \subseteq Cn(A) \div \gamma$ , for all  $\gamma \in A \setminus Cn(A) \div \alpha$ . It follows, by  $(\div 9)$  that  $Cn(A) \div \alpha \subseteq Cn(A) \div \beta$ . It remains to prove that  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha$ . Assume by *reductio* that this is not the case. Hence, by  $(\div 9)$ ,  $\alpha \in Cn(A) \div \beta$ . Hence, by *base-reduction*, it follows that  $Cn(A) \div \beta \cap A \vdash \alpha$ . Therefore, by compactness, there exists a finite subset of  $Cn(A) \div \beta \cap A$ ,  $A' = \{\alpha_1, \dots, \alpha_n\}$ , such that  $A' \vdash \alpha$ . Hence, there is some  $\alpha_i \in A'$  such that  $\alpha_i \notin Cn(A) \div \alpha$ . Hence, by  $(\div 1)$ ,  $\nvdash \alpha_i$ . On the other hand,  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha_i$ . But  $\alpha_i \in Cn(A) \div \beta$ . Contradiction, by  $(\div 4)$ . ■

**PROOF OF THEOREM 22.** *From ensconcement-based withdrawal to postulates:* Let  $\div$  be an ensconcement-based withdrawal related to  $(A, \preceq)$  and let  $\leq = \leq_{\preceq}$ . Hence  $\div$  satisfies the postulates for severe withdrawals. It remains to show that  $\div$  satisfies: *base-reduction*, *upper bound* and *lower bound*.

**Upper Bound:** Let  $\div$  be an ensconcement-based withdrawal related to  $(A, \preceq)$ . Let  $X \neq \emptyset$  and  $X \subseteq Cn(A) \setminus Cn(\emptyset)$ . From Observation 9, since  $(A, \preceq)$  is a bounded ensconcement, it follows that  $\leq_{\preceq}$  is bounded. Hence, there exists  $\beta \in X$  such that  $\beta \leq \alpha$  for all  $\alpha \in X$ . We will prove that  $Cn(A) \div \alpha \subseteq Cn(A) \div \beta$  for all  $\alpha \in X$ . Let  $\gamma \in Cn(A) \div \alpha$ . Hence, by definition of  $\div$ ,  $\gamma \in Cn(A)$  and  $\alpha < \gamma$ . By (EE1), since  $\beta \leq \alpha$  and  $\alpha < \gamma$  it follows that  $\beta < \gamma$ . Hence  $\gamma \in Cn(A) \div \beta$ . Therefore  $Cn(A) \div \alpha \subseteq Cn(A) \div \beta$ .

**Lower Bound:** Analogous to *upper bound*.

**Base-reduction:** Let  $Cn(A) \div \alpha \vdash \beta$ . We will prove that  $(Cn(A) \div \alpha) \cap A \vdash \beta$  by cases:

Case 1)  $\vdash \beta$ . Follows trivially.

Case 2)  $\alpha \notin Cn(A)$  or  $\vdash \alpha$ . Follows trivially by  $(R_{\leq})$ .

Case 3)  $\nvdash \beta$ ,  $\alpha \in Cn(A)$  and  $\nvdash \alpha$ . From  $Cn(A) \div \alpha \vdash \beta$  it follows, by  $(R_{\leq})$ , that  $X \vdash \beta$  where  $X = \{\psi \in Cn(A) : cut_{\preceq}(\psi) \subset cut_{\preceq}(\alpha)\}$ . It holds that  $X \setminus Cn(\emptyset) \neq \emptyset$ , since  $\nvdash \beta$ . Let  $\psi \in X \setminus Cn(\emptyset)$ . Assume that  $cut_{\preceq}(\psi) = \emptyset$  and let  $\theta \in Cn(\emptyset)$ . Hence, by (EE5), it follows that  $\psi < \theta$ . Hence, by Observation 9,  $cut_{\preceq}(\theta) \subset cut_{\preceq}(\psi) = \emptyset$ . Contradiction. Hence  $cut_{\preceq}(\psi) \neq \emptyset$ . From Lemma 6, and since  $\preceq$  is bounded, it follows that there exists  $\delta \in cut_{\preceq}(\psi)$  such that  $cut_{\preceq}(\delta) = cut_{\preceq}(\psi)$ . Let  $Y = \{\mu \in A : cut_{\preceq}(\mu) \subset cut_{\preceq}(\alpha)\}$ . Let  $\mu_1 \in Y$  such that  $\mu_1 \preceq \mu$  for all  $\mu \in Y$ . Let  $\lambda \in cut_{\preceq}(\mu_1)$ . Hence



$cut_{\preceq}(\lambda) \subseteq cut_{\preceq}(\mu_1)$ , from which follows that  $cut_{\preceq}(\lambda) \subset cut_{\preceq}(\alpha)$ . Therefore  $\lambda \in Y$ . Let  $\phi \in Y$ . It follows that  $\mu_1 \preceq \phi$ . Hence  $\phi \in cut_{\preceq}(\mu_1)$ . Therefore  $Y = cut_{\preceq}(\mu_1)$ . By Lemma 7  $cut_{\preceq}(\psi) \vdash \psi$ . Hence, since  $cut_{\preceq}(\delta) = cut_{\preceq}(\psi)$  it follows that  $cut_{\preceq}(\delta) \vdash \psi$ . From  $cut_{\preceq}(\delta) \subset cut_{\preceq}(\alpha)$  it follows that  $\delta \in Y$ . Hence  $\mu_1 \preceq \delta$ . Therefore  $cut_{\preceq}(\delta) \subseteq cut_{\preceq}(\mu_1) = Y$ , and so  $Y \vdash \psi$ . Hence, for all  $\psi \in Cn(A) \div \alpha$  it follows that  $Y \vdash \psi$ . Therefore, since  $Cn(A) \div \alpha \vdash \beta$ , it follows that  $Y \vdash \beta$ .  $Y \subseteq (Cn(A) \div \alpha) \cap A$ . Hence  $(Cn(A) \div \alpha) \cap A \vdash \beta$ .

*From postulates to ensconcement-based withdrawal:* Let  $A$  be a belief base and  $\div$  be an operator on  $Cn(A)$  that satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*. Let  $\preceq$  be a binary relation on  $A$  defined by:

$\alpha \preceq \beta$  if and only if  $\alpha \notin Cn(A) \div \beta$  or  $\vdash \beta$ .

We will prove that  $\preceq$  is a bounded ensconcement.

**( $\preceq 1$ )** Let  $\gamma \in A \setminus Cn(\emptyset)$ . We must show that  $H = \{\alpha \in A : \gamma \prec \alpha\} \not\vdash \gamma$ . It is enough to show that  $H \setminus Cn(\emptyset) \not\vdash \gamma$ . Let  $\alpha \in A \setminus Cn(\emptyset)$  and  $\gamma \prec \alpha$ . Then,  $\gamma \preceq \alpha$  and  $\alpha \not\preceq \gamma$ . Hence, by definition of  $\preceq$ , it follows that  $\gamma \notin Cn(A) \div \alpha$ ,  $\alpha \in Cn(A) \div \gamma$  and  $\not\vdash \gamma$ . Then  $H \subseteq Cn(A) \div \gamma$  where,  $\not\vdash \gamma$ . Hence, since by  $(\div 4)$   $Cn(A) \div \gamma \not\vdash \gamma$  it follows that  $H \not\vdash \gamma$ .

**( $\preceq 2$ )** Let  $\alpha, \beta \in A$  be such that  $\not\vdash \alpha$  and  $\vdash \beta$ . From  $\vdash \beta$  it follows, by definition of  $\preceq$ , that  $\alpha \preceq \beta$ . Assume by *reductio* that  $\not\vdash \alpha$ ,  $\vdash \beta$  and  $\beta \preceq \alpha$ . Hence, by definition of  $\preceq$ ,  $\beta \notin Cn(A) \div \alpha$  or  $\vdash \alpha$ . Contradiction, since  $\not\vdash \alpha$  and by  $(\div 1)$   $\beta \in Cn(A) \div \alpha$ .

**( $\preceq 3$ )** Follows trivially by definition of  $\preceq$ .

**( $\preceq$  is transitive)** Let  $\alpha \preceq \beta$  and  $\beta \preceq \gamma$ . Hence, by definition of  $\preceq$ , it follows that  $(\alpha \notin Cn(A) \div \beta \text{ or } \vdash \beta)$  and  $(\beta \notin Cn(A) \div \gamma \text{ or } \vdash \gamma)$ . Hence,  $\alpha \notin Cn(A) \div \beta$  and  $(\beta \notin Cn(A) \div \gamma \text{ or } \vdash \gamma)$  or  $(\vdash \beta \text{ and } (\beta \notin Cn(A) \div \gamma \text{ or } \vdash \gamma))$ . Hence, we have four cases to consider:

Case 1)  $\alpha \notin Cn(A) \div \beta$  and  $\beta \notin Cn(A) \div \gamma$ . From  $(\div 9)$  it follows that  $Cn(A) \div \gamma \subseteq Cn(A) \div \beta$ . Hence,  $\alpha \notin Cn(A) \div \gamma$ . Therefore  $\alpha \preceq \gamma$ , by definition of  $\preceq$ .

Case 2)  $\alpha \notin Cn(A) \div \beta$  and  $\vdash \gamma$ . Then  $\alpha \preceq \gamma$  follows trivially by definition of  $\preceq$ .

Case 3)  $\vdash \beta$  and  $\beta \notin Cn(A) \div \gamma$ . Contradicts  $(\div 1)$ .

Case 4)  $\vdash \beta$  and  $\vdash \gamma$ . Then  $\alpha \preceq \gamma$  follows trivially by definition of  $\preceq$ .

**( $\preceq$  is connected)** Let  $\alpha \not\preceq \beta$ . Hence  $\alpha \in Cn(A) \div \beta$  and  $\not\vdash \beta$ . We will consider two cases:

Case 1)  $\vdash \alpha$ . Hence  $\beta \preceq \alpha$ , by definition of  $\preceq$ .

Case 2)  $\not\vdash \alpha$ . Hence, by  $\div$  *expulsiveness* (Lemma 3),  $\beta \notin Cn(A) \div \alpha$ . Therefore, by definition of  $\preceq$ ,  $\beta \preceq \alpha$ .

**( $\preceq$  is well-ranked)** Let  $X \subseteq A$  a non empty set. We will prove by cases:

Case 1)  $X \subseteq Cn(\emptyset)$ . Trivial.

Case 2)  $X \not\subseteq Cn(\emptyset)$ . Let  $X' = X \setminus Cn(\emptyset)$ . Hence, by  $\div$  *upper bound* there exists  $\beta \in X'$  such that  $Cn(A) \div \alpha \subseteq Cn(A) \div \beta$  for all  $\alpha \in X'$ . By  $(\div 4)$   $\beta \notin Cn(A) \div \alpha$  for all  $\alpha \in X'$ . Hence, by definition of  $\preceq$ , there exists  $\beta \in X'$  such that  $\beta \preceq \alpha$  for all  $\alpha \in X'$ . If  $X = X'$  trivial. Assume now that  $X \neq X'$ . Let  $\gamma \in X \setminus X'$ . Hence  $\vdash \gamma$  and by  $(\preceq 2)$  it follows that  $\beta \preceq \gamma$ . Therefore, there exists  $\beta \in X$  such that  $\beta \preceq \alpha$  for all  $\alpha \in X$ .

**$(\preceq$  is inversely well-ranked)** Let  $X \subseteq A$  a non empty set. We will consider two cases:

Case 1)  $X \cap Cn(\emptyset) \neq \emptyset$ . Let  $\beta \in X \cap Cn(\emptyset)$  hence, by definition of  $\preceq$ ,  $\alpha \preceq \beta$  for all  $\alpha \in X$ .

Case 2)  $X \cap Cn(\emptyset) = \emptyset$ . Hence, by  $\div$  *lower bound*, there exists  $\beta \in X$  such that  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha$ , for all  $\alpha \in X$ . By  $(\div 4)$   $\alpha \notin Cn(A) \div \beta$ , for all  $\alpha \in X$ . Hence, by definition of  $\preceq$  there exists  $\beta \in X$  such that  $\alpha \preceq \beta$ , for all  $\alpha \in X$ .

We have proved that  $\preceq$  is a bounded ensconement. Let  $\leq_{\preceq}$  be the bounded epistemic entrenchment related to  $Cn(A)$  defined from  $\preceq$  as exposed in Observation 9. It remains to show that  $Cn(A) \div \alpha = Cn(A) \div_{\leq_{\preceq}} \alpha$ , where  $\div_{\leq_{\preceq}}$  is defined (as in  $(R_{\leq})$ ) by:

$$Cn(A) \div_{\leq_{\preceq}} \alpha = \begin{cases} Cn(A) \cap \{\psi : \alpha <_{\preceq} \psi\} & \text{if } \alpha \in Cn(A) \text{ and } \not\vdash \alpha \\ Cn(A) & \text{otherwise} \end{cases}$$

According to Lemma 4 and since  $\div$  is a severe withdrawal function, the epistemic entrenchment  $\leq$  on which  $\div$  is based on is such that:  $\alpha \leq \beta$  if and only if  $\alpha \notin Cn(A) \div \beta$  or  $\vdash \beta$ . Thus to prove that  $Cn(A) \div \alpha = Cn(A) \div_{\leq_{\preceq}} \alpha$  it is enough to show that:

$\alpha \leq_{\preceq} \beta$  if and only if  $\alpha \notin Cn(A) \div \beta$  or  $\vdash \beta$ .

$(\Rightarrow)$  Let  $\alpha \leq_{\preceq} \beta$ . Hence, by definition of  $\leq_{\preceq}$ ,  $\alpha \leq_{\preceq} \beta$  if and only if:

i)  $\alpha \notin Cn(A)$ , or ii)  $\alpha, \beta \in Cn(A)$  and  $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$ .

We will prove by cases:

Case 1)  $\alpha \notin Cn(A)$ . Then, by  $(\div 2)$ ,  $\alpha \notin Cn(A) \div \beta$ .

Case 2)  $\alpha, \beta \in Cn(A)$  and  $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$ . It follows trivially if  $\vdash \beta$ . Assume now that  $\not\vdash \beta$ .

$\{\gamma \in A : \{\delta \in A : \gamma \prec \delta\} \not\vdash \beta\} \subseteq \{\gamma \in A : \{\delta \in A : \gamma \prec \delta\} \not\vdash \alpha\}$ .

Hence,  $\{\gamma \in A : \{\delta \in A : (\gamma \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \gamma \text{ and } \not\vdash \gamma) \text{ or } (\vdash \delta \text{ and } \delta \in Cn(A) \div \gamma \text{ and } \not\vdash \gamma)\} \not\vdash \beta\} \subseteq \{\gamma \in A : \{\delta \in A : (\gamma \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \gamma \text{ and } \not\vdash \gamma) \text{ or } (\vdash \delta \text{ and } \delta \in Cn(A) \div \gamma \text{ and } \not\vdash \gamma)\} \not\vdash \alpha\}$ . Therefore according to  $(\div 1)$  and  $(\div 4)$ ,

$X = \{\gamma \in A : \{\delta \in A : (\gamma \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \gamma) \text{ or } (\vdash \delta \text{ and } \not\vdash \gamma)\} \not\vdash \beta\} \subseteq Y = \{\gamma \in A : \{\delta \in A : (\gamma \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \gamma) \text{ or } (\vdash \delta \text{ and } \not\vdash \gamma)\} \not\vdash \alpha\}$

$Cn(A) \div \gamma$ ) or  $(\vdash \delta \text{ and } \not\vdash \gamma)\} \not\vdash \alpha\}$ . Assume by *reductio* that  $\alpha \in Cn(A) \div \beta$ . From  $\alpha \in Cn(A) \div \beta$  it follows, by *base-reduction*, that  $Cn(A) \div \beta \cap A \vdash \alpha$ . By compactness, there exists a finite subset of  $Cn(A) \div \beta \cap A$ ,  $H = \{\alpha_1, \dots, \alpha_n\}$ , such that  $H \vdash \alpha$ . Let us assume that  $H \cap Cn(\emptyset) = \emptyset$ . For all  $\alpha_i \in H$ ,  $\alpha_i \in Cn(A) \div \beta = Cn(A) \div \beta'$ , for some  $\beta' \in A$  (by Observation 21). Hence, by *expulsiveness* (Lemma 3),  $\beta' \notin Cn(A) \div \alpha_i$ . Therefore  $\beta' \notin Y$ , since  $H \subseteq Z = \{\delta \in A : (\beta' \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \beta') \text{ or } (\vdash \delta \text{ and } \not\vdash \beta')\}$ . On the other hand  $\beta' \in X$ , since  $Z \subseteq Cn(A) \div \beta'$ , and by  $(\div 4)$   $Cn(A) \div \beta' \not\vdash \beta$ . Hence  $X \not\subseteq Y$ . Contradiction.

$(\Leftarrow)$  Let  $\alpha \notin Cn(A) \div \beta$  or  $\vdash \beta$ . We will prove by cases:

Case 1)  $\alpha \notin Cn(A)$ . Trivial.

Case 2)  $\alpha \in Cn(A)$ .

Case 2.1)  $\vdash \beta$ . Then  $\alpha, \beta \in Cn(A)$  and  $cut_{\leq}(\beta) \subseteq cut_{\leq}(\alpha)$ .

Case 2.2)  $\alpha \notin Cn(A) \div \beta$  and  $\not\vdash \beta$ . Hence, it follows that  $\beta \in Cn(A)$ ,  $\not\vdash \alpha$  and  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha$ , by  $(\div 3)$ ,  $(\div 1)$  and  $(\div 9)$ , respectively. Let us assume by *reductio* that  $cut_{\leq}(\beta) \not\subseteq cut_{\leq}(\alpha)$ . Hence there exists  $\psi \in A$  such that  $\psi \in cut_{\leq}(\beta)$  and  $\psi \notin cut_{\leq}(\alpha)$ . From which follows that  $\not\vdash \psi$ ,  $C = \{\delta \in A : (\psi \notin Cn(A) \div \delta \text{ and } \delta \in Cn(A) \div \psi) \text{ or } (\vdash \delta \text{ and } \not\vdash \psi)\} \not\vdash \beta$  and  $C \vdash \alpha$ . It holds that  $C \subseteq Cn(A) \div \psi$ . Then  $Cn(A) \div \psi \vdash \alpha$ . Hence, by  $(\div 4)$  and *linearity* (Lemma 3), it follows that  $Cn(A) \div \alpha \subset Cn(A) \div \psi$ . From  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha$  it follows that  $Cn(A) \div \beta \subset Cn(A) \div \psi$ . By  $(\div 9)$ ,  $\beta \in Cn(A) \div \psi$ . Therefore, by *base-reduction*,  $Cn(A) \div \psi \cap A \vdash \beta$ . On the other hand  $Cn(A) \div \psi \cap A \subseteq C$ . Hence  $C \vdash \beta$ . Contradiction. ■

#### PROOF OF COROLLARY 23.

$(\Rightarrow)$  Let  $-$  be an operator on  $A$  that satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion*, *upper bound* and *lower bound*. Then  $-$  is a bounded brutal contraction by Theorem 16. Hence, there exists a bounded ensconcement  $\leq$  such that  $A - \alpha = A -_{\leq} \alpha$ . Therefore, by  $-$  *relative closure* and *inclusion*,  $A -_{\leq} \alpha = Cn(A -_{\leq} \alpha) \cap A$ . By Observation 9  $\leq_{\leq}$  is a bounded epistemic entrenchment related to  $Cn(A)$ . From Theorems 20 and 22,  $Cn(A -_{\leq} \alpha) = Cn(A) \div_{\leq_{\leq}} \alpha$ , where  $\div_{\leq_{\leq}}$  is an operator on  $Cn(A)$  that satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*.

$(\Leftarrow)$  Let  $-$  be an operator on  $A$  such that  $A - \alpha = Cn(A \div \alpha) \cap A$ , where  $\div$  satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*. It remains to prove that  $-$  satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion*, *upper bound* and *lower bound*. From Theorem 22, it follows that there exists a bounded ensconcement  $\leq$  such that  $Cn(A) \div \alpha = Cn(A) \div_{\leq_{\leq}} \alpha$ . From Theorem 19 it follows that  $A -_{\leq} \alpha = (Cn(A) \div_{\leq_{\leq}} \alpha) \cap A$ .

Hence,  $A - \alpha = A -_{\leq} \alpha$ . Therefore  $-$  is a bounded brutal contraction, from which follows that  $-$  satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion*, *upper bound* and *lower bound*, by Theorem 16. ■

PROOF OF COROLLARY 24.

( $\Rightarrow$ ) Let  $\div$  be an operator on  $Cn(A)$  that satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*. Then, by Theorem 22, there exists a bounded ensconcement such that  $\div$  is an ensconcement-based withdrawal related to  $(A, \leq)$ . Hence,  $Cn(A) \div \alpha = Cn(A) \div_{\leq} \alpha$  where  $\leq$  is the epistemic entrenchment with respect to  $Cn(A)$  defined in Observation 9 and  $\div_{\leq}$  is the severe withdrawal on  $Cn(A)$  defined by  $(R_{\leq})$ . By Theorem 19,  $A - \alpha = (Cn(A) \div_{\leq} \alpha) \cap A$ , where  $-$  is the  $\leq$ -based brutal contraction. Therefore, by *base-reduction* and  $(\div 1)$ ,  $Cn(A - \alpha) = Cn((Cn(A) \div_{\leq} \alpha) \cap A) = Cn(A) \div_{\leq} \alpha = Cn(A) \div \alpha$ . By Theorem 16  $-$  satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion*, *upper bound* and *lower bound*.

( $\Leftarrow$ ) Let  $-$  be an operator on  $A$  that satisfies *success*, *inclusion*, *vacuity*, *failure*, *relative closure*, *strong inclusion*, *upper bound* and *lower bound* such that:  $Cn(A) \div \alpha = Cn(A - \alpha)$ . Hence, by Theorem 16,  $-$  is a bounded brutal contraction on  $A$ . By Theorem 20,  $Cn(A - \alpha) = Cn(A) \div_{\leq} \alpha$ , where  $\leq$  is the epistemic entrenchment with respect to  $Cn(A)$  defined in Observation 9, and  $\div_{\leq}$  is the severe withdrawal on  $Cn(A)$ , defined by  $(R_{\leq})$ . Hence, by Theorem 22,  $\div$  satisfies  $(\div 1)$  to  $(\div 4)$ ,  $(\div 9)$ , *base-reduction*, *upper bound* and *lower bound*. ■

PROOF OF OBSERVATION 25.  $(\div 1)$  Follows trivially from the definition.  $(\div 2)$  follows from the definition and  $-$  *inclusion*.  $(\div 3)$  follows from the definition and  $-$  *failure* and  $-$  *vacuity*.  $(\div 4)$  follows from the definition and  $-$  *success*. For  $(\div 9)$ , consider  $\alpha \notin Cn(A) \div \beta$ . Hence, by definition of  $\div$ ,  $\alpha \notin Cn(A - \beta)$ . By  $-$  *strong inclusion* it follows that  $A - \beta \subseteq A - \alpha$ . Therefore,  $Cn(A) \div \beta \subseteq Cn(A) \div \alpha$ . For *base-reduction*, let  $Cn(A) \div \alpha \vdash \beta$ . Hence, by the definition of  $\div$ , it follows that  $\beta \in Cn(A - \alpha)$ . By  $-$  *relative closure* and  $-$  *inclusion*, it follows that  $A \cap Cn(A - \alpha) = A - \alpha$ . Therefore,  $Cn(A - \alpha) \cap A \vdash \beta$ . Hence, by definition of  $\div$ ,  $(Cn(A) \div \alpha) \cap A \vdash \beta$ . ■

PROOF OF OBSERVATION 26. For  $\div$  *upper bound*. Let  $X \subseteq Cn(A) \setminus Cn(\emptyset)$  be a non-empty set. From Observation 17, for all  $\alpha \in X$ , there exists  $\psi \in A$  such that  $A - \alpha = A - \psi$ . Let  $X' = \{\psi \in A : A - \alpha = A - \psi, \text{ for some } \alpha \in X\}$ . Hence  $X'$  is a non-empty set of non tautological formulae. By  $-$  *upper bound* there exists some  $\beta \in X'$  such that  $A - \gamma \subseteq A - \beta$  for all  $\gamma \in X'$ . Hence, there exists some  $\delta \in X$  such that  $A - \gamma \subseteq A - \delta = A - \beta$  for all

$\gamma \in X$ . Hence, by the definition of  $\div$ , there exists some  $\delta \in X$  such that  $Cn(A) \div \gamma = Cn(A - \gamma) \subseteq Cn(A - \delta) = Cn(A) \div \delta$  for all  $\gamma \in X$ .

The proof for  $\div$  lower bound is analogous to the one presented for  $\div$  upper bound. ■

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