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On the Logic of Theory Change: Contraction without Recovery

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To Carlos E. Alchourrón
In memoriam

Abstract. The postulate of *Recovery*, among the six postulates for theory contraction, formulated and studied by Alchourrón, Gärdenfors and Makinson is the one that has provoked most controversy. In this article we construct withdrawal functions that do not satisfy *Recovery*, but try to preserve *minimal change*, and relate these withdrawal functions with the AGM contraction functions.

Key words: Theory change, AGM theory, belief contraction, recovery postulate, Levi identity, Harper identity

1. Introduction

The postulate of *Recovery*, among the six postulates for theory contraction, formulated and studied by Alchourrón, Gärdenfors and Makinson (Alchourrón and Makinson, 1982, 1985; Alchourrón et al., 1985; Makinson, 1987; Gärdenfors, 1988) is the one that has provoked most controversy (Makinson, 1987, 1996; Hansson, 1991, 1993; Nayak, 1994).

There are clearly cases in which the recovery postulate seems to be contrary to intuition. Basically, proposed counter-examples reduce to the following one (Hansson, 1993):

> Let $\mathbf{K}$ be a theory, and let $x, y \in \mathbf{K}$. Suppose we wish to eliminate x and y; and do so by contracting their disjunction $x \vee y$. If later on we are informed that either x or y is actually true, without being told which one is true, we shall expand our beliefs by $x \vee y$. After performing the expansion as sanctioned by the AGM model, the resulting set will restore the whole of $\mathbf{K}$ and thus in particular both x and y, contrary to what is expected.

The problem appears because when we contract $\mathbf{K}$ by $x \vee y$, by recovery the sentence $(x \vee y) \rightarrow (x \wedge y)$ must remain in the result of the contraction.

Contraction functions without recovery have been dubbed *withdrawal functions* (Makinson, 1987).

This note does not attempt to settle the question of the acceptability of the recovery postulate. Its purpose is to define a sensible *withdrawal function* over theories and to establish its connection with AGM revision functions. Our desire is to retain much as possible of the previous theory, but remove the sentences that provoke non-intuitive results.

Another well known point should be recalled: When dealing with standard AGM functions, the Levi and Harper identities allow for the interdefinability of contraction and revision functions. As a direct consequence, we can take either of the two as primitive. There is no identity that takes us from revision to contraction without recovery such as the Harper identity. We will give the connection with AGM revision through an identity similar to the Harper identity.

We recall that in (Hansson and Olsson, 1995), Hansson and Olsson characterized the Levi contractions that satisfy all basic AGM postulates except *Recovery*. But, this is a quite general withdrawal function and not an attempt to preserve any kind of minimal change.

We also recall that in his discussion of examples like the one above, Fuhrmann (1991) proposed meeting them by imposing the following "filtering condition:"

> "If y has been retracted from a base B in order to bar derivations of x from B, then the contraction of Cn(B) by x should not contain any sentences which were in Cn(B) "just because" y was in Cn(B)."

However, this work and later (Williams, 1994; Rott, 1995; Makinson, 1996) use this condition when dealing with *bases*. Then it is necessary to find a conceptual mechanism like the filtering condition to operate with theories, which in effect is what we propose in this paper.

In Section 2 we present the AGM functions. In Section 3 we develop contraction functions, for theories, that do not satisfy the recovery postulate. In Section 4 we relate these contraction functions with the AGM revision functions.

2. Background: The AGM Account (Alchourrón et al., 1985; Gärdenfors, 1988)

In this account the beliefs of a rational agent are represented by a belief set $\mathbf{K}$ and closed under logical consequence Cn, where Cn satisfies the following properties: $\mathbf{K} \subseteq Cn(\mathbf{K})$ for any set $\mathbf{K}$ of propositions, $Cn(Cn(\mathbf{K})) \subseteq Cn(\mathbf{K})$ and $Cn(\mathbf{K}) \subseteq Cn(\mathbf{H})$ when $\mathbf{K} \subseteq \mathbf{H}$. We assume that Cn includes classical logical consequence, satisfies the rule of introduction of disjunction into premises, and is compact. We write $\vdash x$ if $x \in$ Cn($\emptyset$) and $x \equiv y$ if $x \leftrightarrow y \in$ Cn($\emptyset$).

A theory is understood to be any set $\mathbf{K}$ of propositions closed under Cn, i.e. where $Cn(\mathbf{K}) = \mathbf{K}$.

Let $\mathbf{L}$ be the set of all the sentences of the language. Let $\mathcal{K}$ be the set of all theories. Formally, we define the expansion function $+$ from $\mathcal{K} \times \mathbf{L}$ to $\mathcal{K}$, where $(\mathbf{K}+x)$ denotes the expansion of $\mathbf{K}$ by x and is defined by $(\mathbf{K}+x) = Cn(\mathbf{K} \cup \{x\})$.

The six basic postulates for contraction are:

$(\mathbf{K}-1)$	$\mathbf{K}-x$ is a theory whenever $\mathbf{K}$ is a theory (closure)
$(\mathbf{K}-2)$	$\mathbf{K}-x \subseteq \mathbf{K}$ (inclusion)
$(\mathbf{K}-3)$	If $x \notin Cn(\mathbf{K})$, then $\mathbf{K}-x = \mathbf{K}$ (vacuity)
$(\mathbf{K}-4)$	If $x \notin Cn(\emptyset)$, then $x \notin \mathbf{K}-x$ (success)
$(\mathbf{K}-5)$	If $x \leftrightarrow y \in Cn(\emptyset)$ then $\mathbf{K}-x = \mathbf{K}-y$ (preservation)
$(\mathbf{K}-6)$	$\mathbf{K} \subseteq (\mathbf{K}-x)+x$ whenever $\mathbf{K}$ is a theory (recovery).

Clearly from $(\mathbf{K}-2)$ and $(\mathbf{K}-6)$ every contraction function satisfies the following property:

PROPOSITION 1. *Whenever* $\mathbf{K}$ *is a theory, if* $x \in \mathbf{K}$, *then* $\mathbf{K} = (\mathbf{K}-x)+x$.

In Alchourrón et al. (1985) and Gärdenfors (1988) we see that if $\mathbf{K}$ is a theory then the functions $\mathbf{K}-x = \cap S(\mathbf{K} \perp x)$, called *Partial Meet Contraction Functions*, fully characterize the contraction functions that satisfy $(\mathbf{K}-1)-(\mathbf{K}-6)$; where $\mathbf{K} \perp x$ is the set of all inclusion-maximal subsets A of $\mathbf{K}$ such that x is not a logical consequence of A, S is a selection function, such that $S(\mathbf{K} \perp x)$ is a non-empty subset of $\mathbf{K} \perp x$, unless the latter is empty, in which case $S(\mathbf{K} \perp x) = \{\mathbf{K}\}$.

The six basic postulates for revision are:

$(\mathbf{K}*1)$	$\mathbf{K}*x$ is a theory (closure)
$(\mathbf{K}*2)$	$x \in \mathbf{K}*x$ (success)
$(\mathbf{K}*3)$	$\mathbf{K}*x \subseteq \mathbf{K}+x$ (inclusion)
$(\mathbf{K}*4)$	If $\neg x \notin \mathbf{K}$ then $\mathbf{K}+x \subseteq \mathbf{K}*x$ (vacuity)
$(\mathbf{K}*5)$	$\mathbf{K}*x = \mathbf{K}_{\perp}$ iff $\neg x \in Cn(\emptyset)$ (consistency)
$(\mathbf{K}*6)$	If $x \leftrightarrow y \in Cn(\emptyset)$ then $\mathbf{K}*x = \mathbf{K}*y$ (preservation).

2.1. Relation between Contraction and Revision

We have seen that contraction and revision are defined by two different sets of postulates. These postulates are independent in the sense that the postulates of revision do not refer to contraction and vice versa. However it is possible define revision functions in terms of contraction functions, and vice versa, by means of the formulas of Levi and Harper respectively.

DEFINITION 1 (Makinson, 1987). Let $\mathbf{K}$ be a theory, then Levi is the function such that for every operator $-$ for $\mathbf{K}$, Levi($-$) is the operator for $\mathbf{K}$ such that for all x:

$$\mathbf{K}(\text{Levi}(-)x) = (\mathbf{K} - \neg x) + x$$

DEFINITION 2 (Makinson, 1987). Let $\mathbf{K}$ be a theory, then Harper is the function such that for every operator $*$ for $\mathbf{K}$, Harper($*$) is the operator for $\mathbf{K}$ such that for all x:

$$\mathbf{K}(\text{Harper}(*)x) = \mathbf{K} \cap \mathbf{K} * \neg x$$

THEOREM 1 (Alchourrón et al., 1985; Makinson, 1987). *Let* $\mathbf{K}$ *be a theory and* $-$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-5)$. *Then* Levi($-$) *is an operator for* $\mathbf{K}$ *that satisfies the revision postulates* $(\mathbf{K}*1)-(\mathbf{K}*6)$.

Note that postulate ($\mathbf{K} - 6$) is not needed in this theorem.

THEOREM 2 (Alchourrón et al., 1985; Makinson, 1987). *Let* $\mathbf{K}$ *be a theory and* $*$ *an operator for* $\mathbf{K}$ *that satisfies the revision postulates* $(\mathbf{K}*1)-(\mathbf{K}*6)$. *Then* Harper($*$) *is an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-6)$.

In Makinson (1987), Makinson proves the following results:

THEOREM 3. *Let* $\mathbf{K}$ *be a theory and* $-$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-6)$. *Then* Harper(Levi($-$)) $= -$.

THEOREM 4. *Let* $\mathbf{K}$ *be a theory and* $*$ *an operator for* $\mathbf{K}$ *that satisfies the revision postulates* $(\mathbf{K}*1)-(\mathbf{K}*6)$. *Then* Levi(Harper($*$)) $= *$.

3. How to Construct a Contraction Function without Recovery

We can ask first why a contraction function satisfies recovery. The answer is the following:

DEFINITION 3. Let $V(x) = \{x \to y : y \in \mathbf{K}/\mathbf{K} - x\}$, where $-$ is a *partial meet contraction function.*

OBS. 1. $V(x) \subseteq \mathbf{K} - x$

We see clearly that the contraction function satisfies recovery because all the members of $V(x)$ are in $\mathbf{K} - x$. But there are some cases where we do not wish to retain all the members of $V(x)$ (see the example in the introduction) although *partial meet contraction* cannot eliminate any of them. We need a function that preserves minimal change and allows to remove the undesirable elements of $V(x)$.

DEFINITION 4. Let $\mathbf{L}$ be the set of all the sentences of the language. Let $\mathbf{K}$ be the set of all theories. A function $\underline{s} : \mathbf{K} \times \mathbf{L}$ is a semi-contraction function iff there is a contraction function $-$ (satisfying postulates $(\mathbf{K}-1)-(\mathbf{K}-6)$) such that for all $\mathbf{K}\ in \mathbf{K}$ and $x \in \mathbf{L}$:

$$\mathbf{K}\ \underline{s}\ x = \mathbf{K} - x \cap \mathbf{K} - w(x),$$
where $w(x) = x \to y; y = \text{Sel}(\mathbf{K}/\mathbf{K} - x)$.

Here, $\text{Sel}(\mathbf{K}/\mathbf{K} - x)$ selects an element of $\mathbf{K}/\mathbf{K} - x$; this is equivalent to selecting some finite subset of $\mathbf{K}/\mathbf{K} - x$, because if two sentences are in $\mathbf{K}/\mathbf{K} - x$, their conjunction is in $\mathbf{K}/\mathbf{K} - x$ too (the demonstration is trivial). If $\mathbf{K}/\mathbf{K} - x = \emptyset$, then $\text{Sel}(\mathbf{K}/\mathbf{K} - x)$ is put as x.

In the example of the introduction, we do not want to recover $x \wedge y$ when add $x \vee y$. We achieve this by using a semi-contraction function:

$$\mathbf{K}\ \underline{s}\ (x \vee y) = \mathbf{K} - (x \vee y) \cap \mathbf{K} - ((x \vee y) \to (x \wedge y))$$

THEOREM 5. $\mathbf{K}\ \underline{s}\ x$ *defined as Definition 4 satisfies* $(\mathbf{K}-1)-(\mathbf{K}-5)$.

THEOREM 6. $\mathbf{K}\ \underline{s}\ x$ *defined as Definition 4 fails to satisfy* $(\mathbf{K}-6)$ *iff* $x \to y \notin Cn(\emptyset)$.

COROLLARY 1. *Let* $\mathbf{K}$ *be a theory and* $-$ *an operator for* $\mathbf{K}$. *If* $-$ *satisfies* $(\mathbf{K}-1)-(\mathbf{K}-6)$ *then* $-$ *is a semi-contraction function.*

DEFINITION 5 (Failure) (Fuhrmann and Hansson, 1995). Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$. $\mathbf{K} - x$ satisfies *Failure* iff $\mathbf{K} - x = \mathbf{K}$ when $x \in Cn(\emptyset)$.

We recall the well known fact:

OBS. 2. Let $\mathbf{K}$ be a theory and $-$ an operator for $\mathbf{K}$. If $\mathbf{K} - x$ satisfies $(\mathbf{K}-1)$, $(\mathbf{K}-2)$ and $(\mathbf{K}-6)$ then it satisfies Failure.

COROLLARY 2. $\mathbf{K}\ \underline{s}\ x$ *defined as Definition 4 satisfies Failure.*

COROLLARY 3. *Let* $\mathbf{K}$ *be a theory,* $-$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-6)$, *and* $\underline{s}$ *an associated semi-contraction function defined as in Definition 4. If* $\underline{s}$ *satisfies* $(\mathbf{K}-6)$ *then for all* x $\mathbf{K}\ \underline{s}\ x = \mathbf{K} - x$.

We can ask ourselves if the semi-contraction functions characterize the withdrawal functions that *Failure*. The answer is no as we can see in the following counter-example:

Let $\mathbf{K} = Cn(\{a, b, c\})$: $a, b, c \in \mathbf{L}$

Let $\sim$ be defined as:

$$\mathbf{K} \sim x = \begin{cases} \mathbf{K} & \text{if } x \notin \mathbf{K} \\ Cn(\{b\}) & \text{if } x \equiv a \\ Cn(\{a, y \to b \wedge c\}) & \text{if } x \equiv a \to y, \forall y \in \mathbf{K} \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

It is trivial to show that $\mathbf{K} \sim x$ satisfies $(\mathbf{K}-1)-(\mathbf{K}-5)$ and *Failure*. We show that $\mathbf{K} \sim x$ cannot be defined as a semi-contraction function.

Suppose that $\sim$ is a semi-contraction function defined as in Definition 4. Let $-$ be a contraction function for $\mathbf{K}$ that satisfies $(\mathbf{K}-1)-(\mathbf{K}-6)$, such that:

$$\mathbf{K} \sim x = \mathbf{K} - x \cap \mathbf{K} - (x \to y), \quad y \in \mathbf{K}/\mathbf{K} - x$$

Then in particular,

$\mathbf{K} \sim (a \to y) = \mathbf{K} - (a \to y) \cap \mathbf{K} - ((a \to y)) \to z,$

$z \in \mathbf{K}/\mathbf{K} - (a \to y), \quad y \in \mathbf{K}$

$(\mathbf{K} \sim (a \to y)) + (a \to y) = \mathbf{K}$ (by def. of $\sim$)

$z \in (\mathbf{K} \sim (a \to y)) + (a \to y)$

then $(a \to y) \to z \in \mathbf{K} \sim (a \to y)$ (by $(\mathbf{K}-1)$)

then $(a \to y) \to z \in \mathbf{K} - ((a \to y)) \to z$

then $\vdash (a \to y) \to z$ (by $(\mathbf{K}-3)$)

then $\mathbf{K} - ((a \to y) \to z) = \mathbf{K}$ (by Failure)

so $\mathbf{K} \sim (a \to y) = \mathbf{K} - (a \to y)$

$\mathbf{K} \sim a = \mathbf{K} - a \cap \mathbf{K} - (a \to w), \quad w \in \mathbf{K}/\mathbf{K} - a$

$\mathbf{K} \sim a = \mathbf{K} - a \cap \mathbf{K} \sim (a \to w)$

$\mathbf{K} \sim a = \mathbf{K} - a \cap Cn(\{a, w \to b \wedge c\})$

$\mathbf{K} \sim a$ does not satisfy $(\mathbf{K}-\mathbf{6})$, so $a \nvdash w$

Now $b \notin Cn(\{a, w \to b \wedge c\})$

so $b \notin \mathbf{K} \sim a$, contradicting the def. of $\sim$.

So we have reached a clear contradiction, from the supposition that $\sim$ is a semi-contraction function.

4. Relation between $\underline{s}$ and $*$

As the Levi and Harper functions for the AGM model, we need to define new functions to relate $\underline{s}$ and $*$. To define $*$ in terms of a semi-contraction $\underline{s}$ we can use the Levi identity (see Definition 1), i.e.,

$$\mathbf{K}(\text{Levi}(\underline{s}))\ x = (\mathbf{K}\ \underline{s}\ \neg x) + x$$

However, to define a semi-contraction $\underline{s}$ in terms of $*$ we need a new identity:

DEFINITION 6. Let $\mathbf{K}$ be a theory, then Harper′ is the function such that for every operator $*$ for $\mathbf{K}$, Harper′$(*)$ is the operator for $\mathbf{K}$ such that for all x:

$$\mathbf{K}(\text{Harper}'(*))x = \mathbf{K} \cap \mathbf{K} * \neg x \cap \mathbf{K} * \neg w(x),$$
$$\text{where } w(x) = x \to y;\ y = \text{Sel}(\mathbf{K}/\mathbf{K} \cap \mathbf{K} * \neg x) = \text{Sel}(\mathbf{K}/\mathbf{K} * \neg x)$$

THEOREM 7. *Let* $\mathbf{K}$ *be a theory,* $-$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-6)$*, and* $\underline{s}$ *an associated semi-contraction function defined as Definition 4. Then* Levi$(\underline{s})$ = Levi$(-)$.

COROLLARY 4. Levi$(\underline{s})$ *satisfies* $(\mathbf{K} * 1) - (\mathbf{K} * 6)$.

THEOREM 8. *Let* $\mathbf{K}$ *be a theory and* $*$ *an operator for* $\mathbf{K}$ *that satisfies the revision postulates* $(\mathbf{K} * 1) - (\mathbf{K} * 6)$*. Then* Harper′ *is an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K} - 1) - (\mathbf{K} - 5)$.

THEOREM 9. *Let* $\mathbf{K}$ *be a theory,* $-$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates* $(\mathbf{K}-1)-(\mathbf{K}-6)$*, and* $\underline{s}$ *its associated semi-contraction function defined as Definition 4. Then* Harper′(Levi$(\underline{s})$) $= \underline{s}$.

THEOREM 10. *Let* $\mathbf{K}$ *be a theory and* $*$ *an operator for* $\mathbf{K}$ *that satisfies the revision postulates* $(\mathbf{K} * 1) - (\mathbf{K} * 6)$*. Then* Levi(Harper′$(*)$) $= *$.

5. Conclusions

We have defined a contraction function without recovery applicable to theories. We have obtained this function as a combination of two applications of a single standard AGM contraction function.We have related our withdrawal functions with the classical AGM revision functions through the Levi and Harper′ identities and finally have shown that they become reciprocally dual.

Appendix A: *Proofs*

Proof of Obs. 1

Suppose that $y \in \mathbf{K}$ and $y \notin \mathbf{K} - x$
then $\mathbf{K} \neq \mathbf{K} - x$
then $x \in \mathbf{K}$ (by $(\mathbf{K} - 3)$)
then $y \in (\mathbf{K} - x) + x$ (by $(\mathbf{K} - 6)$)
hence $x \to y \in \mathbf{K} - x$.

Proof of Theorem 5

$(\mathbf{K} - 1)$, $(\mathbf{K} - 2)$ are trivial.
$(\mathbf{K} - 3)$: Suppose that $x \notin \mathbf{K}$
then $\mathbf{K} - x = \mathbf{K}$ (by $(\mathbf{K} - 3)$)
then $\mathbf{K}/\mathbf{K} - x = \emptyset$
then $\mathbf{K} - w(x) = \mathbf{K} - (x \to x)$ (by Definition 4)
then $\mathbf{K} - w(x) = \mathbf{K}$ (by Theorem 2)
hence $\mathbf{K} \underline{s}\, x = \mathbf{K} - x \cap \mathbf{K} - w(x) = \mathbf{K} \cap \mathbf{K} = \mathbf{K}$.

$(\mathbf{K} - 4)$ is trivial.
$(\mathbf{K} - 5)$ is trivial since $\mathbf{K}/\mathbf{K} - x = \mathbf{K}/\mathbf{K} - y$ (by $(\mathbf{K} - 5)$).

Proof of Theorem 6

$\Leftarrow)$ Suppose that $\mathbf{K} \underline{s}\, x$ satisfies $(\mathbf{K} - 6)$, i.e., $\mathbf{K} \subseteq (\mathbf{K} \underline{s}\, x) + x$
a) If $\mathbf{K}/\mathbf{K} - x = \emptyset$
then $y = x$ (by Definition 4)
hence $y \in Cn(\{x\})$ (by Definition of Cn).
b) If $\mathbf{K}/\mathbf{K} - x \neq \emptyset$
then $y \in \mathbf{K}$ (by Definition 4)
then $y \in (\mathbf{K} \underline{s}\, x) + x$ (by $(\mathbf{K} - 6)$)
then $x \to y \in \mathbf{K} \underline{s}\, x$
then $x \to y \in \mathbf{K} \cap \mathbf{K} - (x \to y)$ (by Definition 4)
then $x \to y \in \mathbf{K} - (x \to y)$
hence $x \to y \in Cn(\emptyset)$ (by $(\mathbf{K} - 4)$).

$\Rightarrow)$ Suppose that $x \to y \in Cn(\emptyset)$
then $\mathbf{K} - (x \to y) = \mathbf{K}$ (by Theorem 2)
then $\mathbf{K} \underline{s}\, x = \mathbf{K} - x \cap \mathbf{K} = \mathbf{K} - x$, that satisfies $(\mathbf{K} - 6)$.

Proof of Obs. 2

Let $x \in Cn(\emptyset)$

$$\mathbf{K} \;=\; (\mathbf{K} - x) + x \quad \text{(by } (\mathbf{K} - 6)\text{), so}$$

$$\mathbf{K} \;=\; \mathbf{K} - x \quad \text{(by } (\mathbf{K} - 2)\text{)}$$

Proof of Corollary 2

Trivial, since $\underline{s}$ satisfies $(\mathbf{K} - 2)$ and $(\mathbf{K} - 6)$

Proof of Corollary 3

Suppose that $\underline{s}$ satisfies $(\mathbf{K} - 6)$
then $x \rightarrow y \in Cn(\emptyset)$ (by Theorem 6)
then $\mathbf{K} - (x \rightarrow y) = \mathbf{K}$ (by Failure)
then $\mathbf{K}\ \underline{s}\ x = \mathbf{K} - x \cap \mathbf{K} = \mathbf{K} - x$

Proof of Theorem 7

$\mathbf{K}(\text{Levi}(\underline{s}))x = (\mathbf{K}\ \underline{s}\ \neg x) + x$ (by Definition of Levi)
$= (\mathbf{K} - \neg x \cap \mathbf{K} - w(\neg x)) + x$ (by Definition 4)
$= (\mathbf{K} - \neg x) + x \cap (\mathbf{K} - w(\neg x)) + x$
$= (\mathbf{K} - \neg x) + x \cap (\mathbf{K} - w(\neg x)) + (x \wedge w(\neg x))$
(since $x \equiv x \wedge w(\neg x)$)
$= (\mathbf{K} - \neg x) + x \cap ((\mathbf{K} - w(\neg x)) + w(\neg x)) + x$
$= (\mathbf{K} - \neg x) + x \cap \mathbf{K} + x$
(by Proposition 1, because $w(\neg x) \in \mathbf{K}$)
$= (\mathbf{K} - \neg x) + x$
$= \mathbf{K}(\text{Levi}(-))x$ (by Definition of Levi).

where $w(x) = x \rightarrow y$; $y = \text{Sel}(\mathbf{K}/\mathbf{K} \cap \mathbf{K} * \neg x) = \text{Sel}(\mathbf{K}/\mathbf{K} * \neg x)$.

Proof of Theorem 8

$\mathbf{K}(\text{Harper}'(*))x = \mathbf{K} \cap \mathbf{K} * \neg x \cap \mathbf{K} * \neg w(x)$
where $w(x) = x \rightarrow y$; $y = \text{Sel}(\mathbf{K}/\mathbf{K} \cap \mathbf{K} * \neg x)$
(by Definition of Harper′)
$= (\mathbf{K} \cap \mathbf{K} * \neg x) \cap (\mathbf{K} \cap \mathbf{K} * \neg w(x))$
$= \mathbf{K} - x \cap \mathbf{K} - w(x)$
where $w(x) = x \rightarrow y$; $y = \text{Sel}(\mathbf{K}/\mathbf{K} - x)$
(by definition of Harper),
that satisfies $(\mathbf{K} - 1) - (\mathbf{K} - 5)$ (by Theorem 5).

Proof of Theorem 9

$$\begin{aligned}\mathbf{K}(\text{Harper}'(\text{Levi}(\underline{s})))x &= \mathbf{K} \cap \mathbf{K}(\text{Levi}(\underline{s}))\neg x \cap \mathbf{K}(\text{Levi}(\underline{s}))\neg w(x)\\ &\quad \text{where } w(x) = x \to y;\ y = \text{Sel}(\mathbf{K}/\mathbf{K} \cap \mathbf{K}(\text{Levi}(\underline{s}))\neg x)\\ &\quad \text{(by Definition of Harper}')\\ &= \mathbf{K} \cap (\mathbf{K} - x) + \neg x \cap (\mathbf{K} - w(x)) + \neg w(x)\\ &\quad \text{where } w(x) = x \to y;\ y = \text{Sel}(\mathbf{K}/\mathbf{K} - x)\\ &\quad \text{(by Definition of Levi)}\\ &= (\mathbf{K} \cap (\mathbf{K} - x) + \neg x) \cap (\mathbf{K} \cap (\mathbf{K} - w(x)) + \neg w(x))\\ &= \mathbf{K} - x \cap \mathbf{K} - w(x) \text{ (by Theorem 3)}\\ &= \mathbf{K}\ \underline{s}\ x \text{ (by Definition 4).}\end{aligned}$$

Proof of Theorem 10

$$\begin{aligned}\mathbf{K}(\text{Levi}(\text{Harper}'(*))x &= (\mathbf{K}(\text{Harper}'(*))\neg x) + x \text{ (by Definition of Levi)}\\ &= (\mathbf{K} \cap \mathbf{K} * x \cap \mathbf{K} * \neg w(\neg x)) + x\\ &\quad \text{where } w(\neg x) = \neg x \to y;\ y = \text{Sel}(\mathbf{K}/\mathbf{K} \cap \mathbf{K} * x)\\ &\quad \text{(by Definition of Harper}')\\ &= \mathbf{K} + x \cap (\mathbf{K} * x) + x \cap (\mathbf{K} * \neg w(\neg x)) + x\\ &= \mathbf{K} + x \cap \mathbf{K} * x \cap \mathbf{K}_{\perp} \text{ (since } \neg x \in \mathbf{K} * \neg w(\neg x))\\ &= \mathbf{K} * x \text{ (by } (\mathbf{K} - 3)).\end{aligned}$$

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