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THE RENORMALIZATION OF SELF-INTERSECTION LOCAL TIMES I. THE CHAOS EXPANSION

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1. Introduction

The intersections of Brownian motion paths have been investigated since the '40s.[20] One can consider intersections of sample paths with themselves or e.g. with other, independent Brownian motions,[37] one can study simple[5] or n-fold intersections[6] and one can ask all of these questions for linear, planar, spatial or — in general — d-dimensional Brownian motion: Evidently self-intersections become increasingly scarce as the dimension d increases.

Intersection local times of Brownian motion were studied by many authors, see e.g. Refs. 1, 3–11, 15, 18–39. A more systematic review can be found e.g. in the recent Ref. 15.

An informal but rather suggestive definition of self-intersection local time of Brownian motion B is in terms of an integral over Dirac's — or Donsker's — δ-function

$$L \equiv \int d^2t \; \delta(B(t_2) - B(t_1)) \, ,$$

intended to sum up the contributions from each pair of "times" t_1, t_2 for which the Brownian motion B is at the same point.

In Edwards' modeling of polymer molecules by Brownian motion paths, L is used to model the "excluded volume" effect: Different parts of the molecule should not be located at the same point in space. As another application, Symanzik introduced L as a tool for constructive quantum field theory in Ref. 32.

A rigorous definition, such as e.g. through a sequence of Gaussian's approximating the δ-function or in terms of generalized Brownian functionals,[10,31,34] will lead to increasingly singular objects and will necessitate various "renormalizations" as the dimension d increases. For $d > 1$ the expectation will diverge in the limit and

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must be subtracted,[18,33] clearly L will then no longer be positive. For $d > 3, 5, 7, \ldots$ further subtractions have been proposed[34] that will make L a well-defined generalized function of Brownian motion.

For $d = 3$ another renormalization has been constructed by Westwater to make the Gibbs factor $e^{-g \cdot L}$ of the polymer model well-defined,[36] for yet another, recent approach see Ref. 1.

Yor, in Ref. 39, first suppresses the short time accumulation of self-intersections by the regularization

$$\delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \to \delta(\mathbf{B}(t_2) - \mathbf{B}(t_1) + \boldsymbol{\varepsilon})$$

and shows, again for $d = 3$, that a multiplicative renormalization

$$r(\varepsilon)\,(L_\varepsilon - E(L_\varepsilon))$$

gives rise to another, independent Brownian motion as the weak limit of regularized and subtracted approximations to L.

In this paper we study similar limits, for arbitrary $d \geq 3$, using a Gaussian regularization of the δ-function for which the chaos expansion of the corresponding regularized L_ε is available.[10] For a suitably subtracted and renormalized local time, each term in this expansion converges in law to a Brownian motion.

We prepare and state these results in Sec. 2, in Sec. 3 we give their proofs. In a forth coming paper[3] we extend these results to the corresponding series.

2. Definitions and Main Results

2.1. *White noise analysis and local times*

We reproduce here some white noise analysis concepts as introduced in Ref. 10, referring to Ref. 14 for a systematic presentation.

Brownian motions $B_i, i = 1, \ldots, d$, have version in terms of white noise ω_i via

$$B_i(t) = \langle \omega_i, 1_{[0,t]} \rangle = \int_0^t \omega_i(s) ds \,.$$

Hence we consider independent d-tuples of Gaussian white noise $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, \ldots, f_d) \in S(R, R^d)$, and introduce the following notation:

$$\mathbf{n} = (n_1, \ldots, n_d), \qquad n = \sum_1^d n_i, \qquad \mathbf{n}! = \prod_1^d n_i!$$

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt \; f_i^2(t) \,,$$

$$\langle F_{\mathbf{n}}, \mathbf{f}^{\otimes \mathbf{n}} \rangle = \int d^n t \; F_{\mathbf{n}}(t_1, \ldots, t_n) \; \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \ldots, t_n)$$

and similarly for $\langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle$ where for d-tuples of white noise the Wick product[14] $: \ldots :$ generalizes to

$$: \boldsymbol{\omega}^{\otimes \mathbf{n}} := \bigotimes_{i=1}^{d} : \omega_i^{\otimes n_i} : \,.$$

The vector valued white noise $\boldsymbol{\omega}$ has the characteristic function

$$C(\mathbf{f}) = E(e^{i\langle \boldsymbol{\omega}, \mathbf{f}\rangle}) = \int_{S^*(R,R^d)} d\mu\,[\boldsymbol{\omega}]\, e^{i\langle \boldsymbol{\omega}, \mathbf{f}\rangle} = e^{-\frac{1}{2}\langle \mathbf{f}, \mathbf{f}\rangle}\,, \tag{1}$$

where $\langle \boldsymbol{\omega}, \mathbf{f}\rangle = \sum_{i=1}^d \langle \omega_i, f_i\rangle$ and $f_i \in S(R, R)$.

The Hilbert space

$$(L^2) \equiv L^2(d\mu)$$

is canonically isomorphic to the d-fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq \left(\bigoplus_{k=0}^{\infty} \mathrm{Sym}\ L^2(R^k, k!d^k t) \right)^{\otimes d} \equiv \mathfrak{F}\,, \tag{2}$$

for the general element of (L^2), this implies the chaos expansion:

$$\varphi(\boldsymbol{\omega}) = \sum_{\mathbf{n}=0}^{\infty} \langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_{\mathbf{n}} \rangle \tag{3}$$

with kernel functions F in $\mathfrak{F}$.

It is desirable to introduce regularizations for the intersection local time, with a view towards the construction of well-defined, "renormalized" intersection local times in higher dimensions where the latter do not exist without subtractions. A computationally simple regularization is, for $\varepsilon > 0$,

$$L_\varepsilon \equiv \int_0^t dt_2 \int_0^{t_2} dt_1\, \delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1))\,,$$

with

$$\delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \equiv (2\pi\varepsilon)^{-d/2} e^{-\frac{|\mathbf{B}(t_2)-\mathbf{B}(t_1)|^2}{2\varepsilon}}\,. \tag{4}$$

It has the following chaos expansion, which we quote here only for $d \geq 3$:

Theorem 2.1.[10] *For any $\varepsilon > 0$, $L_\varepsilon - E(L_\varepsilon)$ has kernel functions $F \in \mathfrak{F}$ given by*

$$\begin{aligned} F_{\varepsilon,\mathbf{n}}(s_1, \ldots, s_n) = (-1)^{\frac{n}{2}} \left(\varkappa(\varkappa+1)(2\pi)^{d/2}\, 2^{\frac{n}{2}} \left(\frac{\mathbf{n}}{2}\right)! \right)^{-1} \\ \cdot \Theta(u)\Theta(t-v) \cdot ((v-u+\varepsilon)^{-\varkappa} + (t+\varepsilon)^{-\varkappa} \\ -(v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}) \end{aligned} \tag{5}$$

if all n_i are even, and zero otherwise, with $v(s_1, \ldots, s_n) \equiv \max(s_1, \ldots, s_n)$, $u(s_1, \ldots, s_n) \equiv \min(s_1, \ldots, s_n)$, and $\varkappa \equiv (n+d)/2 - 2$. Θ is the Heaviside function.

Each kernel function is thus the sum of four terms. The first one gives rise to

Definition 2.1.

$$\begin{aligned} M_t(d,\mathbf{n},\varepsilon) &\equiv \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} : \omega^{\otimes \mathbf{n}}(s) : \\ &= \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i)\cdots\omega_i(s_{n_i}^i) : \\ &= \sum_{i=1}^{d}\sum_{m=1}^{n_i}\int_0^t ds_m^i \int_0^{s_m^i} d^{n-1}s(v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i)\cdots\omega_i(s_{n_i}^i) : \\ &= \sum_{i=1}^{d} n_i \int_0^t dB_i(\tau)\int_0^{\tau} d^{n-1}s(\tau-u+\varepsilon)^{-\varkappa} : \omega^{\otimes \mathbf{n}-\boldsymbol{\delta}_\mathbf{i}}(s) : \\ &= \sum_{k=1}^{d}\int_0^t dB_k(v) m_k(v) \\ &\equiv \sum_{k=1}^{d} M_{k,t}\,. \end{aligned} \tag{6}$$

The others give

$$N_t(d,\mathbf{n},\varepsilon) \equiv \int_{[0,t]^n} d^n s((t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}) : \omega^{\otimes \mathbf{n}}(s) : . \tag{7}$$

All the above processes are continuous.

Definition 2.2. We denote the **n**th order contribution to the regularized local time L_ε by

$$K_t(d,\mathbf{n},\varepsilon) \equiv (-1)^{\frac{n}{2}} \left(\varkappa(\varkappa+1)(2\pi)^{d/2}\, 2^{\frac{n}{2}} \left(\frac{\mathbf{n}}{2}\right)!\right)^{-1} (M_t(d,\mathbf{n},\varepsilon) + N_t(d,\mathbf{n},\varepsilon))\,. \tag{8}$$

Remark 2.1. Our key observation is that, as ε goes to zero, M is more singular than N, and that it is a Brownian martingale.

2.2. *The main theorems*

Theorem 2.2. *For $d \geq 3$, the renormalized M_i converge in distribution to independent Brownian motions β_i :*

$$(rM_i, ; i = 1,\ldots,d) \xrightarrow[\varepsilon\to+0]{\mathfrak{L}} \left(\sqrt{\frac{n_i}{n}} k_n \beta_i ; i = 1,\ldots,d\right)$$

with

$$k_n^2 = \mathbf{n}! \begin{cases} n(n-1) & \text{if } d = 3\,, \\ \dfrac{n!(d-4)!}{(n+d-5)!} & \text{if } d > 3 \end{cases} \tag{9}$$

if

$$r(\varepsilon) = \begin{cases} |\ln\varepsilon|^{-1/2} & \text{for } d = 3\,, \\ \varepsilon^{(d-3)/2} & \text{for } d > 3\,. \end{cases}$$

Theorem 2.3. *For $d \geq 3$, the renormalized* $\mathbf{n}$*th order contributions to the regularized local time L_ε converge in distribution to Brownian motions β*

$$rK(d, \mathbf{n}, \varepsilon) \underset{\varepsilon\to+0}{\overset{\mathfrak{L}}{\longrightarrow}} c_{\mathbf{n},d}\beta_{\mathbf{n}}$$

with

$$c_{\mathbf{n},d}^2 = k_n^2 \left(\varkappa(\varkappa+1)(2\pi)^{d/2}\, 2^{\frac{n}{2}} \left(\frac{\mathbf{n}}{2}\right)!\right)^{-2}\,.$$

3. Proofs

Proposition 3.1. *$M_{k,t}$ are orthogonal Brownian martingales.*

Proof. Orthogonality is obvious. For the martingale property see Refs. 13 and 2, it is a consequence of the fact that the kernel functions of M_t in (6) do not depend on t (except through the limit of integrations). □

Their limiting behavior, as $\varepsilon \to +0$, is studied in the following lemma (from now on we shall consider only the situations which require renormalisation, i.e. $d \geq 3$).

Lemma 3.1. *As $\varepsilon \to +0$,*

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} k_n^2\,(t + o(1)) \begin{cases} |\ln\varepsilon| & \text{for } d = 3\,, \\ \varepsilon^{-(d-3)} & \text{for } d > 3\,. \end{cases}$$

Proof.

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n}\mathbf{n}!\|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2\,.$$

For $d > 3$

$$\begin{aligned} \|(v-u+\varepsilon)^{-\varkappa}\|_{L^2([0,t]^n)}^2 &= \int_0^t d^n s (v-u+\varepsilon)^{-2\varkappa} \\ &= n(n-1)\int_0^t dv \int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{2\varkappa}} \\ &= n(n-1)\varepsilon^{3-d}\int_0^t dv \int_0^{v/\varepsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \\ &= \varepsilon^{3-d} t\left(\frac{n!(d-4)!}{(n+d-5)!} + o(1)\right) \end{aligned}$$

while for $d = 3$

$$\|(v-u+\varepsilon)^{-\varkappa}\|^2_{L^2([0,t]^n)} = n(n-1)t|\ln\varepsilon|(1+o(1))\,. \qquad \square$$

To show convergence of these martingales by Theorem VIII.3.11 of Ref. 16 we must show convergence of characteristics. Since the processes M are continuous, this reduces to showing convergence in probability of $\langle rM_i, rM_k\rangle_t$ as $\varepsilon \to +0$. This will be taken care of by Proposition 3.1 which we prepare now:

$$\langle rM_i, rM_k\rangle_t = r^2\delta_{ik}\int_0^t dv(m_i(v))^2$$

and we need to estimate

$$\int_0^t dv(m_i(v))^2 \equiv \sum_{\mathbf{k}}\langle :\omega^{\otimes\mathbf{k}}:, G^{(i)}_{\mathbf{k}}\rangle\,.$$

Let us note that

$$E(\langle rM_i, rM_k\rangle_t) = E(r^2 M_i M_k) = \delta_{ik} r^2 \frac{n_k}{n}\mathbf{n}!\|(v-u+\varepsilon)^{-\varkappa}\|^2_{L^2([0,t]^n)}\,,$$

$$E(\langle rM_i, rM_k\rangle_t) = \delta_{ik}\frac{n_k}{n}k_n^2(t+o(1))\,.$$

Next we intend to show that the rest of $\langle rM_i, rM_k\rangle_t$ goes to zero. The kernel of the highest order is

$$\begin{aligned} &r^2 G^{(i)}_{2\mathbf{n}-2\boldsymbol{\delta}_i}(s_1, s_1';,\ldots;s_{n-1}, s_{n-1}') \\ &\quad = r^2\int_0^t d\tau\Theta\,(\tau - v\vee v')\,(\tau-u+\varepsilon)^{-\varkappa}(\tau-u'+\varepsilon)^{-\varkappa} \end{aligned} \tag{10}$$

$v^{(\prime)}$ and $u^{(\prime)}$ are the largest and the smallest of $s_i^{(\prime)}$ and $(\boldsymbol{\delta}_i)_k \equiv \delta_{ik}$.

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s^{(\prime)}$.)

Remark 3.1. The integral (10) can be calculated in closed form (using e.g. Nos. 2.15 and 2.263.4 of Ref. 12), but one gets a useful approximation by introducing the following auxiliary functions:

$$\begin{aligned} &H_{2n-2}(s_1,\ldots,s_{n-1}; s_1',\ldots,s_{n-1}';\varepsilon;d) \\ &\quad \equiv (v\vee v' - u\vee u' + \varepsilon)^{-(n+d)/2+3}(v\vee v' - u\wedge u' + \varepsilon)^{-(n+d)/2+2}\,, \end{aligned}$$

where $v \equiv \max(s_i), u \equiv \min(s_i)$ and $v' \equiv \max(s_i'), u' \equiv \min(s_i')$. These functions majorize the kernel functions.

Lemma 3.2. *For $n > 2$ and $\varkappa > 1$*

$$\begin{aligned} 0 &\le G^{(i)}_{2\mathbf{n}-2\boldsymbol{\delta}_i}(s_1, s_1';,\ldots;s_{n-1}, s_{n-1}') \\ &\le \frac{1}{\varkappa - 1}H_{2n-2}(s_1,\ldots,s_{n-1}; s_1',\ldots,s_{n-1}';\varepsilon;d)\,. \end{aligned} \tag{11}$$

Proof.

$$\begin{aligned}
0 &\leq \int_0^t d\tau \Theta(\tau - v \vee v')(\tau - u + \varepsilon)^{-\varkappa}(\tau - u' + \varepsilon)^{-\varkappa} \\
&\leq \int_{v \vee v'}^t d\tau (\tau - u \vee u' + \varepsilon)^{-\varkappa}(v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa} \\
&\leq \frac{1}{\varkappa - 1}(v \vee v' - u \vee u' + \varepsilon)^{-\varkappa+1}(v \vee v' - u \wedge u' + \varepsilon)^{-\varkappa}\,. \qquad \square
\end{aligned}$$

This estimate is sufficient to show

Lemma 3.3. *For* $2\mathbf{n} - 2\boldsymbol{\delta}_i \neq 0$

$$r^2 G^{(i)}_{\mathbf{2n}-2\boldsymbol{\delta}_i} \to 0 \quad in \quad L^2(R^{2n-2})$$

as ε *goes to* $+0$.

Proof. Consider first $n \geq 3$. By the above estimate it is sufficient to show that

$$\lim_{\varepsilon \to +0} r^4 \|H_{2n-2}\|^2_{L^2([0,t]^{2n-2})} = 0\,.$$

$$\begin{aligned}
\|H_{2n-2}\|^2_{L^2(R^{2n-2})} &= \int d^{n-1}s \int d^{n-1}s' (v \vee v' - u \vee u' + \varepsilon)^{2-2\varkappa}(v \vee v' - u \wedge u' + \varepsilon)^{-2\varkappa} \\
&= c_n \int_0^t dv \int_0^v dv' \int_0^v du \int_0^{v'} du' \\
&\quad \cdot \frac{(v-u)^{n-3}(v'-u')^{n-3}}{(v \vee v' - u \vee u' + \varepsilon)^{n+d-6}(v \vee v' - u \wedge u' + \varepsilon)^{n+d-4}} \\
&\leq c_n \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^y dx \int_0^{y'} dx' \\
&\quad \cdot \frac{1}{(y - x \vee x' + 1)^{d-3}(y - x \wedge x' + 1)^{d-1}}
\end{aligned}$$

and then decompose the x-integration into the following two domains

$$x' < x \qquad \text{and} \qquad x < x'\,.$$

In the first case

$$\begin{aligned}
I &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \int_{x'}^y dx \frac{1}{(y - x + 1)^{d-3}(y - x' + 1)^{d-1}} \\
&= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \frac{1}{(y - x' + 1)^{d-1}} \int_{x'}^y dx \frac{1}{(y - x + 1)^{d-3}}\,.
\end{aligned}$$

We first estimate the last integral

$$\int_{x'}^{y} \frac{1}{(y-x+1)^{d-3}} dx \leq \begin{cases} y - x' & \text{if } d = 3\,, \\ \ln(y - x' + 1) & \text{if } d = 4\,, \\ \dfrac{1}{d-4} & \text{if } d > 4\,, \end{cases}$$

and one finds

$$I = \begin{cases} O(1) & \text{if } d = 3 \\ O(\varepsilon^{7-2d}) & \text{if } d > 3\,. \end{cases}$$

Hence, in both cases, $r^4 I$ vanishes as $\varepsilon \to 0$.

The second case is

$$I = \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{1}{(y-x+1)^{d-1}} \int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}}\,. \tag{12}$$

Estimating the last integrand by 1 we find

$$\int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \leq \frac{(y-x)^2}{2}\,.$$

Substituting these estimates into the integrals over x and y gives for $d \geq 3$

$$I \leq \text{const.}\ \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{(y-x)^2}{(y-x+1)^{d-1}} = \begin{cases} O(1) & \text{if } d = 3 \\ O(\varepsilon^{-1} \ln \varepsilon) & \text{if } d = 4 \\ O(\varepsilon^{7-2d}) & \text{if } d > 4 \end{cases}$$

so in that $r^4 I$ vanishes as $\varepsilon \to 0$. For $n = 2$ it is sufficient to use the estimate

$$0 \leq G_2 = \int_0^t d\tau \Theta(\tau - u \vee u')(\tau - u + \varepsilon)^{1-d/2}(\tau - u' + \varepsilon)^{1-d/2}$$

$$\leq \int_{u \vee u'}^t d\tau (\tau - u \vee u' + \varepsilon)^{1-d/2}(u \vee u' - u \wedge u' + \varepsilon)^{1-d/2} = H_2(u, u')$$

and to verify that

$$\lim_{\varepsilon \to +0} r^4 \|H_2\|^2_{L^2([0,t]^2)} = 0\,. \qquad \square$$

With this lemma we have established that the highest order term of the (renormalized) quadratic variation goes to zero in quadratic mean for any $t > 0$. The kernels G_k of the other terms are obtained by integrating over pairs of s, s' such as e.g. in

$$\text{Sym} \int_0^t ds G^{(i)}_{2(\mathbf{n}-\boldsymbol{\delta}_i)}(s, s; s_2, s_2'; \ldots; s_{n-1}, s_{n-1}')\,,$$

these new functions are in fact also bounded by an expression like (10), and hence by (11), so that for all $\varepsilon > 0$,

$$\|G_k\| \leq \text{const.}\, \|H_k\|\,.$$

With this goal we show:

Lemma 3.4. *Let $n \geq 2$ and $m > 1$, and let $F_{n,m}$ be a function symmetric in the variables s and in the variables s', with*

$$0 \leq F_{n,m}(s_1, \ldots, s_{n-1}, s_n, s'_1, \ldots, s'_{n-1}, s'_n)$$
$$\leq c \int_0^t d\tau \Theta\Big(\tau - \max_{i \leq n}(s, s')\Big)\Big(\tau - \min_{i \leq n}(s_i) + \varepsilon\Big)^{-m}\Big(\tau - \min_{i \leq n}(s'_i) + \varepsilon\Big)^{-m}. \quad (13)$$

Then $\exists\, c_m < \infty$ such that

$$0 \leq \int_0^t ds F_{n,m}(s_1, \ldots, s_{n-1}, s, s'_1, \ldots, s'_{n-1}, s)$$
$$\leq c_m \int_0^t d\tau \Theta\Big(\tau - \max_{i<n}(s, s')\Big)\Big(\tau - \min_{i<n}(s_i) + \varepsilon\Big)^{-m+1/2}$$
$$\cdot \Big(\tau - \min_{i<n}(s'_i) + \varepsilon\Big)^{-m+1/2}. \quad (14)$$

Proof. Under the assumption of the lemma

$$\int_0^t ds F_{n,m}(s_1, \ldots, s_{n-1}, s, s'_1, \ldots, s'_{n-1}, s)$$
$$\leq c \int_0^t ds \int_0^t d\tau \Theta\Big(\tau - \max_{1 \leq i \leq n-1}(s_i, s'_i, s)\Big)$$
$$\cdot \Big(\tau - \min_{1 \leq i \leq n-1}(s_i, s) + \varepsilon\Big)^{-m}\Big(\tau - \min_{1 \leq i \leq n-1}(s'_i, s) + \varepsilon\Big)^{-m}.$$

Using

$$u = \min_{1 \leq i \leq n-1} s_i, \quad u' = \min_{1 \leq i \leq n-1} s'_i,$$
$$v = \max_{1 \leq i \leq n-1} s_i, \quad v' = \max_{1 \leq i \leq n-1} s'_i.$$

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s_1^{(\prime)}$.) Assuming without loss of generality that $u < u'$ we can decompose the s-integration of our estimate as follows:

$$c \int_0^t ds \int_0^t d\tau \Theta(\tau - \max(v, v', s))(\tau - \min(u, s) + \varepsilon)^{-m}(\tau - \min(u', s) + \varepsilon)^{-m}$$
$$= c \int_0^t d\tau \left(\int_0^u + \int_u^{u'} + \int_{u'}^{v \vee v'} + \int_{v \vee v'}^t\right) ds$$
$$\cdot \Theta(\tau - \max(v, v', s))(\tau - \min(u, s) + \varepsilon)^{-m}(\tau - \min(u', s) + \varepsilon)^{-m}$$
$$= c \int_{v \vee v'}^t d\tau \bigg(\int_0^u ds(\tau - s + \varepsilon)^{-2m} + \int_u^{u'} ds(\tau - u + \varepsilon)^{-m}(\tau - s + \varepsilon)^{-m}$$
$$+ \int_{u'}^{v \vee v'} ds(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m} + \int_{v \vee v'}^{\tau} ds(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}\bigg).$$

We now show that each of the four terms obeys the postulated estimate (14)

$$\int_{v\vee v'}^{t} d\tau \int_0^u ds(\tau - s + \varepsilon)^{-2m} \leq \frac{1}{2m-1}\int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-2m+1}$$

$$\leq \frac{1}{2m-1}\int_0^t d\tau\Theta(\tau - v\vee v')(\tau - u + \varepsilon)^{-m+\frac{1}{2}}(\tau - u' + \varepsilon)^{-m+\frac{1}{2}}$$

since $u < u'$ and $m > 1/2$.

The second term

$$\int_{v\vee v'}^{t} d\tau \int_u^{u'} ds(\tau - u + \varepsilon)^{-m}(\tau - s + \varepsilon)^{-m}$$

$$\leq \frac{1}{m-1}\int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m+1}$$

$$\leq \frac{1}{m-1}\int_0^t d\tau\Theta(\tau - v\vee v')(\tau - u + \varepsilon)^{-m+\frac{1}{2}}(\tau - u' + \varepsilon)^{-m+\frac{1}{2}}$$

using again $u < u'$. The third term

$$\int_{v\vee v'}^{t} d\tau \int_{u'}^{v\vee v'} ds(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}$$

$$= \int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}(v\vee v' - u')$$

$$\leq \int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}(\tau - u')$$

$$\leq \int_0^t d\tau\Theta(\tau - v\vee v')(\tau - u + \varepsilon)^{-m+\frac{1}{2}}(\tau - u' + \varepsilon)^{-m+\frac{1}{2}}\,.$$

Finally

$$\int_{v\vee v'}^{t} d\tau \int_{v\vee v'}^{\tau} ds(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}$$

$$= \int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}(\tau - v\vee v')$$

$$\leq \int_{v\vee v'}^{t} d\tau(\tau - u + \varepsilon)^{-m}(\tau - u' + \varepsilon)^{-m}(\tau - u')$$

$$\leq \int_0^t d\tau\Theta(\tau - v\vee v')(\tau - u + \varepsilon)^{-m+\frac{1}{2}}(\tau - u' + \varepsilon)^{-m+\frac{1}{2}}\,.$$

□

Combining this lemma with the previous one we conclude that for all kernel functions G_k with $k \geq 2$ arguments

$$\lim_{\varepsilon\to+0} r^4 \,\|\,\mathrm{Sym}\ G_k\|^2 \leq \lim_{\varepsilon\to+0} r^4\,\|G_k\|^2 \leq \lim_{\varepsilon\to+0} r^4\,\|H_k\|^2 = 0\,,$$

i.e. we have shown

Proposition 3.2.

$$ms - \lim_{\varepsilon \to +0} \langle rM_i, rM_k \rangle_t = \delta_{ik} \frac{n_k}{n} k_n^2 t\,.$$

Proof of Theorem 2.2. The above limit is clearly (up to constants) the quadratic variation of a Brownian motion. Theorem 2.2 is then a consequence of e.g. Theorem VIII.3.11 in Ref. 16, which in the present case of continuous martingales requires convergence in probability of quadratic variations for a dense set of t.

To control the remaining terms $N_t(d, \mathbf{n}, \varepsilon)$ in the chaos expansion we observe that

$$\begin{aligned}\|N_t(d, \mathbf{n}, \varepsilon)\|^2_{(L^2)} &= \mathbf{n}! \|(t+\varepsilon)^{-\varkappa} - (v+\varepsilon)^{-\varkappa} - (t-u+\varepsilon)^{-\varkappa}\|^2_{L^2([0,t]^n)} \\ &\le \mathbf{n}!(\|(t+\varepsilon)^{-\varkappa}\|^2_{L^2} + \|(v+\varepsilon)^{-\varkappa}\|^2_{L^2} + \|(t-u+\varepsilon)^{-\varkappa}\|^2_{L^2})\,.\end{aligned}$$

The first of these three norms is equal to $t^n(t+\varepsilon)^{-2\varkappa}$, i.e. $O(1)$. The second one is

$$\int_{[0,t]^n} d^n s (v+\varepsilon)^{-2\varkappa} = n \int_0^t dv \frac{v^{n-1}}{(v+\varepsilon)^{n+d-4}} = n\varepsilon^{4-d} \int_0^{t/\varepsilon} dx \frac{x^{n-1}}{(x+1)^{n+d-4}}$$

$$= \begin{cases} O(1) & \text{for } d = 3 \\ O(\ln \varepsilon) & \text{for } d = 4 \\ O(\varepsilon^{4-d}) & \text{for } d > 4 \end{cases}$$

which are suppressed by the renormalization

$$r^2(\varepsilon) = \begin{cases} |\ln \varepsilon|^{-1} & \text{for } d = 3\,, \\ \varepsilon^{d-3} & \text{for } d > 3\,. \end{cases}$$ □

A similar estimate holds for the third term of N, so that we have shown

Lemma 3.5.

$$ms - \lim_{\varepsilon \to +0} r(\varepsilon) N_t(d, \mathbf{n}, \varepsilon) = 0.$$

In fact the convergence is uniform in any finite t-interval. Next we show

Lemma 3.6. *The processes* $\{r(\varepsilon)N\cdot(d, \mathbf{n}, \varepsilon) : \varepsilon > 0\}$, $\{r(\varepsilon)M.(d, \mathbf{n}, \varepsilon) : \varepsilon > 0\}$ *and their linear combinations are tight.*

Proof. A criterion for tightness of M (following p. 64 of Ref. 17) is

$$\sup_{\varepsilon > 0} E|rM_t - rM_s|^\alpha \le C_T(t-s)^{1+\beta}\,, \tag{15}$$

$\forall\, T > 0$ and $0 \le s < t \le T$ and for some positive constants α, β and C_T.

As a first step we show

$$\sup_{\varepsilon>0} E|rM_t - rM_s|^2 \le C_T(t-s) \tag{16}$$

by direct calculation:

$$E|M_t - M_s|^2 = \mathbf{n}! n \int_s^t dv \int_0^v d^{n-1}s(v-u+\varepsilon)^{-2\varkappa}\,.$$

The second integral may be estimated as follows:

$$\begin{aligned}\int_0^v d^{n-1}s(v-u+\varepsilon)^{-2\varkappa} &= (n-1)\int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{n+d-4}} \\ &\le (n-1)\varepsilon^{3-d}\int_0^{v/\varepsilon} dx \frac{1}{(x+1)^{d-2}} \\ &= (n-1)\begin{cases}\ln(v+\varepsilon) - \ln\ \varepsilon & \text{for } d=3\,,\\ O(\varepsilon^{3-d}) & \text{for } d>3\,.\end{cases}\end{aligned}$$

Renormalization of this estimate by the factor r^2 makes it bounded on $[0,T]$, and the integral over v gives the desired estimate (16), i.e.

$$\|rM_t - rM_s\|_2^2 \le c_T(t-s)\,. \tag{17}$$

Note that the kernel functions of K are all dominated by those of M. Hence we have also, possibly with a larger constant c_T, the estimate

$$\|rK_t - rK_s\|_2^2 \le c_T(t-s)\,. \tag{18}$$

By the hypercontractivity of the Ornstein–Uhlenbeck semigroup (see e.g. p. 235 of Ref. 14), one has for nth order white noise monomials $\varphi \in (L^2)$, and any $\alpha > 2$

$$\|\varphi\|_\alpha^\alpha \le c_{n,\alpha}\|\varphi\|_2^\alpha\,.$$

For $\varphi = rK_t - rK_s$ and using the above estimate for the 2-norm, we get

$$E|rK_t - rK_s|^\alpha \le C_T(t-s)^{\alpha/2}$$

as required to ensure tightness. The estimates for rN etc. are of the same kind. □

Proof of Theorem 2.3. We need to consider

$$rK = rM + rN$$

knowing that, as $\varepsilon \to +0$, the rK are tight by Lemma 3.6, the rM converge in law, and the rN_t go to zero in mean square. The latter two facts are sufficient, via the Cramér–Wold device (see e.g. p. 61 of Ref. 17), for finite dimensional convergence of rK; tightness then implies convergence in law.

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