

A NOTE ON GLOBAL STABILITY IN THE PERIODIC LOGISTIC MAP

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(Communicated by Christian Pötzsche)

ABSTRACT. In this paper, the dynamics of the celebrated p –periodic one-dimensional logistic map is explored. A result on the global stability of the origin is provided and, under certain conditions on the parameters, the local stability condition of the p –periodic orbit is shown to imply its global stability.

1. Introduction. One of the most known and studied difference equation (or discrete dynamical system) is the celebrated logistic map, a polynomial mapping of degree 2, often cited for being a simple non-linear dynamical equation with complex and chaotic dynamics.

The logistic map was popularized by the biologist Robert May [14] in 1976, and may be seen as a discrete-time demographic model, with a continuous version given by the logistic equation, introduced by Pierre F. Verhulst [19], more than a hundred years before.

The logistic equation in its canonical form is given by, for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$,

$$x_{n+1} = rx_n(1 - x_n), \quad (1)$$

where $x_n \in [0, 1]$ and $r \in (0, 4]$.

The dynamics of the logistic equation is well known and may be found in any book of discrete dynamical systems or difference equations (cf., e.g., [1, 4, 6, 7, 17], among others).

Taking a sequence of parameters $\{r_n, n \in \mathbb{N}\}$ in the interval $(0, 4]$, the non-autonomous logistic equation is obtained:

$$x_{n+1} = r_n x_n (1 - x_n), \quad (2)$$

2020 *Mathematics Subject Classification.* Primary: 37B55, 39A23; Secondary: 37C25, 37C75, 39A30.

Key words and phrases. Difference equations, periodic logistic map, periodic orbits, local stability, global stability.

The first and second authors are partially supported by FCT/Portugal through the projects UID/MAT/04459/2019 and UID/MAT/00006/2019, respectively.

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where $x_n \in [0, 1]$, as before. Here, the dynamics of the equation is given by the composition of maps f_0, f_1, f_2, \dots , where

$$f_n(x) = r_n x(1 - x). \quad (3)$$

If the sequence of parameters r_n is periodic, i.e., if there exists an integer $p > 1$ (the period) such that $r_{n+p} = r_n$, for all $n \in \mathbb{N}$, then $f_{n+p} = f_n$, for all $n \in \mathbb{N}$. This is equivalent to say that Equation (2) is non-autonomous p -periodic or simply p -periodic. In this case, the dynamics of Equation (2) is determined by the following composition map

$$\Phi_p^i = f_{p+i-1} \circ \dots \circ f_{i+1} \circ f_i. \quad (4)$$

A fixed point of Φ_p^i generates a periodic cycle in the sequence defined by Equation (2). It follows from [2] that this fixed point generates a periodic cycle of minimal period p of the equation (2). Let us represent this p -periodic cycle by the following ordered set of points

$$C_p = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{p-1}\}. \quad (5)$$

Since $\Phi_p^i(\bar{x}_i) = \bar{x}_i$, the dynamics of Equation (2) is completely determined by the composition operator $\Phi_p^0 = \Phi_p$. Throughout this paper we study Equation (2) when the sequence of maps f_n is p -periodic.

The local dynamics of the periodic logistic map has been investigated in the last decades. The composition of two quadratic maps was already considered by M. Kot and W. M. Schaffer [10] in 1984, in the framework of seasonal populations. In this work, the authors study the effects of seasonality on the dynamics of a bivoltine population with discrete, nonoverlapping generations and conclude that large seasonality is inevitably destabilizing but that mild seasonality may have a pronounced stabilizing effect.

In 1992, Li [12] studied a periodically fluctuating environment in a population-modelling process that generates non-autonomous periodic difference equations. In this work, the existence and uniqueness of periodic solutions are studied, a sufficient condition for existence and a necessary condition for uniqueness are obtained, and the stability of the periodic solutions is also investigated. Results are illustrated with several examples including the periodic logistic. Later, Grinfeld *et. al.* in 1996 [8] explored the logistic equation modified by a periodic time dependence. AlSharawi *et. al.* [2], in 2006, proved that the p -periodic logistic equation has cycles (periodic solutions) of minimal periods $1, p, 2p, 3p, \dots$. The authors have extended Singer's theorem to periodic difference equations, and used it to show that the p -periodic logistic equation has at most p stable cycles. The paper also includes computational methods to explore the stable cycles in the cases $p = 2$ and $p = 3$ and the determination of bifurcation curves.

Our main focus in this paper will be the global stability of C_p , defined in (5). We will follow the techniques (partially) employed by Eduardo Liz in [13]. In this groundbreaking paper, Liz showed global stability of the positive periodic cycle of the non-autonomous 2-periodic Ricker equation (and 3-periodic as well) given by

$$x_{n+1} = x_n e^{p_n - x_n}, \quad p_n > 0 \text{ for all } n = 0, 1, 2, \dots$$

Particularly, we will explore the conditions under which the local stability of a fixed point of Φ_p implies its global stability, which is the same to say that Equation (2) has a globally asymptotically stable p -periodic cycle. It should be mentioned that in [11] the authors used different tools and studied the boundedness and the

periodicity of non-autonomous periodic logistic map (3) when the sequence of parameters is periodic such that $0 < r_n \leq 2$ for all n , and that sufficient conditions are given to support the existence of asymptotically stable and unstable p -periodic orbits, in this case.

In this paper we show that the global stability condition may be extended for a larger set of parameters, as long as the parameters are such that $\prod_{i=0}^{p-1} r_i > 1$ and that the absolute value of the derivative along the periodic orbit is less or equal than 1. Under these conditions and an assumption on the critical points of the composition operator Φ_p , we establish global stability of the nontrivial fixed point of Φ_p in Theorem 4.2. (or global stability of the p -periodic cycle of the p -periodic logistic equation (2)), which is the main goal of this work.

2. General results in stability. Let us consider the difference equation given by

$$x_{n+1} = f_n(x_n), \quad n = 0, 1, 2, \dots, \quad (6)$$

where, for each n , $f_n : X \rightarrow X$, for some topological space X , is a C^3 map. Here, the orbit of a point x_0 is generated by the composition of the sequence of maps

$$f_0, f_1, f_2, \dots$$

Let $f_{n+p} = f_n$ for all $n = 0, 1, 2, \dots$ and define the composition operator Φ as follows

$$\Phi_m^i = f_{m+i-1} \circ \dots \circ f_{i+1} \circ f_i, \quad m = 1, 2, \dots, \quad i = 0, 1, 2, \dots \quad (7)$$

Notice that we represent the n -th composition of Φ_p^i by $\Phi_p^{i,n}$ and when $i = 0$ we write

$$\Phi_p^0 = \Phi_p = f_{p-1} \circ \dots \circ f_1 \circ f_0.$$

Let $\bar{x}_{i \bmod r}$ to be a fixed point of Φ_s^i (guaranteed by Brouwer's fixed point theorem [15]). In other words we have that

$$\Phi_s^i(\bar{x}_{i \bmod r}) = \bar{x}_{i \bmod r}.$$

Notice that a nontrivial fixed point of Φ_s^i generates a periodic cycle in Equation (6). In order to study the behaviour of the composition map Φ_p^i , we follow the definition of stability present in Elaydi [7], page 19. However, this definition of stability may not be the most practical tool to show the stability of a fixed point of Φ_p^i . There exists a simple but powerful criterion that guarantees the local stability of fixed points that we recall here in the following two results.

Theorem 2.1 (Local stability criteria, Elaydi [7], page 25). *Let \bar{x}_i be a hyperbolic fixed point of a map Φ_p^i , i.e., a fixed point such that $|(\Phi_p^i)'(\bar{x}_i)| \neq 1$, where Φ_p^i is continuously differentiable at \bar{x}_i . The following statements hold true:*

1. *If $|(\Phi_p^i)'(\bar{x}_i)| < 1$, then \bar{x}_i is locally asymptotically stable.*
2. *If $|(\Phi_p^i)'(\bar{x}_i)| > 1$, then \bar{x}_i is unstable.*

The stability criteria for nonhyperbolic fixed points are more complex and may be found in Elaydi's book [7], pages 28-30.

The precedent result gives conditions that guarantee local stability. In 1955 W. Coppel stated the following result concerning global dynamics:

Theorem 2.2 (Global dynamics, Coppel [3]). *Let $I = [a, b] \subseteq \mathbb{R}$ and $f : I \rightarrow I$ be a continuous map. If the equation $f(f(x)) = x$ has no roots except the roots of the equation $f(x) = x$, then every orbit under the map f converges to a fixed point.*

Coppel's theorem ensures global stability only when the map f has a unique fixed point. However, solving analytically the equation $f(f(x)) = x$ can be a hard work, if not impossible, especially in periodic equations.

The following result is a consequence of Singer's results [18]:

Theorem 2.3 (Global stability, Singer's theorem [18]). *If a unimodal map with negative Schwarzian derivative has a unique fixed point x^* which is locally asymptotically stable, then x^* is globally stable.*

Theorem 2.3 is a powerful result in global stability. However, when dealing with composition of maps, only in particular cases the composition map Φ_p^i is unimodal. In 2008, El-Morshedy and Jiménez López presented the following result concerning global stability, a consequence of Theorem 2.3:

Theorem 2.4 (Global stability, El-Morshedy & Jiménez López [5]). *Let $a \geq 0$ and $b > a$ ($b = +\infty$ is allowed) and $f : (a, b) \rightarrow [a, b]$ be a continuous map with a unique fixed point x^* in (a, b) such that*

$$(f(x) - x)(x - x^*) < 0, \text{ for all } x \neq x^*.$$

Assume that there are numbers c and d with $a \leq c < x^ < d \leq b$ such that $f|_{(c, d)}$ has at most one turning point, and (whenever it makes sense) $f(x) \leq f(c)$ for every $x \leq c$, and $f(x) \geq f(d)$ for every $x \geq d$. In addition, if f is decreasing at x^* , assume additionally that $Sf(x) < 0$ for all $x \in (c, d)$ except at most one critical point of f , and $-1 \leq f'(x^*) < 0$. Then x^* is globally stable.*

3. Trivial solution. In this section we establish the global stability of the trivial fixed point of Equation (2), i.e, equation

$$x_{n+1} = r_n x_n (1 - x_n), \quad (8)$$

when $\{r_n, n \in \mathbb{N}\}$ is a p -periodic sequence of numbers of the interval $(0, 4]$. Notice that $x^* = 0$ is a fixed point of Equation (8).

Define $\Phi_0(x) = x$, and for $p \geq 1$ let

$$\Phi_p = f_{p-1} \circ \dots \circ f_1 \circ f_0. \quad (9)$$

A condition that guarantees the local stability of $x^* = 0$ is given by $\prod_{i=0}^{p-1} r_i \leq 1$. In order to see this, notice that

$$\Phi_p'(x) = \prod_{i=0}^{p-1} f_i'(\Phi_i(x)), \quad f_i'(x) = r_i(1 - 2x),$$

where $p > 1$ and $\Phi_0(x) = x$. Hence $\Phi_p'(0) = |\Phi_p'(0)| = \prod_{i=0}^{p-1} r_i$ and from the first statement of Theorem 2.1 the inequality $\prod_{i=0}^{p-1} r_i < 1$ follows, while the equality $\prod_{i=0}^{p-1} r_i = 1$ follows from Theorem 1.15 in Elaydi [7, page 29] since

$$\Phi_p''(0) = -2 \prod_{i=0}^{p-1} r_i \left(1 + \sum_{j=0}^{p-2} \prod_{i=0}^j r_i \right) < 0.$$

Now we show that this local stability condition implies global stability.

Proposition 1. *If $\prod_{i=0}^{p-1} r_i \leq 1$, then the origin is a globally asymptotically stable fixed point of Equation (8).*

Proof. If $1 - \Phi_i(x)$, $i = 0, 1, 2, \dots$ is either 0 or 1, then we are done. Suppose $\Phi_i(x) \neq 1$ and $\Phi_i(x) \neq 0$, $i = 0, 1, 2, \dots$. Since $1 - \Phi_i(x) \in (0, 1)$ for all $x \in (0, 1)$, the relation $\prod_{i=0}^{p-1} r_i \leq 1$ implies that

$$\prod_{i=0}^{p-1} r_i (1 - \Phi_i(x)) < 1.$$

Multiplying both sides of this last inequality by $x \in (0, 1]$ we obtain

$$x \prod_{i=0}^{p-1} r_i (1 - \Phi_i(x)) < x,$$

which is equivalent to have

$$\Phi_p(x) < x, \text{ for all } x \in (0, 1].$$

Hence, the origin is the unique fixed point of Φ_p in the unit interval.

Moreover, $\Phi_p(\Phi_p(x)) < \Phi_p(x) < x$, for all $x \in (0, 1]$. Thus, the unique solution of the equation $\Phi_p(\Phi_p(x)) = x$ in the unit interval is $x^* = 0$. Thus, applying Coppel's Theorem it follows that every orbit under the map Φ_p , in the unit interval, converges to $x^* = 0$. \square

4. Nontrivial solution. In this section we prove the global stability of the p -periodic solution of the periodic logistic equation (8). There are two cases to consider: (i) the composition map Φ_p has a unique critical point (unimodal map) and (ii) the composition map Φ_p has more than one critical point.

The critical points of Φ_p are the solutions of the equation $\Phi'_p(x) = 0$, or equivalently, of the equation $\prod_{i=0}^{p-1} f'_i(\Phi_i(x)) = 0$. Since $\frac{1}{2}$ is the unique critical point of each individual map f_i , $0 \leq i \leq p-1$, it follows that the critical points of Φ_p are the solutions of the following p equations

$$\Phi_i(x) = \frac{1}{2}, \quad i = 0, 1, 2, \dots, p-1.$$

Since $\Phi_i(x) = f_{i-1}(\Phi_{i-1}(x)) = r_{i-1}\Phi_{i-1}(x)(1 - \Phi_{i-1}(x))$, the p equations are equivalent to

$$x = \frac{1}{2} \text{ or } \Phi_{i-1}(x) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{2}{r_{i-1}}}, \quad i = 1, 2, \dots, p-1. \quad (10)$$

It is clear that if $r_{i-1} \leq 2$ for all $i = 1, 2, \dots, p-1$, then $x = \frac{1}{2}$ is the unique critical point of Φ_p .

It is not possible to write the solutions of the last $p-1$ equations explicitly for all cases. However, it is not hard to see that the number of solutions is an odd number not greater than $\sum_{j=1}^p 2^{j-1} = 2^p - 1$. For instance, when $p = 3$ we obtain the equations

$$\Phi_0(x) = \frac{r_0 \pm \sqrt{r_0(r_0 - 2)}}{2r_0} \text{ or } \Phi_1(x) = \frac{r_1 \pm \sqrt{r_1(r_1 - 2)}}{2r_1}.$$

Since $\Phi_0(x) = x$ and $\Phi_1(x) = f_0(x)$, we obtain the following six potential solutions:

$$x^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{2}{r_0}} \text{ or } \bar{x}^{\pm, \pm} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{2}{r_0} \left(1 \pm \sqrt{1 - \frac{2}{r_1}} \right)}.$$

1. If $r_0 > 2$ and $r_1 \leq 2$, then Φ_3 has three critical points, $\frac{1}{2}$ and x^\pm .

2. If $r_0 \leq 2$ and $r_1 > 2$, then, since $r_0 < 2 \frac{r_1 + \sqrt{r_1(r_1-2)}}{r_1}$, the composition operator has either one or three critical points, either $\frac{1}{2}$ or $\frac{1}{2}$ and $\bar{x}^{\pm,-}$.
3. If $r_0 > 2$ and $r_1 > 2$, then Φ_3 has either five or seven critical points, i.e., either $\frac{1}{2}$, x^{\pm} and $\bar{x}^{\pm,-}$ or $\frac{1}{2}$, x^{\pm} and $\bar{x}^{\pm,\pm}$.

4.1. Unimodal map. In this subsection we study the p -periodic solution of the periodic logistic equation when the composition map Φ_p is unimodal.

So let us assume the following condition on the parameters

$$\prod_{i=0}^{p-1} r_i > 1 \text{ and } r_i \leq 2 \text{ for all } i = 0, 1, 2, \dots, p-1. \quad (11)$$

Notice that the condition in the product is needed in order to find a positive fixed point of Φ_p , i.e., a p -periodic cycle of the difference equation (8).

We shall now prove the following result:

Theorem 4.1. *Under the conditions on the parameters r_i , $i = 0, \dots, p-1$, given by (11), the p -periodic logistic equation (8) has a globally asymptotically stable p -periodic cycle.*

Proof. The individual maps f_i are increasing on $(0, \frac{1}{2})$ and their positive fixed points are $x_i^* = \frac{r_i-1}{r_i} < \frac{1}{2}$, $i = 0, 1, \dots$. Moreover, $\frac{1}{2}$ is the unique and common critical point of all individual maps. Notice that conditions (10) and (11) together imply that $\frac{1}{2}$ is also the unique critical point of the composition map Φ_p . Let us take the interval $[m, M]$, where $m = \min_{0 \leq i \leq p-1} \{x_i^*\}$ and $M = \max_{0 \leq j \leq p-1} \{x_j^*\}$. Clearly, $\Phi_p(m) \geq m$ and $\Phi_p(M) \leq M$. Thus $[m, M]$ is invariant under Φ_p . Moreover, it is compact and a convex subset of \mathbb{R} and Φ_p is continuous. Hence, from Brouwer's fixed-point theorem [9] the composition map Φ_p has a nontrivial fixed point in the interval $[m, M]$.

Under conditions (11), it is clear that the origin is an unstable fixed point. So there is a neighborhood of the origin where $\Phi_p(x) > x$. A simple induction shows that $\Phi_p(\frac{1}{2}) \leq \frac{1}{2}$. Moreover, $\Phi_p(0) = \Phi_p(1) = 0$ and $\Phi_p(x)$ is a polynomial of degree 2^p with a negative leading coefficient. The composition function $\Phi_p(x)$ is thus a concave down function in $[0, 1]$ with a unique critical point $\frac{1}{2}$. More precisely, $\Phi_p(x)$ is increasing in $(0, \frac{1}{2})$ since it is a composition of increasing functions on $(0, \frac{1}{2})$ and decreasing in $(\frac{1}{2}, 1)$ since it is the composition of decreasing functions on $(\frac{1}{2}, 1)$. In other words, the composition map Φ_p is a unimodal map in $[0, 1]$. Moreover, $\Phi_p(x)$ has a unique positive fixed point x^* in $(0, 1)$ in which $x^* \leq \frac{1}{2}$. Thus, x^* is global attractor. \square

4.2. Non-unimodal map. In this sub-section we study the p -periodic logistic equation when the composition operator $\Phi_p(x)$ is not unimodal and it has a unique positive fixed point but has no periodic points. This scenario happens when $\prod_{i=0}^{p-1} r_i > 1$ and some of the parameters r_i , $i = 0, 1, \dots, p-1$ are greater than 2. We will follow the techniques employed by Liz in [13] for a p -periodic Ricker map.

Assumption H1: Let us assume that Φ_p has more than one critical point, the sequence of parameters satisfy $\prod_{i=0}^{p-1} r_i > 1$, $i = 0, 1, \dots, p-1$, and Φ_p has a unique positive real fixed point x^* such that

$$|\Phi_p'(x^*)| = \left| \prod_{i=0}^{p-1} r_i (1 - 2\Phi_i(x^*)) \right| \leq 1. \quad (12)$$

The discussion that follows depends on the number of critical points of the map Φ_p and on the period of the composition map. Moreover, the analysis for $p = 2$ and $p = 3$ gives us the guidelines for the proof of the general case.

4.2.1. *When $p = 2$.* In this case, the map Φ_2 has three critical points, $c_2 = \frac{1}{2}$ (local minimum) and the solutions of the equation $f_0(x) = \frac{1}{2}$ (local maxima). Let c_1 and c_3 to be the left and the right solution of this equation, respectively. Notice that $c_1 < \frac{1}{2} < c_3$. There are two cases:

Case 1. The map Φ_2 is decreasing at x^* . If the fixed point x^* belongs to $(c_3, 1)$, we choose $a = 0$, $c = c_3$ and $b = d = 1$, and all the conditions of Theorem 2.4 are satisfied. If the fixed point x^* belongs to (c_1, c_2) , we choose $a = 0$, $c = c_1$ and $d = c_2$ and $b = c_3$. It is clear that the interval $I = [a, b]$ is invariant and attracts all the orbits. Moreover, the map $\Phi_2|_I$ satisfies all the conditions of Theorem 2.4. Hence, x^* is a globally asymptotically stable fixed point under Assumption H1.

Case 2. The map Φ_2 is increasing at x^* . Either, $x^* \in (0, c_1)$ or $x^* \in (c_2, c_3)$. It is clear that $\Psi = \Phi_2|_{I_k}$ is invariant, where $I_1 = [0, c_1]$ and $I_2 = [c_2, c_3]$, and attracting (the map Ψ is increasing where $\Psi(x) > x$ for all $x < x^*$ and $\Psi(x) < x$ for all $x > x^*$). Hence, the global stability follows by monotonicity of Ψ in the interior of I_k .

4.2.2. *When $p = 3$.* The map Φ_3 has either three, five or seven critical points as follows:

(i) $c_2 = \frac{1}{2}$, c_1 and c_3 (solutions of the equation $f_0(x) = \frac{1}{2}$) such that $c_1 < c_2 < c_3$,
(ii) $c_3 = \frac{1}{2}$, c_2 and c_4 (solutions of the equation $f_0(x) = \frac{1}{2}$), and c_1 and c_5 (solutions of the equation $\Phi_2(x) = \frac{1}{2}$) such that $c_i < c_{i+1}$, $i = 0, 1, \dots, 4$ (iii) $c_4 = \frac{1}{2}$, c_2 and c_6 (solutions of the equation $f_0(x) = \frac{1}{2}$) and c_1, c_3, c_5 , and c_7 as solutions of the equation $\Phi_2(x) = \frac{1}{2}$ such that $c_i < c_{i+1}$, $i = 0, 1, \dots, 6$. The exact values of these critical points were presented in the beginning of this section.

Case 1. The map Φ_3 has 3 critical points. Similar to $p = 2$.

Case 2. The map Φ_3 has 5 critical points. We have to split our analysis according the map Φ_3 is decreasing or increasing at x^* :

1. The map Φ_3 is decreasing at x^* . There are 3 cases: (i) if $x^* \in (c_5, 1)$, then we choose $a = 0$, $c = c_5$, $b = d = 1$ and apply Theorem 2.4; (ii) if $x^* \in (c_1, c_2)$, then we choose $a = 0$, $c = c_1$, $d = c_2$ and $b = c_5$ and, considering the invariant interval $[0, b]$, we apply Theorem 2.4, and (iii) if $x^* \in (c_3, c_4)$, then we choose $a = c_2$, $c = c_3$, $b = c_5$ and $d = c_4$ and as invariant interval the set $[a, b]$ and apply Theorem 2.4. In conclusion, x^* is a globally asymptotically stable fixed point under Assumption H1.
2. The map Φ_3 is increasing at x^* . There are 3 cases, either $x^* \in (0, c_1)$, or $x^* \in (c_2, c_3)$, or $x^* \in (c_4, c_5)$. Since $\Psi = \Phi_3|_{I_k}$ is invariant, where $I_1 = [0, c_1]$, $I_2 = [c_2, c_3]$ and $I_3 = [c_4, c_5]$ and attracting (the map Ψ is increasing with $\Psi(x) > x$ for all $x < x^*$ and $\Psi(x) < x$ for all $x > x^*$), the global stability follows from the monotonicity of Ψ in the interior of I_k .

Case 3. The map Φ_3 has 7 critical points. Similarly to the precedent case, it is necessary to split the analysis into two situations:

1. The map Φ_3 is increasing at x^* . In this case, there are four situations to consider, depending on the fixed point being in one of the following sub-intervals: $(0, c_1)$, (c_2, c_3) , (c_4, c_5) or (c_6, c_7) . We argue as before in

the restriction $\Psi = \Phi_3|_{I_k}$ where $I_1 = [0, c_1]$, $I_2 = [c_2, c_3]$, $I_3 = [c_4, c_5]$ and $I_4 = [c_6, c_7]$.

2. The map Φ_3 is decreasing at x^* . Again, there are four cases to consider in order to apply Theorem 2.4: (i) if $x^* \in (c_7, 1)$, then we choose $a = 0$, $c = c_7$, $b = d = 1$; (ii) if $x^* \in (c_1, c_2)$, then we choose $a = 0$, $c = c_1$, $d = c_2$ and $b = c_3$; (iii) if $x^* \in (c_3, c_4)$, then we choose $a = 0$, $c = c_3$, $d = c_4$ and $b = c_5$ and (iv) if $x^* \in (c_5, c_6)$, then we choose $a = 0$, $c = c_5$, $d = c_6$ and $b = c_7$. In conclusion, x^* is a globally asymptotically stable fixed point under Assumption **H1**.

4.2.3. *General period p .* Let k be the number of critical points of Φ_p , defined in (9), and $c_i < c_{i+1}$, $i = 1, 2, \dots, k$, be two consecutive critical points of Φ_p , such that c_{2i} is a local minimum for $i = 1, 2, \dots, (k-1)/2$, and c_{2i-1} is a local maximum, for $i = 1, 2, \dots, (k+1)/2$. Here too we have two cases to consider:

Case 1. The map Φ_p is increasing at x^* . If $x^* \in (0, c_1)$, then we consider the restriction $\Psi = \Phi_p|_{J_1}$, where $J_1 = [0, c_1]$. Notice that Ψ is an invariant and attracting map since $\Psi(x) > x$ for all $x \in (0, x^*)$ and $\Psi(x) < x$ for all $x \in (x^*, c_1)$. If $x^* \in (c_{2i}, c_{2i+1})$, then we consider $J_1 = [c_{2i}, c_{2i+1}]$ as invariant sub-interval. Notice that in all cases we have $\Psi(x) > x$ for all $x \in (c_{2i}, x^*)$ and $\Psi(x) < x$ for all $x \in (x^*, c_{2i+1})$.

Case 2. The map Φ_p is decreasing at x^* . Let $\mathcal{C} = \max_{i=1,2,\dots,(k+1)/2} \{c_{2i-1}\}$. If $x^* \in (\mathcal{C}, 1)$, then we choose $a = 0$, $c = \mathcal{C}$ and $b = d = 1$ and apply Theorem 2.4. Now let us assume that $x^* \in (c_{2i-1}, c_{2i})$ for some $i = 1, 2, \dots, (k-1)/2$. Since the critical points of the map Φ_p play a central role in the global stability, it is not possible to write a general conclusion. However, in certain scenarios it is possible to use Theorem 2.4 and conclude the global stability. To do that consider the following:

Assumption H2: Let $[a, b]$ the minimum invariant interval containing x^* where

$$a = \max_{j \in \{1, \dots, i-1\}} \{c_{2j} : c_{2j} \text{ is an absolute minimum of } \Phi_p\}$$

and

$$b = \min_{j \in \{i, \dots, (k-1)/2\}} \{c_{2j+1} : c_{2j+1} \text{ is an absolute maximum of } \Phi_p\}.$$

Suppose that $\Phi_p(c_{2i-1}) \leq b$ and $\Phi_p(c_{2i}) \geq a$. It is clear that putting $c = c_{2i-1}$ and $d = c_{2i}$, all conditions of Theorem 2.4 are satisfied.

We now summarize these ideas in the following result:

Theorem 4.2. *Under Assumptions **H1** and **H2**, every locally asymptotically stable fixed point of Φ_p is a globally asymptotically stable fixed point.*

Since a fixed point of Φ_p generates a periodic point of period p in the periodic logistic equation (2), from Theorem 4.2 the following corollary follows:

Corollary 1. *Under Assumptions **H1** and **H2**, a locally asymptotically stable p -periodic cycle of the p -periodic logistic equation $x_{n+1} = r_n x_n (1 - x_n)$, $r_{n+p} = r_n$ for all n , is globally asymptotically stable.*

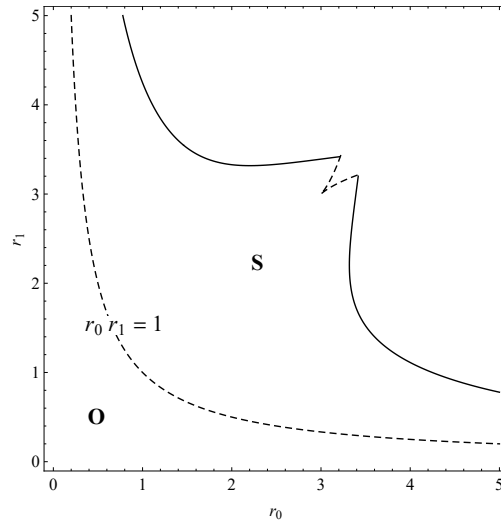


FIGURE 1. Regions of stability, in the parameter space r_0Or_1 , of the fixed points of $f_1 \circ f_0$, with $f_i(x) = r_i x(1 - x)$, $i = 0, 1$.

5. Parameter space for Φ_2 . Following the techniques employed in [2], one can find the region of local stability of the fixed points of the composition map $\Phi_2 = f_1 \circ f_0$ (and of Φ_3 as well) by calculating the solution of the equation $|\Phi'_2(x^*)| = 1$ which is equivalent to the following set of equations:

$$\begin{cases} f_1(f_0(x^*)) = x^* \\ f'_1(f_0(x^*))f'_0(x^*) = 1 \end{cases} \quad (13)$$

and

$$\begin{cases} f_1(f_0(x^*)) = x^* \\ f'_1(f_0(x^*))f'_0(x^*) = -1 \end{cases} \quad (14)$$

The implicit solution, in the parameter space r_0Or_1 , of System (13) is

$$-r_0^6 r_1^3 (r_0 r_1 - 1)^2 (r_1^2 r_0^2 - 4r_1 r_0^2 - 4r_1^2 r_0 + 18r_1 r_0 - 27) = 0 \quad (15)$$

while the implicit solution of System (14) is

$$-r_0^6 r_1^3 (r_0 r_1 + 1) (r_1^3 r_0^3 - 4r_1^2 r_0^3 - 4r_1^3 r_0^2 + 15r_1^2 r_0^2 + 12r_1 r_0^2 + 12r_1^2 r_0 - 85r_1 r_0 + 125) = 0. \quad (16)$$

Drawing implicitly the curves where the two previous equations are satisfied, we are able to find the region where the stability of the fixed points of Φ_2 occurs. This region is represented in Figure 1, in the parameter space r_0Or_1 . The solutions of Equation (15) are represented by the dashed curves while the solutions of Equation (16) are represented by the solid curves.

If the parameters r_0 and r_1 belong to the region **O**, then the origin is a fixed point globally asymptotically stable. Once the parameters cross the dashed curve, from Region **O** to Region **S**, a bifurcation occurs, known as saddle-node bifurcation. The fixed point $x^* = 0$ becomes unstable and a new globally asymptotically stable fixed point of Φ_2 is born. This fixed point is, in fact, a 2-periodic cycle of the 2-periodic

equation (2). Now if the parameters r_0 and r_1 cross the dashed pitchfork, a saddle-node bifurcation occurs. In this case, the 2-periodic cycle becomes unstable and a new locally asymptotically stable 2-periodic cycle is born.

At the solid curve, a new type of bifurcation occurs known as a period-doubling bifurcation. Hence, when the parameters cross the solid curve, the globally 2-periodic cycle of equation (2) becomes unstable and a new locally asymptotically stable 4-periodic cycle is born.

Before ending this section, it should be mentioned that similar conclusions can be drawn for the triples (r_0, r_1, r_2) in the case of $p = 3$.

REFERENCES

- [1] K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos. An Introduction to Dynamical Systems*, Textbooks in Mathematical Sciences, Springer-Verlag, New York, 1997.
- [2] Z. AlSharawi and J. Angelos, [On the periodic logistic equation](#), *Appl. Math. Comput.*, **180** (2006), 342–352.
- [3] W. A. Coppel, [The solution of equations by iteration](#), *Proc. Cambridge Philos. Soc.*, **51** (1955), 41–43.
- [4] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Studies in Nonlinearity, second edition, Westview Press, Boulder, CO, 2003.
- [5] H. A. El-Morshedy and V. J. López, [Global attractors for difference equations dominated by one-dimensional maps](#), *J. Difference Equ. Appl.*, **14** (2008), 391–410.
- [6] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, third edition, Springer, New York, 2005.
- [7] S. N. Elaydi, *Discrete Chaos. With Applications in Science and Engineering*, Second edition, Chapman & Hall/CRC, Boca Raton, FL, 2008.
- [8] M. Grinfeld, P. A. Knight and H. Lamba, [On the periodically perturbed logistic equation](#), *J. Phys. A*, **29** (1996), 8035–8040.
- [9] R. B. Kellogg, T. Y. Li and J. Yorke, [A constructive proof of the Brouwer fixed-point theorem and computational results](#), *SIAM J. Numer. Anal.*, **13** (1976), 473–483.
- [10] M. Kot and W. M. Schaffer, [The effects of seasonality on discrete models of population growth](#), *Theoret. Population Biol.*, **26** (1984), 340–360.
- [11] C. P. Li and M. Zhao, [On the periodic logistic map](#), *Acta Math. Sin. (Engl. Ser.)*, **34** (2018), 891–900.
- [12] J. Li, [Periodic solutions of population models in a periodically fluctuating environment](#), *Math. Biosci.*, **110** (1992), 17–25.
- [13] E. Liz, [On the global stability of periodic Ricker maps](#), *Electron. J. Qual. Theory Differ. Equ.*, **76** (2016), 8 pp.
- [14] R. M. May, Simple mathematical models with very complicated dynamics, *Nature*, **261** (1976), 459–467.
- [15] J. R. Munkres, *Topology: A First Course*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1975.
- [16] R. J. Sacker and G. R. Sell, [Lifting properties in skew-product flows with applications to differential equations](#), *Mem. Amer. Math. Soc.*, **11** (1977).
- [17] H. Sedaghat, *Nonlinear Difference Equations. Theory with Applications to Social Models*, Mathematical Modelling: Theory and applications, vol. 15, Kluwer Academic Publishers, Dordrecht, 2003.
- [18] D. Singer, [Stable orbits and bifurcation maps of the interval](#), *SIAM J. Appl. Math.*, **35** (1978), 260–267.
- [19] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, *Correspondance Math. Phys.*, **10** (1838), 113–121.

Received July 2019; revised October 2019.

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