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Non-autonomous periodic systems with Allee effects

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A new class of maps called unimodal Allee maps are introduced. Such maps arise in the study of population dynamics in which the population goes extinct if its size falls below a threshold value. A unimodal Allee map is thus a unimodal map with three fixed points, a zero fixed point, a small positive fixed point, called threshold point, and a bigger positive fixed point, called the carrying capacity. In this paper, the properties and stability of the three fixed points are studied in the setting of non-autonomous periodic dynamical systems or difference equations. Finally, we investigate the bifurcation of periodic systems/difference equations when the system consists of two unimodal Allee maps.

Keywords: unimodal Allee maps; threshold point; carrying capacity; composition map; stability; bifurcation

1. Introduction

The Allee effect is a phenomenon in population dynamics attributed to the biologist Allee [1]. Allee proposed that the per capita birth rate declines at low density or population sizes. In the languages of dynamical systems or difference equations, a map representing the Allee effect must have three fixed points, an asymptotically stable zero fixed point, a small unstable fixed point, called the threshold point, and a bigger positive fixed point, called the carrying capacity, which is asymptotically stable at least for smaller values of the parameters.

Recently, there has been a surge in research activities on models with Allee effect and a publication of a book dedicated solely to this phenomenon [5].

Some of the relevant work may be found in Allen et al. [2], Cushing [6], Dennis [7], Li et al. [14], Luís et al. [15], Yakubu [17,18], Sacker and Elaydi [19], Schreiber [21] and Sophia and Jang [23].

Our main interest in this paper is to study non-autonomous periodic difference equations/discrete dynamical systems in which the maps of the system are unimodal Allee maps. Such systems model population with fluctuating habitat, and they are commonly called periodically forced systems.

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2. Preliminaries

Consider the set $\mathcal{F} = \{f_0, f_1, \ldots, f_{p-1}\}$ of continuous maps on $I = [0, b]$, where $b \leq \infty$. The set $\mathcal{F}$ generates the non-autonomous p-periodic difference equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}^+, \tag{1}$$

where $\mathbb{Z}^+ := \{0, 1, 2, 3, \ldots\}$ and $f_{n+p} = f_n, \ \ \forall n \in \mathbb{Z}^+$. Here, the orbit of a point x_0 is generated by the composition of the sequence of maps $\{f_n\}$. Explicitly,

$$\mathcal{O}(x_0) = \{x_0, f_0(x_0), f_1(f_0(x_0)), f_2(f_1(f_0(x_0))), \ldots\} = \{x_0, x_1, x_2, \ldots\}.$$

Though the non-autonomous periodic difference equation (1) does not generate a discrete dynamical system [9], one may speak about the non-autonomous p-periodic dynamical system $\mathcal{F}$. One of the most effective ways of converting the non-autonomous difference equation (1) into a genuine discrete dynamical system is the construction of the associated skew-product system as described in a recent paper by Elaydi and Sacker [10]. It is noteworthy to mention that this idea was originally used to study non-autonomous differential equations by Sacker and Sell [20]. However, since the focus here will be on the case when $\mathcal{F}$ consists of two maps, we will not utilize the skew-product construction as it is more appropriate for more complicated setting.

We now present few basic definitions that will be used in the sequel.

Definition 2.1. *A point x^* is a fixed point of equation* (1) *or the systems $\mathcal{F}$ if $f_n(x^*) = x^*$ for all $n \in \mathbb{Z}^+$. In other words, x^* is a fixed point of all the maps in $\mathcal{F}$.*

Definition 2.2. *Let $C_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be an ordered set in I. Then, C_r is called an r-periodic cycle if*

$$f_{(i+nr)\bmod p}(\bar{x}_i) = \bar{x}_{(i+1)\bmod r}, \ \ n \in \mathbb{Z}^+.$$

In particular,

$$f_i(\bar{x}_i) = \bar{x}_{i+1}, \quad 0 \leq i \leq r-2,$$

and

$$f_t(\bar{x}_{t \bmod r}) = \bar{x}_{(t+1)\bmod r}, \ \ r-1 \leq t \leq p-1.$$

To this end, we have talked about general continuous maps on an interval. The focus in this paper will be on special types of map that we call unimodal Allee maps. A definition of these maps now follows.

Definition 2.3. *Let $I = [0, b] \subset \mathbb{R}^+$. A continuous function $f : I \to I$ is called an* Allee map *if the following hold:*

- $f(0) = 0$, *and there are positive points A_f and K_f such that*
- $f(x) < x$ *for* $x \in (0, A_f) \bigcup (K_f, b)$ *and* $f(x) > x$ *for* $x \in (A_f, K_f)$.

If, in addition, the map is unimodal, then it is called a *unimodal Allee map*. Explicitly, we require the following:

- $f(b) = 0$ when b is finite or $\lim_{x \to +\infty} f(x) = 0$ otherwise.
- There exists a unique critical point C_f of f, where $f(x)$ is strictly increasing on $[0, C_f)$ and strictly decreasing on (C_f, b) (or $(C_f, +\infty)$ when $b = +\infty$).

Figure 1 depicts a prototype of unimodal Allee maps.

From Definition 2.3, it follows that A_f and K_f are positive fixed points. We call the smaller positive fixed point the threshold point A_f of the map f, and the greater positive fixed point K_f the carrying capacity of the map f. It is easy to verify that $x^* = 0$ is an attracting fixed point and $[0, A_f) \bigcup (\tilde{A}_f, b) \subset \mathcal{B}_f(0)$, where $\mathcal{B}_f(0)$ is the basin of attraction of zero and $\tilde{A}_f = f^{-1}(A_f)$, i.e. $f(\tilde{A}_f) = A_f$, with $\tilde{A}_f > K_f$. Note that the threshold point A_f is always repelling, while the carrying capacity K_f may or not be stable.

Next, we define a unimodal Allee map f by two maps, a left map f_l and a right map f_{r}. Thus, we have

$$f(x) = \begin{cases} f_l(x) & \text{if } 0 \leq x \leq A_f, \\ f_{\mathrm{r}}(x) & \text{if } A_f < x \leq b. \end{cases} \tag{2}$$

It follows that $f(0) = f_l(0) = f_{\mathrm{r}}(b) = 0$ (or $\lim_{x \to \infty} f_{\mathrm{r}}(x) = 0$). Since f is continuous in $\mathbb{R}^+$, it follows that $f(A_f) = f_l(A_f) = f_{\mathrm{r}}(A_f) = A_f$ and $f(K_f) = f_{\mathrm{r}}(K_f) = K_f$.

To facilitate our study, we introduce two zones, the threshold zone and the carrying capacity zone.

DEFINITION 2.4.

(1) *The square that contains the origin and the points* $(A_f, 0), (0, A_f)$ *and* (A_f, A_f) *will be called the* threshold zone.

(2) *The rectangle that contains the points* (A_f, A_f), $(A_f, f(C_f))$, $(\tilde{A}_f, A_f)$ *and* $(\tilde{A}_f, f(C_f))$ *will be called the* carrying capacity zone.

Consider now the non-autonomous periodic system $\mathcal{W} = \{f, g\}$, where f and g are unimodal Allee maps with $f(x) > g(x)$ for all x on $(0, b)$. We note that under this hypothesis, we have $0 < A_f < A_g < K_g < K_f$. Henceforth, we assume that the right end point b of I is fixed for all the unimodal Allee maps.

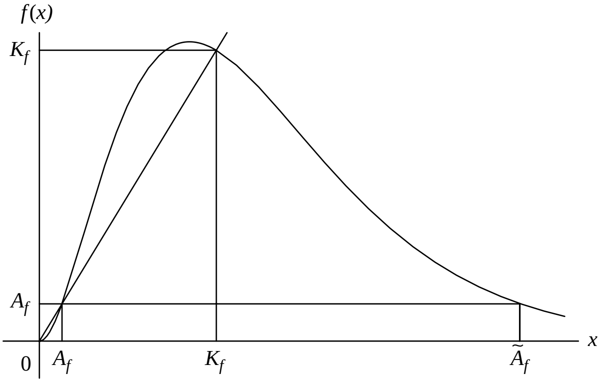

Figure 1. An instance of one unimodal Allee map f.

The composition map $f \circ g$ may be written as follows:

$$f \circ g = \begin{cases} f(g_l(x)) & \text{if} \quad 0 \leq x \leq A_g, \\ f(g_{\mathrm{r}}(x)) & \text{if} \quad A_g < x \leq b. \end{cases} \tag{3}$$

The first branch of (3) may be written as

$$f(g_l(x)) = \begin{cases} f_l(g_l(x)) & \text{if} \quad 0 \leq g_1(x) \leq A_f \wedge 0 \leq x \leq A_g, \\ f_{\mathrm{r}}(g_l(x)) & \text{if} \quad A_f \leq g_1(x) < b \wedge 0 \leq x \leq A_g, \end{cases}$$

hence

$$f(g_l(x)) = \begin{cases} f_l(g_l(x)) & \text{if} \quad 0 \leq x \leq A_f^-, \\ f_{\mathrm{r}}(g_l(x)) & \text{if} \quad A_f^- < x \leq A_g, \end{cases} \tag{4}$$

where A_f^- represents the left preimage of A_f under the map g, i.e. $g(A_f^-) = A_f$, or equivalently

$$f_l(g_l(A_f^-)) = f_{\mathrm{r}}(g_l(A_f^-)) = f(A_f) = A_f.$$

The second branch of (3) may be written as

$$f(g_{\mathrm{r}}(X)) = \begin{cases} f_{\mathrm{r}}(g_{\mathrm{r}}(x)) & \text{if} \quad A_g < x < A_f^+, \\ f_l(g_{\mathrm{r}}(x)) & \text{if} \quad A_f^+ \leq x \leq b, \end{cases} \tag{5}$$

where A_f^+ represents the right preimage of A_f under the map g, i.e. $g(A_f^+) = A_f$, or equivalently

$$f_{\mathrm{r}}(g_{\mathrm{r}}(A_f^+)) = f_l(g_{\mathrm{r}}(A_f^+)) = f(A_f) = A_f.$$

From (4) and (5), we obtain

$$f \circ g = \begin{cases} f_l(g_l(x)) & \text{if} \quad 0 \leq x \leq A_f^-, \\ f_{\mathrm{r}}(g_l(x)) & \text{if} \quad A_i^- < x \leq A_g, \\ f_{\mathrm{r}}(g_{\mathrm{r}}(x)) & \text{if} \quad A_g < x < A_f^+, \\ f_l(g_l(x)) & \text{if} \quad A_f^+ \leq x \leq b. \end{cases} \tag{6}$$

Similarly,

$$g \circ f = \begin{cases} g_l(f_l(x)) & \text{if} \quad 0 \leq x \leq A_f, \\ g_l(f_{\mathrm{r}}(x)) & \text{if} \quad A_f < x < A_g^-, \\ g_{\mathrm{r}}(f_{\mathrm{r}}(x)) & \text{if} \quad A_g^- \leq x \leq A_g^+, \\ g_l(f_{\mathrm{r}}(x)) & \text{if} \quad A_g^+ < x \leq b, \end{cases} \tag{7}$$

where A_g^- and A_g^+ represent the left and the right preimages of A_g under the map f, respectively, i.e. $f(A_g^-) = f(A_g^+) = A_g$. In other words, we have

$$g_l(f_{\mathrm{r}}(A_g^-)) = g_{\mathrm{r}}(f_{\mathrm{r}}(A_g^-)) = g(A_g) = A_g$$

and

$$g_{\mathrm{r}}(f_{\mathrm{r}}(A_g^+)) = g_l(f_{\mathrm{r}}(A_g^+)) = g(A_g) = A_g.$$

Figure 2 summarizes the above remarks.

LEMMA 2.5. *Let* $f, g \in \mathcal{W}$. *If* $C_f > A_g$, *where* C_f *is the unique critical point of* f *and* A_g *is the threshold point of* g, *then* f *and* g, *both, are homeomorphisms on* $[0, A_g]$.

3. Threshold points of the composition map

In this section, we prove the existence of the fixed points, called threshold points, of the composition map. In addition, we establish an order relation between these fixed points.

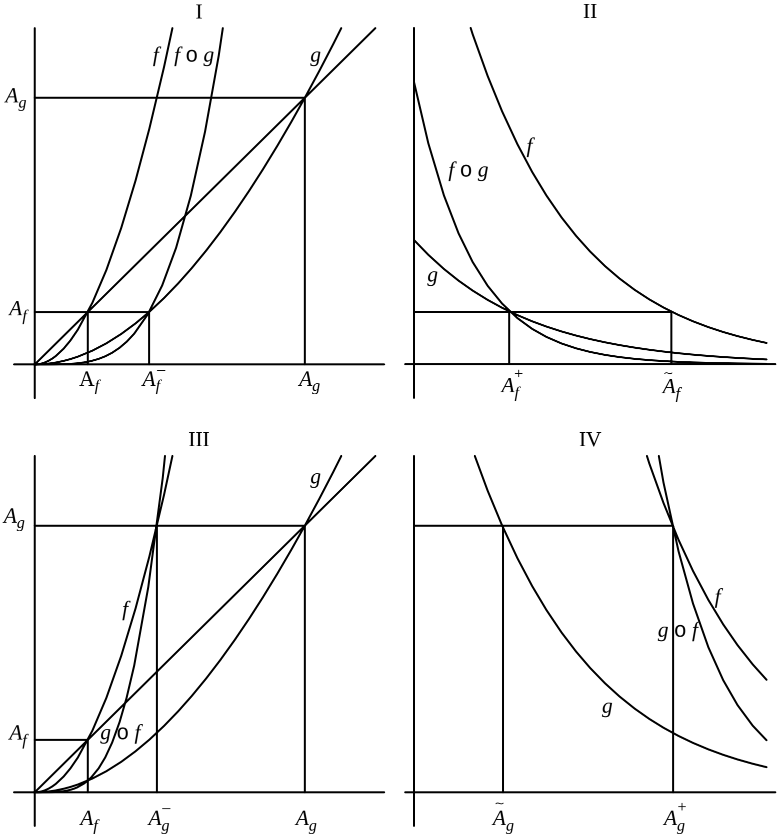

Figure 2. Parts I and II depict the left and the right preimages of A_f under g, while parts III and IV depict the left and the right preimages of A_g under the map f.

Hereafter, we assume that A_f and A_g are the threshold points of the unimodal Allee maps f and g, respectively. We also assume that A_f^- and A_g^- are, respectively, the first preimage of A_f by the map g and the first preimage of A_g by the map f.

THEOREM 3.1. *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Then, both $f \circ g$ and $g \circ f$ have threshold points that we denote by A_{fg} and A_{gf}, respectively. Moreover, $A_f^- < A_{fg} < A_g$ and $A_f < A_{gf} < A_g^-$.*

Proof. Assume that f and g satisfy the conditions of the hypothesis of the theorem. First, let us prove the existence of A_{fg} and A_{gf}. We know that $g_l(A_f) < A_f$, and on $[0, A_f]$, f is increasing. This implies that $f_l(g_l(A_f)) < f_l(A_f) = A_f$. On the other hand, $f_{\mathrm{r}}(g_l(A_g)) = f_{\mathrm{r}}(A_g) > g_l(A_g) = A_g$. Hence, the function $f \circ g(x) - x$ changes sign on (A_f, A_g). Then, there exits $x \in (A_f, A_g)$ such that $f \circ g(x) = x$, i.e. $A_f < A_{fg} < A_g$. In the same way, we prove that $A_f < A_{gf} < A_g$.

To prove that $A_{fg} \in (A_f^-, A_g)$, first we will prove that $A_{fg} \notin [A_f, A_f^-]$. Let $x \in [A_f, A_f^-]$. We know that $f \circ g(A_f) < A_f$ and $f \circ g(A_f^-) = A_f < A_f^-$. When $x \in (A_f, A_f^-)$, we have $g_l(x) < A_f$ and so $f_l(g_l(x)) < A_f < x$. Therefore, $A_{fg} \notin [A_f, A_f^-]$.

Now, let $x \in (A_f^-, A_g)$. On the one side, we have $f \circ g(A_g) = f(A_g) > A_g > x$ and on the other side, $f \circ g(A_f^-) = A_f < x$, consequently, there exits $y \in (A_f^-, A_g)$ such that $f \circ g(y) = y$, i.e. $A_{fg} \in (A_f^-, A_g)$. Following the same reasoning we prove that $A_f < A_{gf} < A_g^-$. ☐

Next, we establish an order relation between these two threshold points of $f \circ g$ and $g \circ f$, respectively.

THEOREM 3.2. *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Suppose that in the threshold region, i.e. on $J = [0, A_g]$, these two maps are convex, f is increasing and $f'(x) > g'(x)$, $\forall x \in J$. Suppose also that*

$$f'(A_g) + g'(A_g) \leq f'(A_f) g'(A_g). \tag{8}$$

Then, $A_g^- \leq A_f^-$. Moreover $A_{gf} < A_{fg}$.

Proof. Since f and g are unimodal Allee maps such that $f(x) > g(x)$, for all $x \in (0, b)$, we have that $A_g - A_f = \varepsilon > 0$ and $\forall x \in J = [0, A_g]$, $f(x) - g(x) = \delta(x) > 0$ such that $\delta(x)$ is increasing. We need to prove that the firstpreimage of both A_f and A_g satisfies the relation $A_g^- \leq A_f^-$ or $f^{-1}(A_g) \leq g^{-1}(A_f)$, which is equivalent to

$$A_g \leq f \circ g^{-1}(A_f). \tag{9}$$

By Taylor's series we know that

$$g^{-1}(A_f) = g^{-1}(A_g - \varepsilon) = g^{-1}(A_g) - g^{-1'}(A_g)\varepsilon + O(\varepsilon^2) = A_g - \frac{\varepsilon}{g'(A_g)} + O(\varepsilon^2).$$

Substituting the previous relation in (9), we get

$$A_g \leq f\left(A_g - \frac{\varepsilon}{g'(A_g)} + O(\varepsilon^2)\right)$$

and again by Taylor's series

$$A_g = f(A_g) - \frac{f'(A_g)}{g'(A_g)}\varepsilon + O(\varepsilon^2),$$

that is

$$\frac{f'(A_g)}{g'(A_g)}\varepsilon \leq f(A_g) - A_g + O(\varepsilon^2).$$

Once $f(A_g) - A_g = f(A_g) - g(A_g) = \delta$, we get

$$\frac{f'(A_g)}{g'(A_g)}\varepsilon \leq \delta + O(\varepsilon^2) \approx \delta. \tag{10}$$

So, relation (9) is equivalent to relation (10).

f is convex and therefore

$$f'(A_f) < \frac{f(A_g) - f(A_f)}{A_g - A_f} = f'(M) < f'(A_g),$$

where $M \in \,]A_f, A_g[$. So, $f'(M) = (f(A_g) - A_g + A_g - A_f)/\varepsilon$ and therefore $f'(M) = (\delta + \varepsilon)/\varepsilon$.

From (8), we have $f'(A_g) + g'(A_g) \leq f'(A_f)g'(A_g)$, which is equivalent to $(f'(A_g)/g'(A_g)) + 1 \leq f'(A_f)$. But, $f'(A_f) < f'(M)$ so

$$\frac{f'(A_g)}{g'(A_g)} + 1 \leq f'(M) = \frac{\delta + \varepsilon}{\varepsilon}.$$

Multiplying both the members of the last relation by ε, we get

$$\frac{f'(A_g)}{g'(A_g)}\varepsilon \leq \delta,$$

which is equivalent to relation (10), and therefore this part of the theorem is proved.

From Theorem 3.1 and by the fact that $A_g^- \leq A_f^-$, it follows that $A_{gf} < A_{fg}$. □

Hypothesis (8) requires that f and g stay sufficiently far apart to avoid the collapse of the interval where the threshold points of $f \circ g$ and $g \circ f$ belong.

4. The carrying capacity of the composition map

In this section, we study the existence, the location and the properties of the carrying capacity of the composition map.

Note that if f and g are two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$, then the critical points of $f \circ g$ are the solutions of the equation $f'(g(x))g'(x) = 0$. This implies that C_g is a critical point of both g and $f \circ g$. The equation $f'(g(x)) = 0$ has a solution if and only if the equation $g(x) = C_f$ has a solution. Thus, either $g^{-1}(C_f) = \emptyset$ or $g^{-1}(C_f)$ consists of two points, one on the left side of C_g and the other on the right side of C_g. Let us represent the points by C_{fg}^- (respectively C_{fg}^+), the critical point of the composition map $f \circ g$ on the left (respectively on the right) side of C_g.

So, if C_{fg}^- and C_{fg}^+ exist, then the composition map $f \circ g$ has four intervals of monotonicity (otherwise $f \circ g$ has two intervals of monotonnicity). Explicitly, $f \circ g$ is strictly increasing on $[0, C_{fg}^-] \bigcup [C_g, C_{fg}^+]$ and is strictly decreasing on $[C_{fg}^-, C_g] \bigcup [C_{fg}^+, b]$. The same analysis can be made for the map $g \circ f$.

Note that the threshold point of the composition map $f \circ g$ (respectively $g \circ f$) belongs always to the first interval where the composition map is increasing.

Recall from the previous sections that K_f and K_g are the carrying capacities of f and g, respectively, and A_f^+ (respectively A_g^+) is the right positive preimage of A_f (respectively A_g) under the map g (respectively f). We also follow the notation about the critical points of the composition map that we described above.

Theorem 4.1. *Let* f *and* g *be two unimodal Allee maps such that* $f(x) > g(x)$ *for all* x *on* $(0, b)$*. Then, both* $f \circ g$ *and* $g \circ f$ *have carrying capacities that we denote by* K_{fg} *and* K_{gf}*, respectively. Moreover,* $K_g < K_{fg} < A_f^+$ *and* $A_g^- < K_{gf} < A_g^+$*.*

Proof. It is clear that $0 < A_f < A_g < K_g < K_f$, $A_f^+ > K_f$ and $A_g^- < A_g < K_g < A_g^+$.

We can see that $f \circ g(K_g) = f(K_g) > g(K_g) = K_g$ and $f \circ g(A_f^+) = f(A_f) = A_f < A_f^+$. Therefore, the map $f \circ g(x) - x$ changes sign on (K_g, A_f^+). Hence, there exists $x \in (K_g, A_f^+)$ such that $f \circ g(x) = x$.

Note that $C_{fg}^+ < A_f^+$. To see this fact, suppose by contradiction that $C_{fg}^+ \geq A_f^+$ or equivalently $g^{-1}(C_f) \geq g^{-1}(A_f)$. We know that $C_{fg}^+, A_f^+ > C_g$ and g is decreasing on (C_g, b). Consequently, applying g on both sides of the last inequality, we get $C_f \leq A_f$, which is impossible. Similarly, we prove that $C_{gf}^+ < A_g^+$.

Since $C_{fg}^+ < A_f^+$, the carrying capacity of $f \circ g$, K_{fg}, is the greater root of $f \circ g(x) = x$ on (K_g, A_f^+). We can also see that $g \circ f(A_g^+) = g(A_g) < A_g^+$ and $g \circ f(A_g^-) = A_g > A_g^-$. So, the map $g \circ f(x) - x$ changes sign on (A_g^-, A_g^+) and therefore there exists $K_{gf} \in (A_g^-, A_g^+)$ such that $g \circ f(K_{gf}) = K_{gf}$ since $C_{gf}^+ < A_g^+$. □

Remark 1. Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. If $f \circ g(C_g) > C_g$ (respectively $g \circ f(C_f) > C_f$), then the map $f \circ g$ (respectively $g \circ f$) has exactly two positive fixed points, the threshold point and the carrying capacity.

Corollary 4.2. *Let* f *and* g *be two unimodal Allee maps such that* $f(x) > g(x)$ *for all* x *on* $(0, b)$*. If* $C_g, C_f > K_f$*, then* $f \circ g(x) > g \circ f(x)$*,* $\forall x \in [K_g, K_f]$*. Moreover,* $K_g < K_{gf} < K_{fg} < K_f$*.*

Proof. If $C_g, C_f > K_f$, we have f and g increasing on $[K_g, K_f]$. The composition of increasing maps is an increasing map. The interval $[K_g, K_f]$ is invariant under composition because $f \circ g(K_g) > K_g$, $f \circ g(K_f) < K_f$ and $g \circ f(K_f) < K_f$, $g \circ f(K_g) > K_g$. So, the map $f \circ g(x) - x$ (respectively $g \circ f(x) - x$) changes sign on $[K_g, K_f]$. We know that $f(K_g) > K_g$ and therefore $g \circ f(K_g) < f(K_g) = f \circ g(K_g)$ $(g(x) < x, \forall x > K_g)$. On the other hand, we know that $g(K_f) < K_f$, so $f \circ g(K_f) > g(K_f) = g \circ f(K_f)$ $(f(x) > x, \forall x \in]A_f, K_f[)$. Consequently, $f \circ g(x) > g \circ f(x)$, $\forall x \in [K_g, K_f]$. Once $f \circ g(C_g) < C_g$ (respectively $g \circ f(C_f) < C_f$) from Remark 1, it follows that K_{fg} (respectively K_{gf}) is the unique fixed point of $f \circ g$ (respectively $g \circ f$) on $[K_g, K_f]$. The relation order between K_{fg} and K_{gf} is an immediate consequence of the relation order between the composition maps. □

COROLLARY 4.3. *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Then, following statements hold true*

(1) *if $C_f > K_f$ and $C_g > K_g$, then K_{fg}, $K_{gf} \in (K_g, K_f)$.*

(2) *if $C_f < K_f$, then we have*

   (a) *$K_{fg} \in (K_g, K_f)$, if $f \circ g(K_f) < K_f$ and $f \circ g(C^+_{fg}) < C^+_{fg}$ (in the case where C^+_{fg} does not exist, we have $K_{fg} \in (K_g, K_f)$, if $f \circ g(K_f) < K_f$);*

   (b) *$K_{fg} \in (K_f, A^+_f)$, if $f \circ g(K_f) > K_f$;*

   (c) *$K_{gf} \in (A^-_g, K_g)$, if $g \circ f(K_g) < K_g$ and $g \circ f(C^+_{gf}) < C^+_{gf}$;*

   (d) *$K_{gf} \in (K_g, K_f)$, if $g \circ f(K_g) > K_g$ and $g \circ f(C^+_{gf}) < C^+_{gf}$;*

   (e) *$K_{gf} \in (C^+_{gf}, A^+_g)$, if $g \circ f(C^+_{gf}) > C^+_{gf}$.*

(3) *if $C_f > K_f$ and $C_g < K_g$, then we have*

   (a) *$K_{fg} \in (K_g, K_f)$;*

   (b) *the situation of K_{gf} is similar to (2)(c)–(e).*

From the previous corollary, it is possible, in certain cases, to establish an order relation between the two carrying capacities K_{fg} and K_{gf} of the composition maps $f \circ g$ and $g \circ f$. In particular, we are interested in an order when such fixed points are between the carrying capacities of the individual maps. The next result provides this information, respectively.

THEOREM 4.4. *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Suppose that $C_f < K_f$, $C_g < K_g$, $f \circ g(K_f) < K_f$, $f \circ g(C^+_{fg}) < C^+_{fg}$, $g \circ f(K_g) > K_g$ and $g \circ f(C^+_{gf}) > C^+_{gf}$. Let $y \in J = [k_g, k_f]$ and suppose that $g(y) > K^-_f$, $\forall y \in J$, where K^-_f is the left preimage of K_f by the map f. Then, $g \circ f(y) < f \circ g(y)$, $\forall y \in J$. Moreover, $K_g < K_{gf} < K_{fg} < K_f$.*

Proof. Let K^-_f be the left preimage of K_f by the map f, i.e. $f(K^-_f) = K_f$. Then, $A_f < K^-_f < K_f$. Note that g is decreasing on $J = [k_g, k_f]$, $g(y) < y$, $\forall y \in J$, and $f(y) > y$, $\forall y \in J$.

From the hypothesis, we have $g(y) > K^-_f$ and therefore $f \circ g(y) > K_f$, $\forall y \in J$. On the other hand, $f(y) > K_f > y > K_g$ and then $g \circ f(y) < f(y) < K_g$, $\forall y \in J$. Consequently, $g \circ f(y) < f \circ g(y)$, $\forall y \in J$.

From the hypothesis of the theorem and Remark 1, the theorem is established. □

5. Stability

The first objective in this section is to formulate, in a more precise form, definitions of stability in the settings of general periodic difference equations of the form

$$x_{n+1} = f_n(x_n), \quad f_{n+p} = f_n, \quad n \in \mathbb{Z}^+. \tag{11}$$

Equivalently, one may speak about the stability in the settings of the non-autonomous periodic dynamical systems $\mathcal{F} = \{f_0, f_1, \ldots, f_{p-1}\}$.

Stability notions for periodic difference equations have been investigated by many authors, including, to cite few, AlSharawi and Angelos [3], AlSharawi et al. [4], Henson [13], Oliveira and D'Aniello [16], Yakubu [17] and Selgrade and Roberds [22].

It is our hope that our definitions will standardize the notion and terminology in the area of non-autonomous systems.

DEFINITION 5.1. $\Phi_i = f_{i-1} \circ \cdots \circ f_1 \circ f_0$ *is called the composition operator with order i associated with equation* (11)*, and* $\tilde{\Phi}_i = f_0 \circ f_1 \circ \cdots \circ f_{i-1}$ *is called the reverse composition operator with order i associated with equation* (11).

DEFINITION 5.2. *Let* x^* *be a fixed point of* $\mathcal{F}$*, i.e.* x^* *is a fixed point of all the members of the set* $\mathcal{F}$*. Then,*

(1) x^* *is stable if given* $\varepsilon > 0$*, there exits* $\delta > 0$ *such that* $|x_0 - x^*| < \delta$ *implies* $|\Phi_i(x_0) - x^*| < \varepsilon$, $\forall\, i \geq 1$. *Otherwise,* x^* *is unstable.*
(2) x^* *is attracting if there exists* $\eta > 0$ *such that* $|x_0 - x^*| < \eta$ *implies* $\lim_{n\to\infty}\Phi_n(x_0) = x^*$.
(3) x^* *is asymptotically stable if it is both stable and attracting.*
(4) x^* *is globally asymptotically stable if it is asymptotically stable and* $\eta = \infty$.

Many authors use the notion of attractivity of non-hyperbolic fixed point instead of our general definition. The following result provides us the connection between these two notions.

LEMMA 5.3. *Suppose that* $\mathcal{F}$ *is a set of continuously differentiable maps at* x^*. *If* $|\Phi_i'(x^*)| < 1$, $\forall\, i \geq 1$, *then* x^* *is an asymptotically stable fixed point of* $\mathcal{F}$.

Proof. From the hypothesis, we have for all $i \geq 1$

$$|\Phi_i'(x^*)| = |f_{i-1}'(x^*) f_{i-2}'(x^*) \cdots f_1'(x^*) f_0'(x^*)| \leq M < 1.$$

Hence, there exists an open interval $J = (x^* - \varepsilon, x^* + \varepsilon)$ such that $|\Phi_i'(y)| \leq M < 1$, $\forall\, y \in J$. By the mean value theorem, we know that

$$|\Phi_1(x_0) - x^*| = |\Phi_1'(y)|\,|x_0 - x^*| \leq M|x_0 - x^*|,$$

for any $x_0 \in I$ and y between x_0 and x^*. The last inequality implies that $\Phi_1(x_0) \in I$ since $M < 1$. Using the same argument, we get

$$|\Phi_2(x_0) - x^*| \leq M|\Phi_1(x_0) - x^*| \leq M^2|x_0 - x^*|.$$

By mathematical induction, we can prove that

$$|\Phi_n(x_0) - x^*| \leq M^n|x_0 - x^*|, \quad \forall\, n \geq 1. \tag{12}$$

Assuming $\delta = \varepsilon/2$, for any $\varepsilon > 0$, from $|x_0 - x^*| < \delta$ follows that $|\Phi_n(x_0) - x^*| \leq M^n\varepsilon/2 < \varepsilon$, $\forall\, n \geq 1$, since $M < 1$ and consequently x^* is stable. Moreover, $\lim_{n\to\infty}\Phi_n(x_0) = x^*$ and thus x^* is attracting. □

In particular, if $\mathcal{F}$ is a set formed by unimodal Allee maps, we have $x^* = 0$ as a fixed point of $\Phi_i(x)$, for all $i \geq 1$. Since this fixed point is asymptotically stable for each map, we have $|\Phi_i'(0)| < 1$ and thus from the previous lemma, $x^* = 0$ is an asymptotically stable fixed point of $\mathcal{F}$.

Now, let us focus our attention on the stability of a periodic cycle for a periodic non-autonomous difference equation. Our definition of stability now follows.

DEFINITION 5.4. *Let* $C_{\mathrm{r}} = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ *be a r-periodic cycle of equation* (11), *where* $f_{n+p} = f_n$, $n \in \mathbb{Z}^+$, $p > 1$ *and* $s = \operatorname{lcm}[r,p]$ *be the least common multiple of p and r.*

1. (1) C_{r} *is stable if given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|x_0 - \bar{x}_0| < \delta$ *implies* $|\Phi_i(x_0) - \Phi_i(\bar{x}_0)| < \varepsilon$, *for all* $i > 0$. *Otherwise,* C_{r} *is said to be unstable.*
2. (2) C_{r} *is attracting if there exits* $\eta > 0$ *such that* $|x_0 - \bar{x}_0| < \eta$ *implies* $\lim_{n\to\infty}\Phi_{sn+i}(x_0) = \bar{x}_i$, $0 \le i \le r-1$.
3. (3) C_{r} *is asymptotically stable if it is both stable and attracting.*
4. (4) C_{r} *is globally asymptotically stable if it is asymptotically stable and* $\eta = \infty$.

An immediate consequence of this definition now follows.

LEMMA 5.5. *Suppose that* $\mathcal{F} = \{f_0, f_1, \dots, f_{p-1}\}$ *is a set of continuously differentiable maps at* $\bar{x}_0$. *An* r*-periodic cycle* C_{r} *of equation* (11) *is*

1. (1) *asymptotically stable if* $|\Phi_s'(\bar{x}_0)| < 1$,
2. (2) *unstable if* $|\Phi_s'(\bar{x}_0)| > 1$,

where $f_{n+p} = f_n$, $n \in \mathbb{Z}^+$, $p > 1$ *and* $s = \operatorname{lcm}[r,p]$ *is the least common multiple of p and r.*

Proof. Note that by the chain rule, we have

$$|\Phi_s'(\bar{x}_0)| = \left|\prod_{i=0}^{s} f'_{i \bmod p}(\bar{x}_{i \bmod r})\right|.$$

(1) Suppose that $\left|\prod_{i=0}^{s} f'_{i \bmod p}(\bar{x}_{i \bmod r})\right| < 1$. Following the same argument that we used in the proof of Lemma 5.3, we get

$$|\Phi_{ns}(x_0) - \bar{x}_0| \le M^{ns}|x_0 - \bar{x}_0|, \quad n \ge 1. \tag{13}$$

This implies that $\lim_{n\to\infty}\Phi_{ns}(x_0) = \bar{x}_0$ since $M < 1$. By continuity (the composition of continuous maps is a continuous map), the following statement yields

$$\Phi_j\Phi_{ns}(x_0) = \Phi_{ns+j}(x_0) \underset{n\to\infty}{\longrightarrow} \Phi_{j \bmod p}(\bar{x}_0) = \bar{x}_{j \bmod r}, \quad 0 \le j \le s-1,$$

and thus C_{r} is attracting.

To see the stability of C_{r}, let us take $\delta < \varepsilon/2$, for any given $\varepsilon > 0$. Since $\bar{x}_0 \in C_{\mathrm{r}}$, we have $\bar{x}_0 = \Phi_s(\bar{x}_0) = \Phi_{sn}(\bar{x}_0)$, $n \ge 1$, and thus from (13), we have

$$|\Phi_{sn+j}(x_0) - \Phi_{sn+j}(\bar{x}_0)| \le M^{ns+j}|x_0 - \bar{x}_0| < M^{ns+j}\frac{\varepsilon}{2} < \varepsilon, \quad 0 < M < 1.$$

Once $n \ge 1$ and $0 \le j \le s-1$, we have $|\Phi_i(x_0) - \Phi_i(\bar{x}_0)| < \varepsilon$, $\forall\, i \ge 1$.

(2) If $\left|\prod_{i=0}^{s} f'_{i \bmod p}(\bar{x}_{i \bmod r})\right| > 1$, we get

$$|\Phi_{ns}(x_0) - \bar{x}_0| \geq M^{ns}|x_0 - \bar{x}_0|, \quad M > 1,$$

and thus instability. □

In the particular case, when $\mathcal{F}$ is a periodic set formed by unimodal Allee maps such that $f_i < f_{i+1}$, $\forall\, i \in \{0, 1, \ldots, p\}$, the threshold point A_{Φ_p} of Φ_p is unstable since the map $\Phi_p(x)$ is increasing on $[0, A_{\Phi_p}]$ and $\Phi_p(x) < x$, $\forall\, x \in (0, A_{\Phi_p})$. The same happens for $\tilde{\Phi}_p$ on $[0, A_{\tilde{\Phi}_p}]$.

Remark 1. The above theorems cover the hyperbolic case when $|\Phi'_p(x^*)| \neq 1$. When $|\Phi'_p(x^*)| = 1$ or -1, the fixed point is called neutral. A complete analysis of these non-hyperbolic cases for the autonomous system or single maps can be found in Elaydi's book 'Discrete Chaos' [8, p. 33].

6. Bifurcation

The study of various notions of bifurcation in the setting of non-autonomous systems is still in its infancy stage. The main contribution in this area are the papers by AlSharawi and Angelos [3], Henson [13] and Oliveira and D'Aniello [16]. Our main goal here is to give precise and complete definitions and notions for the various bifurcation notions in the setting of non-autonomous systems. Though our focus here will be on 2-periodic systems, the ideas presented can be easily extended to the general periodic case.

We start our exposition by presenting the following theorem due by Henson [13], where the idea is to perturb the parameters.

THEOREM 6.1. *Suppose that* $F(\alpha, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is non-linear in x, one to one in* α, C^2 *in both x and* α, *and for a specific real number* $\tilde{\alpha}$, *the autonomous difference equation* $x_{n+1} = F(\tilde{\alpha}, x_n)$ *has an attracting r-periodic cycle* $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$, *with a minimal period r. Then, for sufficiently small* $\varepsilon > 0$, *if* $\alpha_0, \alpha_1, \ldots, \alpha_{p-1} \in (\tilde{\alpha} - \epsilon, \tilde{\alpha} + \epsilon)$, *then the p-periodic difference equation* $x_{n+1} = F(\alpha_n, x_n)$, $\alpha_{n+p} = \alpha_n$, *has t attracting periodic solutions of minimal period s, where* $t = \gcd(p, r)$ (*greatest common divisor of p and r*) *and* $s = \operatorname{lcm}(p, r)$ (*least common multiple of p and r*).

Proof. See [13]. □

Consider the 2-periodic system $\mathcal{F} = \{f_0, f_1\}$, $f_0 \neq f_1$, where both maps arise from a one-parameter family of maps f_α in which $f_0 = f_{\alpha_0}$ and $f_1 = f_{\alpha_1}$.

Table 1. Number of asymptotically stable *r*-periodic cycles in the 2-periodic non-autonomous system $\mathcal{F} = \{f_0, f_1\}, f_0 \neq f_1$, where both maps arise from a one-parameter family of maps.

r	gcd (r, 2)	lcm (r, 2)	Conclusion
1	1	2	$\mathcal{F}$ has one GAS 2-periodic cycle
2	2	2	$\mathcal{F}$ has two AS 2-periodic cycle
4	2	4	$\mathcal{F}$ has two AS 4-periodic cycle
⋮	⋮	⋮	⋮
$2m$	2	$2m$	$\mathcal{F}$ has two AS $2m$-periodic cycle

GAS, globally asymptotically stable; AS, asymptotically stable.

Let $C_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be an r-periodic cycle of $f_{\tilde{\alpha}}$. In Table 1, we summarize the ideas present in Theorem 6.1 for sufficiently small ϵ and $\alpha_0, \alpha_1 \in (\tilde{\alpha} - \epsilon, \tilde{\alpha} + \epsilon)$.

Table 1 motivates the following result.

THEOREM 6.2. *Assume that the one-parameter family $F(\alpha, x)$ is one to one in α. Let $f_n(x_n) = F(\alpha_n, x_n)$. Then, if the p-periodic difference equation, with minimal period p,*

$$x_{n+1} = f_n(x_n), \tag{14}$$

has a non-trivial periodic cycle of minimal period r, then $r = tp$, $t \in \mathbb{Z}^+$.

Proof. Suppose that equation (14) has a periodic cycle $C_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ of period $r \neq p$, and let $d = \gcd(r,p)$, $s = \operatorname{lcm}[r,p]$, $m = p/d$ and $\ell = s/p$, then $d < p$. By [11], the graphs of the maps $f_0, f_d, \ldots, f_{(m-1)d}$ must intersect at the points $(\bar{x}_0, \bar{x}_1), (\bar{x}_d, \bar{x}_{d+1}), \ldots, (\bar{x}_{(\ell-1)d}, \bar{x}_{(\ell-1)d+1})$.

Since $F(\alpha, x)$ is one to one in α, the maps $f_0, f_d, \ldots, f_{(m-1)d}$ do not intersect, unless they are all equal. Similarly, one may show that $f_i = f_{i+d} = \cdots = f_{i+(m-1)d}$. This shows that equation (14) is of minimal period d, a contradiction. Hence, $r = p$ or a multiple of p. □

In the rest of this section, we assume that the maps f_0 and f_1 arise from a one-parameter family of maps such that $f_0 = f_{\alpha_0}$ and $f_1 = f_{\alpha_1}$ with $\alpha_1 = q\alpha_0$ for some real number $q > 0$. Thus, one may write, without loss of generality, our system as $\mathcal{F} = \{f_0, f_1\}$.

Moreover, we assume that the one-parameter family of maps is one to one with respect to the parameter. Let $C_{\mathrm{r}} = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be an r-periodic cycle of $\mathcal{F}$. Then, by Theorem 6.2, the latter assumption implies that $r = 2m$, $m \geq 1$.

With $\Phi_2 = f_1 \circ f_0$, one may write the orbit of $\bar{x}_0$ as (see Figure 3)

$$\begin{aligned} \mathcal{O}(\bar{x}_0) &= \left\{\bar{x}_0, f_0(\bar{x}_0), \Phi_2(\bar{x}_0), f_0 \circ \Phi_2(\bar{x}_0), \Phi_4(\bar{x}_0), \ldots, \Phi_{2(m-1)}(\bar{x}_0), f_0 \circ \Phi_{2(m-1)}(\bar{x}_0)\right\} \\ &= \{\bar{x}_0, \Phi_1(\bar{x}_0), \Phi_2(\bar{x}_0), \ldots, \Phi_{2m-1}(\bar{x}_0)\}. \end{aligned} \tag{15}$$

Equivalently, one may write the sequence of points given in (15) as

$$\begin{aligned} \mathcal{O}(\bar{x}_1) &= \left\{f_1 \circ \tilde{\Phi}_{2(m-1)}(\bar{x}_1), \bar{x}_1, f_1(\bar{x}_1), \tilde{\Phi}_2(\bar{x}_1), f_1 \circ \tilde{\Phi}_2(\bar{x}_1), \ldots, \tilde{\Phi}_{2(m-1)}(\bar{x}_1)\right\} \\ &= \{\tilde{\Phi}_{2m-1}(\bar{x}_1), \bar{x}_1, \tilde{\Phi}_1(\bar{x}_1), \tilde{\Phi}_{2m}(\bar{x}_1), \ldots, \tilde{\Phi}_{2m-2}(\bar{x}_1)\}, \end{aligned} \tag{16}$$

where $\tilde{\Phi}_2 = f_0 \circ f_1$. Hence, the order of the composition is irrelevant to the dynamics of the system.

The dynamics of $\mathcal{F}$ depends very much on the parameter, and the qualitative structure of the dynamical system changes as the parameter changes. These qualitative changes in the dynamics of the system are called *bifurcation* and the parameter values at which they occur are called bifurcation points. For autonomous systems or single maps, the bifurcation analysis may be found in Elaydi [8].

In one-dimensional systems generated by a one-parameter family of maps f_α, bifurcation at a fixed point x^* occurs when $(\partial f/\partial x)(\alpha^*, x^*) = 1$ or -1 at a bifurcation point α^*. The former case leads to a saddle-node bifurcation, while the latter case leads to a period-doubling bifurcation.

Our objective here is to extend this analysis to 2-periodic difference equations or $\mathcal{F} = \{f_0, f_1\}$. To simplify the notation, we write $\Phi_2(\alpha, x)$ instead of $\Phi_2(x)$. Let $C_{\mathrm{r}} =$

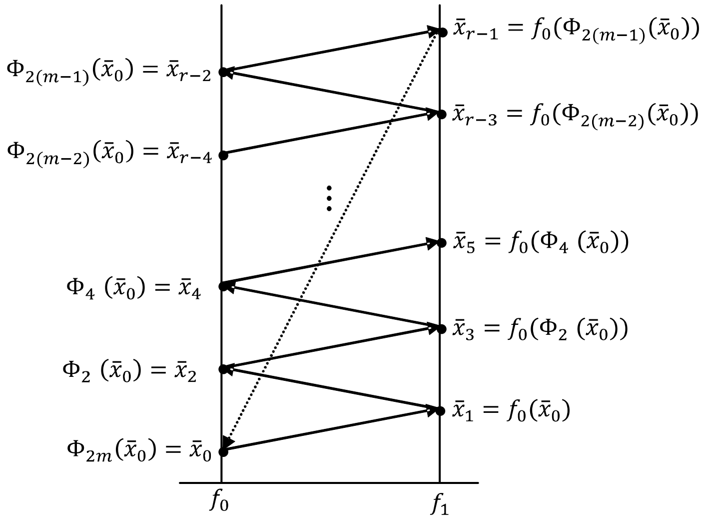

Figure 3. Sequence of the periodic points $\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ in the 2-periodic system $\mathcal{F} = \{f_0, f_1\}$ is illustrated in the fibres, where $\Phi_2 = f_1 \circ f_0$ and $r = 2m, m \geq 1$.

$\{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ be an r-periodic cycle of $\mathcal{F}$ and suppose that $2m = r$, $\Phi = f_1 \circ f_0$ and $\tilde{\Phi} = f_0 \circ f_1$, then $\Phi_{2m}(\bar{x}_{2i}) = \bar{x}_{(2i)\bmod r}$ and $\tilde{\Phi}_{2m}(\bar{x}_{2i+1}) = \bar{x}_{(2i+1)\bmod r}$, $1 \leq i \leq m$. In general, we have $\Phi_{2nm}(\bar{x}_{2i}) = \bar{x}_{(2i)\bmod r}$ and $\tilde{\Phi}_{2nm}(\bar{x}_{2i+1}) = \bar{x}_{(2i+1)\bmod r}$, $n \geq 1$.

Assuming $(\partial\Phi_{2m}/\partial x)(\bar{\alpha}, \bar{x}_0) = 1$ at a bifurcation point $\bar{\alpha}$, by the chain rule, we have

$$\frac{\partial\Phi_2}{\partial x}(\bar{\alpha}, \bar{x}_{2m-2})\frac{\partial\Phi_2}{\partial x}(\bar{\alpha}, \bar{x}_{2m-4}) \cdots \frac{\partial\Phi_2}{\partial x}(\bar{\alpha}, \bar{x}_2)\frac{\partial\Phi_2}{\partial x}(\bar{\alpha}, \bar{x}_0) = 1$$

or

$$f_1'(\bar{x}_{2m-1})f_0'(\bar{x}_{2m-2})f_1'(\bar{x}_{2m-3})f_0'(\bar{x}_{2m-4}) \cdots f_1'(\bar{x}_3)f_0'(\bar{x}_2)f_1'(\bar{x}_1)f_0'(\bar{x}_0)) = 1. \tag{17}$$

Applying f_0 on both sides of the identity, $\Phi_{2m}(\bar{\alpha}, \bar{x}_0) = \bar{x}_0$ yields $\tilde{\Phi}_{2m}(\bar{\alpha}, \bar{x}_1) = \bar{x}_1$. Differentiating both sides of this equation yields

$$\frac{\partial\tilde{\Phi}_2}{\partial x}(\bar{\alpha}, \bar{x}_{2m-1})\frac{\partial\tilde{\Phi}_2}{\partial x}(\bar{\alpha}, \bar{x}_{2m-3}) \cdots \frac{\partial\tilde{\Phi}_2}{\partial x}(\bar{\alpha}, \bar{x}_3)\frac{\partial\tilde{\Phi}_2}{\partial x}(\tilde{\alpha}, \bar{x}_1) = 1$$

or equivalently

$$f_0'(\bar{x}_0)f_1'(\bar{x}_{2m-1})f_0'(\bar{x}_{2m-2})f_1'(\bar{x}_{2m-3}) \cdots f_0'(\bar{x}_4)f_1'(\bar{x}_3)f_0'(\bar{x}_2)f_1'(\bar{x}_1) = 1. \tag{18}$$

Hence, equation (17) is equivalent to equation (18). More generally, the following relation holds

$$\frac{\partial\Phi_{2m}}{\partial x}(\bar{\alpha}, \bar{x}_{2j}) = \frac{\partial\tilde{\Phi}_{2m}}{\partial x}(\bar{\alpha}, \bar{x}_{2j-1}), \quad j \in \{1, 2, \dots, m\}. \tag{19}$$

The next result gives the conditions for the saddle-node bifurcation.

THEOREM 6.3. [SADDLE-NODE BIFURCATION FOR 2-PERIODIC SYSTEMS]. *Let* $C_{\mathrm{r}} = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ *be a periodic r-cycle of* $\mathcal{F}$. *Suppose that both* $\partial^2\Phi_2/\partial x^2$ *and* $\partial^2\Phi_2/\partial^2$ *exist and are continuous in a neighbourhood of a periodic orbit such that* $(\partial\Phi_{2m}/\partial x)$

$(\bar{\alpha}, \bar{x}_0) = 1$ for the periodic point $\bar{x}_0$. Assume also that

$$A = \frac{\partial \Phi_{2m}}{\partial \alpha}(\bar{\alpha}, \bar{x}_0) \neq 0 \quad \textit{and} \quad B = \frac{\partial^2 \Phi_{2m}}{\partial x^2}(\bar{\alpha}, \bar{x}_0) \neq 0,$$

then there exists an interval J around the periodic orbit and a C^2-map $\alpha = h(x)$, where $h : J \to \mathbb{R}$ such that $h(\bar{x}_0) = \bar{\alpha}$ and $\Phi_{2m}(x, h(x)) = x$. Moreover, if $AB < 0$, the periodic points exist for $\alpha > \bar{\alpha}$ and if $AB > 0$, the periodic points exist for $\alpha < \bar{\alpha}$.

Proof. The proof is similar to the proof of Theorem 2.5 in [8, p. 86] and will be omitted. □

When $(\partial \Phi_{2m}/\partial x)(\bar{\alpha}, \bar{x}_0) = 1$ but $(\partial \Phi_{2m}/\partial \alpha)(\bar{\alpha}, \bar{x}) = 0$, two types of bifurcations appear. The first is called transcritical bifurcation which occurs when $(\partial^2 \Phi_{2m}/\partial x^2)(\bar{\alpha}, \bar{x}_0) \neq 0$ and the second is called pitchfork bifurcation which appears when $(\partial^2 \Phi_{2m}/\partial x^2)(\bar{\alpha}, \bar{x}_0) = 0$. For more details about these two types of bifurcation see table 2.1 in [8, p. 90] and [16]. In the former work, the author presents many cases for autonomous maps, while in the latter article, the authors study the pitchfork bifurcation for non-autonomous 2-periodic systems in which the maps have negative Schwarzian derivative.

The next result gives the conditions for the period-doubling bifurcation.

THEOREM 6.4. [PERIOD-DOUBLING BIFURCATION FOR 2-PERIODIC SYSTEMS]. *Let $C_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be a periodic r-cycle of $\mathcal{F}$. Assume that both $\partial^2 \Phi_2/\partial x^2$ and $\partial \Phi_2/\partial \alpha$ exist and are continuous in a neighbourhood of a periodic orbit, $(\partial \Phi_{2m}/\partial x)(\bar{\alpha}, \bar{x}_0) = -1$, for the periodic point $\bar{x}_0$ and $(\partial^2 \Phi_{4m}/\partial \alpha \partial x)(\bar{\alpha}, \bar{x}_0) \neq 0$. Then, there exists an interval J around the periodic orbit and a function $h : J \to \mathbb{R}$ such that $\Phi_{2m}(x, h(x)) \neq x$ but $\Phi_{4m}(x, h(x)) = x$.*

Proof. The proof is similar to the proof of Theorem 2.7 in [8, p. 89] and is omitted. □

Note that if $\mathcal{W}$ is a periodic set formed by unimodal Allee maps, neither the zero fixed point nor the threshold point can contribute to bifurcation, since the former is always asymptotically stable and the latter is always unstable. Hence, bifurcation may only occur at the carrying capacity of $\mathcal{W}$.

Now, we apply the above results to study the bifurcation of the system $\mathcal{W} = \{f_0, f_1\}$, where $f_i(x) = a_i x^2(1 - x)$, $i = 0, 1$, $x \in [0, 1]$ and $a_i > 0$.

For an individual map f_i, the dynamics is interesting but predictable. For $a_i < 4$, we have a globally asymptotically stable zero fixed point and no other fixed point. A new unstable fixed point is born at $a_i = 4$, after which f_i becomes a unimodal map with an Allee effect. Henceforth, we assume that $a_0, a_1 > 4$.

Since 0 is the only fixed point under the system $\mathcal{W}$, we focus our attention on 2-periodic cycles $\{\bar{x}_0, \bar{x}_1\}$ with $f_0(\bar{x}_0) = \bar{x}_1$ and $f_1(\bar{x}_1) = \bar{x}_0$.

To determine the two main types of bifurcation, we solve the equations

$$\begin{cases} \bar{x}_0 = f_1(f_0(\bar{x}_0)), \\ f_1'(f_0(\bar{x}_0)) f_0'(\bar{x}_0) = 1, \end{cases} \tag{20}$$

and

$$\begin{cases} \bar{x}_0 = f_1(f_0(\bar{x}_0)), \\ f_1'(f_0(\bar{x}_0))f_0'(\bar{x}_0) = -1. \end{cases} \tag{21}$$

Using the command 'resultant'[4] in Mathematica or Maple Software, we eliminate the variable $\bar{x}_0$ in both systems. Equation (20) yields

$$\begin{aligned} &16777216 + 16384a_0a_1 - 576000a_0^2a_1 + 84375a_0^3a_1 - 576000a_0a_1^2 + 914a_0^2a_1^2 \\ &\quad - 350a_0^3a_1^2 + 84375a_0a_1^3 - 350a_0^2a_1^3 + 19827a_0^3a_1^3 - 2916a_0^4a_1^3 - 2916a_0^3a_1^4 \\ &\quad + 432a_0^4a_1^4 = 0, \end{aligned}$$

while equation (21) yields

$$\begin{aligned} &100000000 - 120000a_0a_1 - 2998800a_0^2a_1 + 453789a_0^3a_1 - 2998800a_0a_1^2 - 4598a_0^2a_1^2 \\ &\quad + 2702a_0^3a_1^2 + 453789a_0a_1^3 + 2702a_0^2a_1^3 + 89765a_0^3a_1^3 - 13500a_0^4a_1^3 \\ &\quad - 13500a_0^3a_1^4 + 2000a_0^4a_1^4 = 0. \end{aligned}$$

For each one of these two equations, we invoke the implicit function theorem to plot, in the (a_0,a_1)-plane, the bifurcation curves (see Figure 4).

The black curves are the solution of the former equation at which saddle-node bifurcation occurs, while the grey curves are the solution of the latter equation at which period-doubling bifurcations occur. The black cusp is the curve of pitchfork bifurcation.

In the regions identified by letters, one can conclude the following:

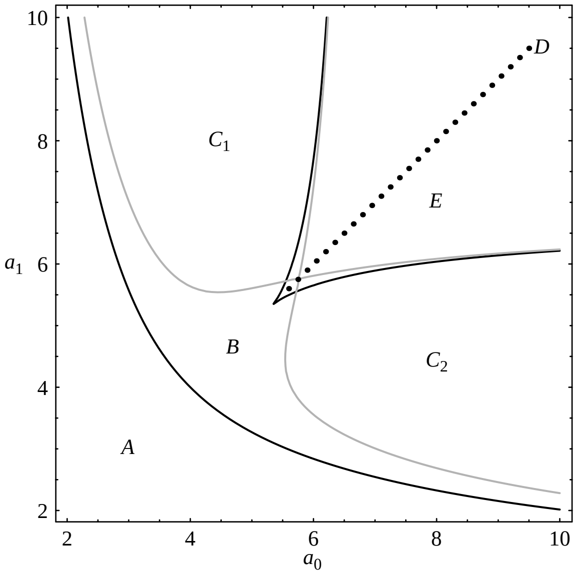

Figure 4. Bifurcations curves for the 2-periodic non-autonomous difference equation with Allee effects $x_{n+1} = a_nx_n^2(1 - x_n)$, $a_{n+2} = a_n$ and $x_{n+2} = x_n$ in the (a_0,a_1)-plane.

- If $a_0, a_1 \in A$, then the fixed point $x^* = 0$ is globally asymptotically stable.
- If $a_0, a_1 \in B \backslash D$, then there are two 2-periodic cycles, one attracting and one unstable.
- If $a_0, a_1 \in D$, then there are two attracting 2-periodic cycles (from the pitchfork bifurcation) and two unstable 2-periodic cycles.
- If $a_0, a_1 \in (C_1 \cup C_2) \backslash (D_1 \cup D_2)$, then there is an attracting 4-periodic cycle (from the period doubling bifurcation) and two unstable 2-periodic cycles.
- If $a_0, a_1 \in D_1 \cup D_2$, then there are two attracting 4-periodic cycles (from pitchfork bifurcation) and two unstable 2-periodic cycles.
- If $a_0, a_1 \in E$, then there are two attracting 8-periodic cycles (from period doubling bifurcation), two attracting 4-periodic cycles (from pitchfork bifurcation) and four unstable 2-periodic cycles.

It should be noted here that the bifurcation curves for the system $\mathcal{W}$ shown in Figure 4 are incomplete. If we want to draw more bifurcation curves in the space of the parameters, we must do the same for 4-periodic cycles, 8-periodic cycles and so on. Finding the implicit solutions of these new equations involves horrendous computations. The command 'resultant' does not produce answers after certain values of the degree of the polynomial. So, for the system $\mathcal{W}$, unfortunately we are unable to draw these curves for the 4-periodic cycle. However, it should be noted that AlSharawi and Angelos [3] have used the command 'resultant' to investigate the bifurcations of the periodically forced logistic map, and they were able to draw these curves for the 4-periodic cycles of the 2-periodic system. Moreover, these authors drew the bifurcation surfaces for the 3-periodic cycle of the 3-periodic system in the three-dimensional space of the parameters.

Finally, we should mention that Grinfeld et al. [12] have used the command 'resultant' much earlier to study the bifurcation of 2-periodic logistic systems.

Notes

1. Email: selaydi@trinity.edu
2. Email: holiv@math.ist.utl.pt
3. This work is part of the first author's PhD dissertation.
4. The command 'resultant' is a powerful tool that helps us in finding the implicit solutions for polynomial equations with low degree. We are not aware of similar techniques that work for non-polynomial equations such as the Ricker map $R_p(x) = x\,\mathrm{e}^{p-x}$, $p,\ \ x > 0$.

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