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An economical model with Allee effect

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We formulate a mathematical model based on Marx theory of economics. The profit rate r is considered as a function of both the exploitation rate e and the organic composition of the capital k. This model possesses a new property, commonly used in biology, called the Allee effect, in which the profit rate declines to zero if it falls below a certain threshold. It is represented by the difference equation $r_{n+1} = f_a(r_n)$, which is a family of unimodal maps depending on the parameter a, where a measures the relative growth of the exploitation rate when the profit rate is zero. Moreover, the model predicts a period-doubling bifurcation scenario as the parameter a increases. Finally, we allow the parameter a fluctuate periodically which leads to a periodic non-autonomous difference equations.

Keywords: profit rate; organic composition of the capital; exploitation rate; Allee effect; non-autonomous periodic systems; bust and boom in the profit rate

1. Model formulation

In [8] the authors developed a dynamical system for an economic model based on Marx's theory. In this paper we develop a new dynamical Marx model with an Allee effect. The Allee effect has been known in the biology literature for over half a century. However, it is perhaps the first time that this effect is explicitly used in modeling economics phenomena. Several papers in the mathematical biology literature addressing the impact of the Allee effect have recently appeared, among them the papers by Yakubu [12], Sophia and Jang [11], Li and all [5], Elaydi and Sacker [9]. But the concept is due to Warder Clyde Allee who introduced it in 1938 [1].

Note that the question of bust and boom or attenuance and resonance (see [2]) is central in both disciplines of economics and biology. In this paper we will investigate this question in connection with our model. Now a formulation of the model follows.

Marx defined the 'organic composition of capital k' as the ratio of what he called *constant capital to variable capital*. It is important to verify that constant capital is *not* what we today call fixed capital, but circulating capital, such as raw materials. Marx's 'variable capital' is defined as advances to labour, that is, total wage payments, or heuristically, $v = wL$ (where w is wages and L is labour employed). From the definition we have

$$k = \frac{c}{v}, \tag{1}$$

where c is the constant capital and v is the variable capital.

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The *profit rate*, according to Marx, is defined as

$$r = \frac{s}{v + c}, \tag{2}$$

where r represent the profit, s is the surplus, and $v + c$ are the total advances (constant and variable). The surplus, s, is the amount of total output produced above total advances, or $s = y - (v + c)$, where y is total output. It is important to note that, for Marx, only labour produces surplus value. Marx called the ratio of surplus to variable capital the 'exploitation rate e', that is

$$e = \frac{s}{v} \tag{3}$$

(surplus produced for every dollar/euro spent on labour).

Dividing the numerator and the denominator of (2) by v we get the equation

$$r = \frac{e}{1 + k}. \tag{4}$$

To study the profit rate using the discrete dynamical system approach, let r_n be the profit rate at the time unit n. We assume that the exploitation rate and the organic composition of the capital at time $n + 1$ depend on the profit rate at time n, that is

$$e_{n+1} = E(r_n), k_{n+1} = K(r_n). \tag{5}$$

In this case the profit rate depends also on the profit rate of the previous time unit, that is

$$r_{n+1} = \frac{E(r_n)}{1 + K(r_n)}. \tag{6}$$

The specific model that we propose here is based on the following assumptions:

(1) We first assume that the economy will not suffer any losses, that is, a negative profit rate does not occur. In fact, we think that in all of the economic activity there exists a nonnegative balance (possibly zero) of the profit. On the other hand, the profit rate cannot be unlimited because that contradicts the fact that the total quantity of 'money that exists' in the planet is finite.

(2) When the profit rate is low, the exploitation rate tends to increase. On the other hand, when the profit rate is high, the pressure on workers tends to decrease and therefore the exploitation rate tends to decrease to values near zero. This leads to the equation

$$E_a(r_n) = \frac{ar_n}{1 + r_n^2} \tag{7}$$

as a model for the exploitation rate, where $a > 0$ is constant (Figure 1). Note that the function E_a has the following properties

$$E_a(0) = E_a(\infty) = 0, E'_a(0) = a. \tag{8}$$

Hence $a > 0$ measures the relative growth of the exploitation rate when the profits are low.

(3) When the profit is low we have a relative amount for investment that decreases as long as the profit increases. If the economic system has a low profit rate, the tendency will be to incorporate more capital (invest) and, on the other hand, to decrease the human capital, through dismissals, which will lead to an increase of the organic composition of the

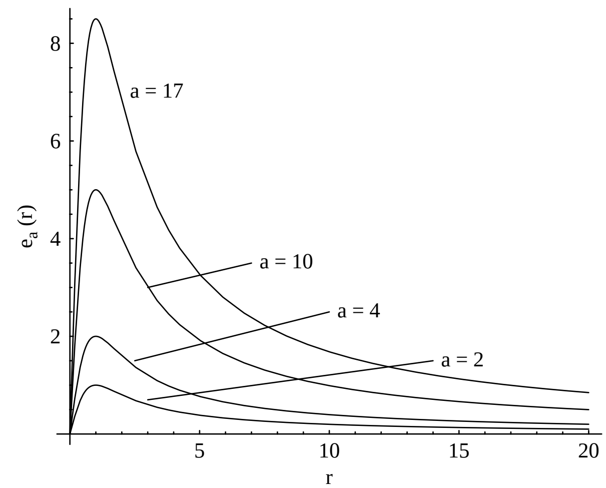

Figure 1. Progress of the exploitation rate.

capital. If the profit rate is very high, the tendency will be to reinvest in the capital: human capital (technical formation) and constant capital (technological innovation). A model that describes this reality is given by an incorporation of the capital function depending on the former profit rate, that is,

$$K_{b,d}(r_n) = \frac{e^{r_n - b}}{(r_n + d)^2}, \tag{9}$$

where b, $d > 0$ are constant. Note that the function $K_{b,d}$ has the following properties

$$K_{b,d}(0) = \frac{1}{d^2 e^b} > 0, K_{b,d}(\infty) = \infty, K'_{b,d}(0) = \frac{d-2}{d^3 e^b}. \tag{10}$$

This means that $1/(d^2 e^b)$ gives the initial amount, the organic composition of the capital as a function of the profit, grows unlimitedly, which is not realistic, but it is used to make the model approachable and $(d-2)/(d^3 e^b)$ measures the growth for the organic composition of the capital when the profits are low.

Using equations (7) and (9), the model equation (6) for the profit rate becomes

$$r_{n+1} = \frac{a r_n (r_n + d)^2}{(1 + r_n^2)[(r_n + d)^2 + e^{r_n - b}]}. \tag{11}$$

Now we are going to study the effect of varying the parameters b and d on K in equation (9).

When we fix the parameter b and increase d, we observe that the qualitative behaviour of the organic composition of the capital is similar for various values of d. However, the variation of d leads to a decrease of the organic composition of the capital, when the profit rate is low. This means that the main impact of the parameter d is on the initial value of the organic composition of the capital, that is, the initial investment (see the left plot in Figure 2).

On the other hand, when we fix the parameter d and increase b we observe that the graphs of the corresponding organic composition of the capital in model (9) are similar. Moreover, as the parameter b varies, the organic composition of the capital decreases for low values of the profits, until it stabilizes during a specific profits period after which it increases as profits increase. Note

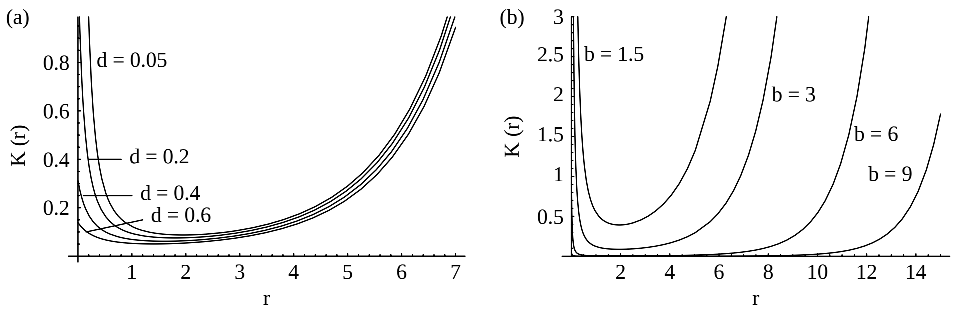

Figure 2. The progression of the organic composition of the capital in the model (9). In the left plot we fix the parameter b and increase d, while in the right plot we fix d and increase b. In both cases, in general, the behaviour of the organic composition of the capital is similar. As according b or d increases, for low profits the tendency is decreases the organic composition of the capital, for high profits, the tendency is increases the organic composition of the capital.

that, when the profits are low, this variation of b, leads to a decrease of the initial amount of the organic composition of the capital (see the right plot in Figure 2).

Hence, for a concrete study we need to choose specific values of b and d. It should be noted that the essence of our results will not change if one chooses other values of b and d. Accordingly, we select $b = 3$ and $d = 1/100$. Note that, the choice of these two values is conditioned by the initial amount of the organic composition of the capital.

We rewrite the model equation (11) as

$$r_{n+1} = \frac{ar_n\left(r_n + \frac{1}{100}\right)^2}{\left(1 + r_n^2\right)\left[\left(r_n + \frac{1}{100}\right)^2 + e^{r_n - 3}\right]} = f_a(r_n). \tag{12}$$

We observe that the function $f_a(r)$ is continuous, for all $a > 0$ and $r \in [0, +\infty[$. Moreover, $f_a(r) \geq 0, f_a(0) = f_a(\infty) = 0$. The function $f'_a(r)$ is also continuous and has only one root for all $r \geq 0$. Therefore $f_a(r)$ attains its maximum at the positive critical point $c \approx 1.10042$ and thus has an upper limit. Furthermore, $f'_a(0) = pa$, where $p \approx 0.00200453$. Thus, the origin is a locally asymptotically stable fixed point if $a < 1/p$ and unstable if $a > 1/p$. When $a = 1/p$ we have $f''_a(0) \approx 397.202$ and consequently, the origin is unstable (see Ref. [3] or [4]).

2. Existence of positive fixed points and their stability

In this section we investigate the existence of positive fixed points of the model (12) and their stability, if they exist.

THEOREM 2.1. *For the model* (12)*, there are two positive fixed points if $a > a_c$ where $a_c \approx 1.64271$, a unique positive fixed point if $a = a_c$ and no positive fixed points if $a < a_c$.*

Proof. To find the positive fixed point of (12) we consider the equation

$$\frac{a\left(r + \frac{1}{100}\right)^2}{(1 + r^2)\left[\left(r + \frac{1}{100}\right)^2 + e^{r-3}\right]} = 1, \tag{13}$$

or, equivalently,

$$\ln(1+r^2)+\ln\left(\left(r+\frac{1}{100}\right)^2+e^{r-3}\right)-2\ln\left(r+\frac{1}{100}\right)=\ln a. \tag{14}$$

Let $g(r)=\ln(1+r^2)+\ln((r+(1/100))^2+e^{r-3})-2\ln(r+(1/100))$. We have

$$g'(r)=\frac{2r}{1+r^2}-\frac{2}{r+\frac{1}{100}}+\frac{2\left(r+\frac{1}{100}\right)+e^{r-3}}{\left(r+\frac{1}{100}\right)^2+e^{r-3}}. \tag{15}$$

Solving $g'(r)=0$, we have a unique positive critical point $r_c\approx 0.478625$ of $g(r)$.

Observe that $g(r)>0$ for all $r\geq 0$. Moreover, for $a\approx 1.64271$, we have the following results:

(1) when $g(r_c)=\ln a$ equation (14) has a unique positive solution,
(2) if $g(r_c)<\ln a$, there exists two positive solution of (14),
(3) if $g(r_c)>\ln a$, equation (14) has no positive solutions.

Figure 3 demonstrates these observations. □

Remark 1. If r_1 and r_2 are two positive fixed points of (12) such that $r_1<r_2$, then $r_1<r_c<r_2$ and $g'(r_1)<g'(r_c)<g'(r_2)$, with $g'(r_c)=0$.

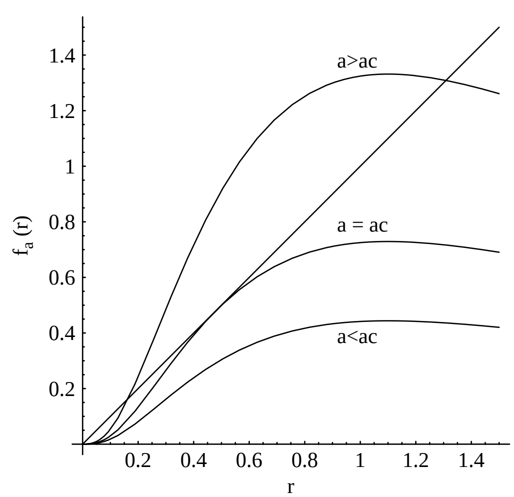

Figure 3. Zero, one and two positive fixed points for the profit rate when a is respectively, less than a_c, equal a_c and greater than a_c, where $a_c\approx 1.64271$.

To investigate the stability of the positive fixed points for the model (12), if they exist, we observe that

$$f'_a(r) = \frac{a\left(r+\frac{1}{100}\right)}{(1+r^2)\left(e^{r-3}+\left(\frac{1}{100}+r\right)^2\right)}$$

$$\left[-r\left(r+\frac{1}{100}\right)\left[\frac{e^{r-3}+2\left(\frac{1}{100}+r\right)}{\left(e^{r-3}+\left(\frac{1}{100}+r\right)^2\right)}+\frac{2r}{1+r^2}\right]+3r+\frac{1}{100}\right].$$

From (13) we know that, if r is a positive fixed point, then it satisfies

$$\frac{a\left(r+\frac{1}{100}\right)^2}{1+r^2} = \left(r+\frac{1}{100}\right)^2 + e^{r-3}. \tag{16}$$

So it follows that

$$f'_a(r) = 1 - r\left[\frac{2r}{1+r^2} - \frac{2}{r+\frac{1}{100}} + \frac{1+r^2}{a}\left(\frac{2}{r+\frac{1}{100}} - 1\right) + 1\right] = 1 - rg'(r). \tag{17}$$

Therefore r is locally asymptotic stable if

$$0 < rg'(r) < 2 \tag{18}$$

and is unstable if

$$rg'(r) < 0 \;\text{ or }\; rg'(r) > 2. \tag{19}$$

To study the stability of each positive fixed point, if they exist, we consider the following two cases:

(1) Suppose that the model (12) has only one positive fixed point $r = r_c$. Then from (17) we have $f'_a(r_c) = 1$. Since $f''_a(r_c) \approx -2.36074 \neq 0$ it follows from [3] or [4] that $r = r_c$ is unstable. More precisely, $r = r_c$ is semi-stable from the right side, if the initial value $r_0 \in\,]r_c, r_3[$, where $f_a(r_3) = r_1$ and $r_3 > c$ where c is the critical point of f_a determined in Section 1 and $r = r_c$ is unstable if the initial value $r_0 \in\,]0, r_c[\bigcup]r_3, \infty[$ (see Figure 4).

(2) Let us now assume that there are two positive fixed points for the profit rate r_1 and r_2 such that $r_1 < r_2$. It follows from Remark 1 that $r_1 g'(r_1) < 0$ and therefore r_1 is always unstable (note that $r_1 \in\,]0, r_c[$). For r_2 we know that $g'(r_2) > 0$, and therefore $r_2 g'(r_2) > 0$. We have two situations:
   - $0 < r_2 g'(r_2) < 2$ if and only if $r_2 \in\,]r_c, r_s[$, where $r_s \approx 3.3976$ and consequently r_2 is locally asymptotic stable,
   - $r_2 g'(r_2) > 2$ if and only if $r_2 \in\,]r_s, +\infty[$ and therefore r_2 is unstable.

In Figure 5 we illustrate these ideas.

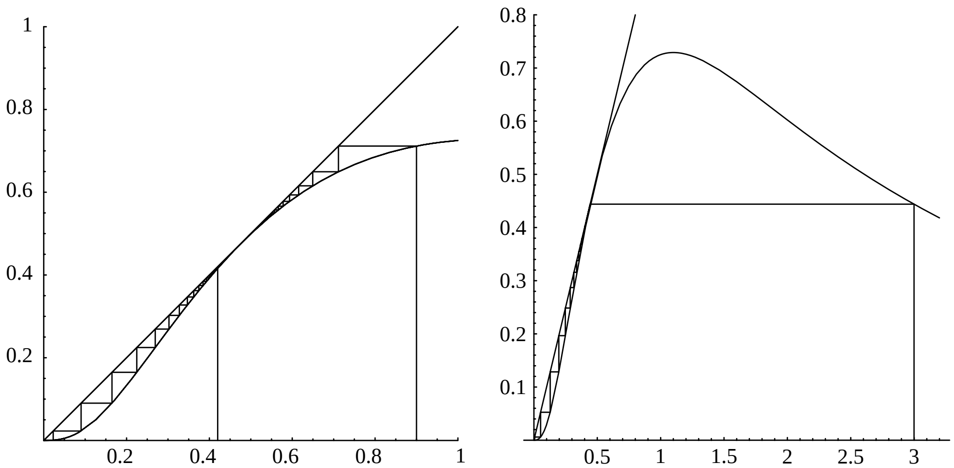

Figure 4. Stability of the unique positive fixed point. In the first case $r = r_c$ is semi-stable on the right side and unstable on the left side ($r_c \approx 0.478625$). In the second case $r = r_c$ is always unstable. In the right plot of this figure the beginning of the line in the stair step diagram is wrong (the line is disconnected). Is suppose it start in 3. This means that you need to put here more to the left side and a little to up.

The following theorem summarizes the above discussion.

THEOREM 2.2. *Consider* r_c *and* a_c *in the same conditions of the proof of Theorem* 2.1. *For* $a = a_c$ *there exists a unique positive unstable fixed point* $r = r_c$. *For* $a > a_c$ *there exists two positive fixed points* r_1 *and* r_2 *such that* $r_1 < r_2$, *where* r_1 *is always unstable and* r_2 *is locally asymptotic stable if* $r_2 \in]r_c, r_s[$, *and is unstable if* $r_2 \in]r_s, +\infty[$, *where* $r_s \approx 3.3976$.

Finally, we give a general definition of the Allee effect (see Ref. [12]).

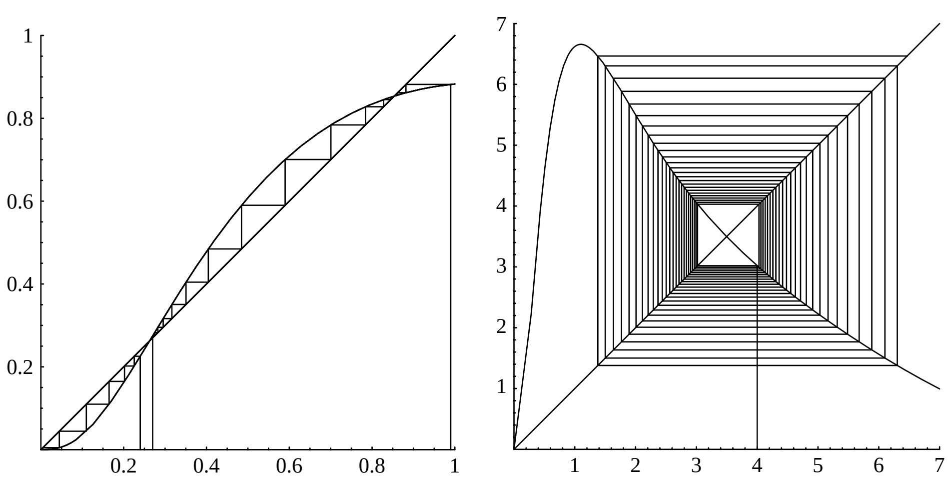

Figure 5. Stability of the two positive fixed points. In the first case the 1st positive fixed point is unstable and the 2nd positive fixed point is locally asymptotic stable. In the second case the 2nd positive fixed point is unstable (the 1st positive fixed point is not visible because it is small, but it exists). The extra line above the figure in the right plot do not exist.

Definition 2.3. *Every model is under an Allee effect whenever $f_a(r)$ supports three fixed points,* 0 *and two positive fixed points, r_1 and r_2, such that $r_1 < r_2$ and* 0 *is globally attracting in $[0,r_1[$, r_1 is repellers and a positive attractor exists in the interval $]r_1, \max I_1[$ where $I_1 = f_a([0,r_2])$.*

We call the smaller positive fixed point, the Allee point, and the greater positive fixed point, the carrying capacity (see Ref. [11]).

3. Chaos in the profit rate

In this section we use the notions of Lyapunov exponents and topological entropy to establish the existence of chaos in the profit rate for certain values of the parameter a.

When we increase the parameter a, model (12) exhibits a typical period-doubling bifurcation.

For values $0 < a < \approx 12.5$, the model (12) has a stable positive equilibrium point. The profit rate tends to approach the bigger positive equilibrium point. When $\approx 12.5 < a < \approx 17.09957$, we have a two-cycle. As the parameter a increases we have more cycles of the profit rate with period-doubling scenario, 4, 8, 14,... This situation of period-doubling becomes more complex until the system becomes completely unpredictable. As in every unimodal map, the chaotic region has windows of odd periods. In particular, for $a = 21.94$, we have a period-five orbit. When the parameter a grows even more, we have again more aperiodic orbits until we fall into a period-three zone (see Ref. [6]). This happens for values $\approx 23.082565 < a < \approx 23.217854$. This means that we have a triennial repetition of the profit rate.

A Lyapunov exponent is a mathematical indicator of the exponential degree of the velocity according to which two arbitrary nearby orbits grow further apart as the number of iterations increases. We can define it as follows:

Definition 3.1. *The Lyapunov exponent $\lambda(r_0)$ for a point r_0 is given by*

$$\lambda(r_0) = \limsup_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|f'_a(r_k)| \tag{20}$$

where $r_k = f_a^k(r_0)$.

In practice, to calculate experimentally the value of the Lyapunov exponent we can use the formula

$$\lambda(r_0) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|f'_a(r_k)|, \tag{21}$$

that is equivalent to

$$\lambda(r_0) = \ln \lim_{n\to+\infty} \sqrt[n]{|(f_a^n(r_0))'|}. \tag{22}$$

If the absolute value of $f'_a(r_k)$ is greater than one, then the Lyapunov exponent is positive, which implies that the system possesses sensitive dependence on initial conditions (see Ref. [4], page 131).

In Figure 6 we can see the progression of the Lyapunov exponent of the function f_a when varying a. We observe that if a is less than ≈ 18.245 the system has no sensitive

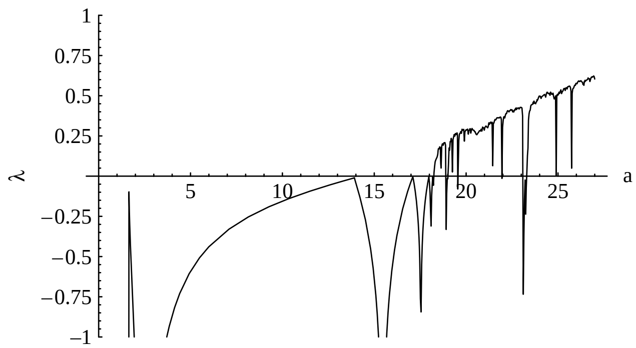

Figure 6. The Lyapunov exponent λ of the function f_a under the parameter a applied to the profit rate.

dependence on initial profit rates. However, when a exceeds ≈ 18.245, the system starts to posses sensitive dependence on initial profit rates and we enter in the chaotic region. The chaotic region has windows of locally stable periodic orbits of odd periods as predicted by Sharkovsky's theorem [12].

Another mathematical invariant which detect the existence of chaos is topological entropy. To introduce this concept we will use kneading theory [10]. For this, it is necessary to define the growth number s for unimodal maps (a function that has only one critical point).

Definition 3.2. *The lap number, $l(f_a^n)$, is the number of maximal intervals of monotonicity of f_a^n (f_a^n is piecewise-monotone). The growth number s is defined as*

$$s(f_a) = \lim_{n \to +\infty} \left(l\left(f_a^n\right) \right)^{1/n}. \tag{23}$$

When the growth of the laps is small (polynomial with the number of iterates) we do not have chaos, but when the growth of the lap number is exponential we have chaos. This happens when the growth number is greater than 1. When the growth number is equal to 1, we can have chaos, but this invariant does not give a precise number.

To calculate the topological entropy, we define the parity function for the turning point $c \approx 1.10042$ as follows:

Definition 3.3. *The parity function for the turning point c is given by*

$$\varepsilon(x) = \begin{cases} 1 & \text{if } x < c \\ 0 & \text{if } x = c \\ -1 & \text{if } x > c \end{cases} \tag{24}$$

It follows a few results of these two definitions:

Theorem 3.4.

(1) *The kneading determinant is a formal series in t given by*

$$Z(t, a) = 1 + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \varepsilon(f_a^n(c)) t^n \right). \tag{25}$$

(2) *In case of periodic orbits of c, Z(t,a) is a polynomial of degree* $(n - 1)$. *The inverse of the smallest root of Z(t,a) in* [0,1] *is the growth number of* f_a.

(3) *The topological entropy* h_t *is given by the relation* $h_t = log_2(s)$.

Proof. See Ref. [10]. □

Our function has two intervals of monotonicity, that is, f_a increases in $]0, c[$ and it decreases in $]c, +\infty[$. For $a = 25$ we have that, the first terms of the kneading determinant are

$$1 - t - t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + t^{10} - t^{11} + \dots$$

The smallest real root of this determinant belongs to [0,1] it is approximately equal to 0.562781. The topological entropy therefore is given by

$$h_t = \log_2(0.562781^{-1}) \approx 0.83064.$$

In Figure 7 we can see the evolution of the topological entropy values for the profit rate.

The fact that the topological entropy increases and is greater than zero (in addition to the fact that the Lyapunov exponent is positive) means that the dynamical system becomes more complex as the parameter a increases. For values of $a > \approx 18.245$, the model exhibits chaos, which may be seen in the bifurcation diagram. One can see clearly in this diagram an a periodic band. From the economic point of view this situation would also result in a huge complexity and instability of the system. This happens when the exploitation rate is very high and the profit rate is very low.

In this model, trying to compensate for low profits at the expense of high exploitation rate leads inevitably to instability and chaos, both from the mathematical point of view and from the common sense point of view (economic chaos). For values of $a > \approx 18.245$ our model can be unrealistic in the short term, because in those circumstances, after a high profit, we will have a low profit. This variation of the profit rate can be explained by some extraordinary factors that

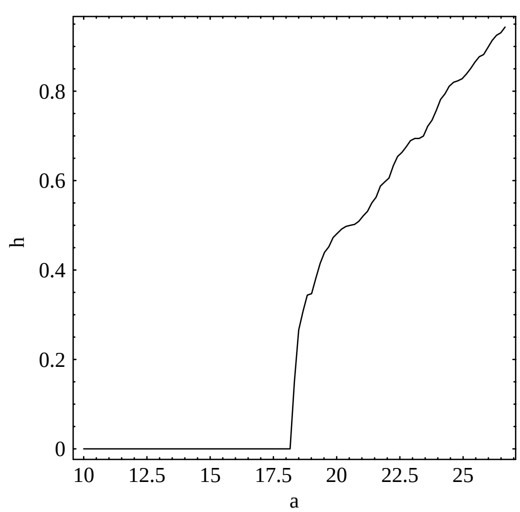

Figure 7. Progress of the topological entropy h_t for the model (12). As the parameter a increases the topological entropy h_t increases for values near 1.

happen in an economy as a whole, like wars, natural catastrophes, pressures of various agents, etc. If it were possible to introduce an exploitation rate so high that we would be led to this situation, that would mean that the system would react by presenting a reasonable profit rate from five to five or three to three units of time (values of a close to ≈ 21.94, in the first case, and ≈ 23.1 in the second case).

Hence, in order to maintain the stability of the system, the exploitation rate must be kept below a certain threshold value for which $a = \approx 12.5$.

4. Non-autonomous periodic systems with Allee effect

So far, we have limited our analysis to the autonomous case in which the parameter a is assumed to be independent of time. This means we have assumed that the exploitation rate E, when the profit rate is close to zero, is fixed for all seasons.

It is, however, more realistic to assume that a various with varying seasons. In this scenario, a depends on seasons. This will lead to assuming that $a = a_n$, $n = 1, 2, \ldots, p$, and a_n is periodic of period p. Under this assumption we have the non-autonomous periodic equation

$$x_{n+1} = f_n(x_n), n \in \mathbb{Z}^+ := \{1, 2, \ldots\} \tag{26}$$

where $f_{n+p} = f_n$, $\forall n \in \mathbb{Z}^+$.

In this section we focus our attention on the case when $p = 2$, that is, we have only two maps $\{f_1, f_2\}$. In other words we have a 2-periodic difference equation.

4.1 Existence of Allee points and carrying capacities

Let us rewrite model (12) as $f_i(x) = a_i g(x)$, $i = 1, 2$. We verify that $g(x) < x, \forall x \in \mathbb{R}^+$. More precisely, $g(x) \in [0, M]$, $\forall x \in \mathbb{R}^+$, where $M \approx 0.443856$ and $g(x)$ has its critical point when $x \approx 1.10042$, that is, the critical point of the map g is the same of the map f. Consequently, g is increasing (or decreasing) in the same interval than f. In other words, g increase on $]0, c_f[$ and decrease on $]c_f, +\infty[$.

Model (12) is under Allee effect when $a \in\ ]a_c, 12.5[$ where $a_c \approx 1.64271$. Let a_1, $a_2 \in\ ]a_c, 12.5[$ such that $a_1 < a_2$. Then $f_1(x) < f_2(x)$, $\forall x \in \mathbb{R}^+$. Let A_1 and A_2 be the Allee points of f_1 and f_2, respectively, and K_1 and K_2 be the carrying capacities of these two maps, respectively, i.e.,

$$f_1(A_1) = a_1 g(A_1) = A_1, f_1(K_1) = a_1 g(K_1) = K_1$$

$$f_2(A_2) = a_2 g(A_2) = A_2, f_2(K_2) = a_2 g(K_2) = K_2.$$

The order relation of these four fixed point is $A_2 < A_1 < K_1 < K_2$.

We are also interested in studying the properties of the Allee points of the composition of the maps f_1 and f_2. We define the Allee region as the square with side length A_1, where A_1 is the bigger of the Allee points.

Let P_1 and P_2 be the first pre-image of A_2 and A_1, by the function f_1 and f_2, respectively. In other words $\exists P_1, P_2 \in\ ]A_2, A_1[: f_1(P_1) = A_2$ and $f_2(P_2) = A_1$. Consequently, $f_2 \circ f_1(P_1) = A_2$ and $f_1 \circ f_2(P_2) = A_1$ (see Figure 8).

Remark 1. We note that both f_1 and f_2 are homeomorphism in their respective Allee regions.

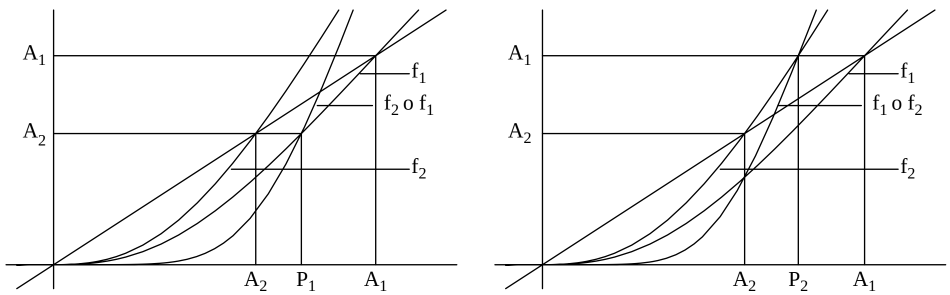

Figure 8. In the left plot we represent the first pre-image of A_2 by the function f_1 while in the right plot we represent the first pre-image of A_1 by the map f_2.

THEOREM 4.1. *$f_2 \circ f_1$ and $f_1 \circ f_2$, both, have Allee points, that we denote by $A_{f_2 \circ f_1}$ and $A_{f_1 \circ f_2}$, respectively. Moreover $P_1 < A_{f_2 \circ f_1} < A_1$ and $A_2 < A_{f_1 \circ f_2} < P_2$.*

Proof. First we will prove the existence of $A_{f_2 \circ f_1}$ and $A_{f_1 \circ f_2}$. For $x \in\,]0, A_2]$ we have $f_1(x) < x$, this implies that $f_2(f_1(x)) < x$. On other hand when $x \in\,]A_1, C[$ where $C = \min\{C_{f_1}, C_{f_2}\}$, C_{f_1} and C_{f_2} are, respectively, the critical points of f_1 and f_2, we have that $f_1(x) > x$ and consequently $f_2(f_1(x)) > x$. Therefore there exists $y \in\,]A_2, A_1[$ such that $f_2 \circ f_1(y) = y$, i.e., $A_2 < A_{f_2 \circ f_1} < A_1$. Similarly, we will show that $A_2 < A_{f_1 \circ f_2} < A_1$.

To prove that $A_{f_2 \circ f_1} \in\,]P_1, A_1[$ first we will prove that $A_{f_2 \circ f_1} \notin [A_2, P_1]$. Since that $f_1(A_2) < A_2$ we get $f_2 \circ f_1(A_2) < A_2$, when $x \in\,]A_2, P_1[$ we have $f_2 \circ f_1(x) < x$, and $f_2 \circ f_1(P_1) = A_2 < P_1$. Therefore $A_{f_2 \circ f_1} \notin [A_2, P_1]$.

Now let $x \in\,]P_1, A_1[$. By one side we have $f_2 \circ f_1(A_1) = f_2(A_1) > A_1 > x$ and on other side $f_2 \circ f_1(P_1) = A_2 < x$, consequently

$$\exists y \in\,]P_1, A_1[\; : f_2 \circ f_1(y) = y = A_{f_2 \circ f_1}.$$

Following the same reasoning we prove that $A_2 < A_{f_1 \circ f_2} < P_2$. □

Remark 2. Let $J = [0, C]$ and $C = \min\{C_{f_1}, C_{f_2}\}$, C_{f_1} and C_{f_2} are, respectively, the critical points of f_1 and f_2. Then on the interval J, $f'_1 < f'_2$.

THEOREM 4.2. *If $g'(A_2) \geq (1/a_1) + (1/a_2)$ then $P_1 \geq P_2$. Moreover, $A_{f_2 \circ f_1} \geq A_{f_1 \circ f_2}$.*

Proof. Suppose that $A_1 - A_2 = \varepsilon > 0$ and $f_2(x) - f_1(x) = \delta(x)$, $\forall x \in J$ ($\delta(x)$ is increasing). We need to prove that the first pre-image of A_2 and A_1, both, satisfy the relation $P_1 \geq P_2$ or $f_2^{-1}(A_1) \leq f_1^{-1}(A_2)$, that is equivalent

$$A_1 \leq f_2 \circ f_1^{-1}(A_2). \tag{27}$$

Developing in Taylor's series we get

$$\begin{aligned} f_1^{-1}(A_2) &= f_1^{-1}(A_1 - \varepsilon) \\ &= f_1^{-1}(A_1) - (f_1^{-1}(A_1))'\varepsilon + O(\varepsilon^2) \\ &= A_1 - \frac{\varepsilon}{f'_1(A_1)} + O(\varepsilon^2). \end{aligned}$$

Substituting the previous relation in (27) we get

$$A_1 \leq f_2\left[A_1 - \frac{\varepsilon}{f_1'(A_1)} + O(\varepsilon^2)\right],$$

and developing again series

$$A_1 \leq f_2(A_1) - \frac{f_2'(A_1)}{f_1'(A_1)}\varepsilon + O(\varepsilon^2),$$

that is

$$\begin{aligned} A_1 &\leq a_2 g(A_1) - \frac{a_2 g'(A_1)}{a_1 g'(A_1)}\varepsilon + O(\varepsilon^2) \\ \Leftrightarrow \frac{a_2}{a_1}\varepsilon &\leq a_2 g(A_1) - a_1 g(A_1) + O(\varepsilon^2) = \delta + O(\varepsilon^2), \end{aligned}$$

and consequently

$$\frac{a_2}{a_1}\varepsilon \leq \delta + O(\varepsilon^2). \tag{28}$$

So relation (28) is equivalent to relation (27).

We know that $g'(A_2) < (g(A_1) - g(A_2))/(A_1 - A_2) = g'(M) < g'(A_1)$, where $M \in]A_2, A_1[$. So

$$\begin{aligned} a_2 g'(M) &= \frac{a_2 g(A_1) - a_1 g(A_1) + a_1 g(A_1) - a_2 g(A_2)}{A_1 - A_2} \\ &= \frac{f_2(A_1) - f_1(A_2) + A_1 - A_2}{A_1 - A_2} = \frac{\delta + \varepsilon}{\varepsilon}. \end{aligned}$$

Therefore $g'(A_2) < g'(M) < (\delta + \varepsilon)/(a_2\varepsilon)$.

By hypothesis we have

$$g'(A_2) \geq \frac{1}{a_1} + \frac{1}{a_2} \Leftrightarrow \frac{1}{a_1} + \frac{1}{a_2} \leq g'(A_2) + O(\varepsilon),$$

and consequently

$$\frac{1}{a_1} + \frac{1}{a_2} \leq \frac{\delta + \varepsilon}{a_2 \varepsilon} + O(\varepsilon).$$

Multiplying both sides of the last relation by $a_2\varepsilon$ we get

$$\frac{a_2}{a_1}\varepsilon + \varepsilon \leq \delta + \varepsilon + a_2 \varepsilon O(\varepsilon)$$

and therefore

$$\frac{a_2}{a_1}\varepsilon \leq \delta + O(\varepsilon^2),$$

that is equivalent to relation (28) and therefore the relation $P_1 \geq P_2$ is proved.

Knowing that $P_1 \geq P_2$ and from Theorem 4.1 it follows that $A_{f_2 \circ f_1} \geq A_{f_1 \circ f_2}$. □

Now we will focus our attention on the carrying capacity of the composition of the two maps f_1 and f_2. We know that $\lim_{x\to+\infty} g(x) = 0$, therefore $\lim_{x\to+\infty} f_2 \circ f_1(x) = \lim_{x\to+\infty} a_2 g(a_1 g(x)) = 0$. Consequently, $\exists S > K_2$: $f_2 \circ f_1(S) = A_1 < K_1$.

THEOREM 4.3. *The composition map $f_2 \circ f_1$ has a carrying capacity greater than its Allee point, that we denote by $K_{f_2\circ f_1}$, such that $K_{f_2\circ f_1} \in\ ]K_1, S[$. Moreover, if $g((a_1/a_2)K_2) < K_2/a_2$ then $K_{f_2\circ f_1} \in\ ]K_1, K_2[$ and if $g((a_1/a_2)K_2) > K_2/a_2$ then $K_{f_2\circ f_1} \in\ ]K_2, S[$.*

Proof. We have $f_2 \circ f_1(K_1) = a_2 g(a_1 g(K_1)) = a_2 g(K_1) = (a_2/a_1)a_1 g(K_1) = (a_2/a_1)K_1 > K_1$ $(a_1 < a_2)$. On the other side, $\exists\, S > K_2$: $f_2 \circ f_1(S) = A_1 < K_1$. Consequently, $\exists q \in]K_1, S[$: $f_2 \circ f_1(q) = q = K_{f_2\circ f_1}$.

To prove the second part of the theorem we note that from $g((a_1/a_2)K_2) < K_2/a_2$ follows $f_2 \circ f_1(K_2) < K_2$ and therefore $K_{f_2\circ f_1} \in]K_1, K_2[$. When $g((a_1/a_2)K_2) > K_2/a_2$ it follows that $f_2 \circ f_1(K_2) > K_2$, so $K_{f_2\circ f_1} \in]K_2, S[$. □

Remark 3. The carrying capacity of $f_2 \circ f_1$ can be greater than the carrying capacity of f_2 or not, but it will never be less than the carrying capacity of the map f_1.

THEOREM 4.4. *The composition map $f_1 \circ f_2$ has a carrying capacity greater than its Allee point, that we denote by $K_{f_1\circ f_2}$, such that $K_{f_1\circ f_2} \in\ ]A_1, K_2[$. Moreover, if $g((a_2/a_1)K_1) < K_1/a_1$ then $K_{f_1\circ f_2} \in\ ]K_1, K_2[$ and if $g((a_2/a_1)K_1) > K_1/a_1$ then $K_{f_1\circ f_2} \in\ ]A_1, K_1[$.*

Proof. We have $f_1 \circ f_2(K_2) = a_1 g(K_2) = (a_1/a_2)a_2 g(K_2) = (a_1/a_2)K_2 < K_2$ and on the other side we know that $f_2(A_1) > A_1$ that implies $f_1 \circ f_2(A_1) > f_1(A_1) = A_1$. Consequently $\exists q \in\ ]A_1, K_2[$: $f_1 \circ f_2(q) = q = K_{f_1\circ f_2}$.

The second part of the theorem is easily obtained because from $g((a_2/a_1)K_1) < K_1/a_1$ follows $f_1 \circ f_2(K_1) > K_1$ and therefore $K_{f_1\circ f_2} \in\ ]K_1, K_2[$. We get the other case knowing that $f_1 \circ f_2(K_1) < K_1$ and that permits to conclude that $K_{f_1\circ f_2} \in\ ]A_1, K_1[$. □

Remark 4. The carrying capacity of $f_1 \circ f_2$ can be greater than the carrying capacity of f_1 or not, but it will never be greater than the carrying capacity of the map f_1.

4.2 Stability

Let us consider an economic system modelled by equation (12). We assume that the unit of time between iterations is six months. The initial condition x_1 will be taken in the 1st of January, x_2 will be taken in the 1st of July and so on. Model (12) may be written as

$$x_{n+1} = ag(x_n).$$

Now, suppose that due to political decisions, i.e., periodic variation of taxes, or the real nature of the economic model, the behaviour of the economy is different in the first half of the year from the second half of the year. In this case we will have a periodic model with period 2 where the model changes every six months. This leads to the non-autonomous 2-periodic system

$$x_{n+1} = f_n(x_n), \quad \text{with } f_1 = a_1 g,\ f_2 = a_2 g, \tag{29}$$

where $f_{2n+1} = f_1$ and $f_{2n} = f_2, n \in \mathbb{Z}^+$ and a_1 will be used for the first half of the year and a_2 for the second half. Since $a_1 = a_2$ reduce equation (29) to the autonomous case, we assume that $a_1 \neq a_2$. Observe that $a_1 < a_2$ and $a_1 > a_2$ exhibit the same dynamics.

The 2-periodic difference equation for model (12) is given by

$$x_1 = f_2(f_1(x_1))$$

that is equivalent

$$x_1 = a_2 g(a_1 g(x_1)). \tag{30}$$

Rewriting equation (30) as

$$G(a_1, a_2, x_1) = a_2 g(a_1 g(x_1)) - x_1 = 0. \tag{31}$$

In the sequel we need the following stability definition.

DEFINITION 4.5. *The periodic* 2*-cycle* $\{f_1(x_1), f_2(f_1(x_1))\}$ *is said:*

(1) *Asymptotically stable if* $|(f_2 \circ f_1)'(x_1)| < 1$,
(2) *Unstable if* $|(f_2 \circ f_1)'(x_1)| > 1$,
(3) *Neutral if* $|(f_2 \circ f_1)'(x_1)| = 1$.

It is worth mentioning that the neutral periodic cycle leads to bifurcation.

We made some computations in Mathematica software to find numerically the solutions of equation (31). The existence of this solution is guaranteed by the Implicit Function Theorem. In Figure 9 we represent in the (a_1, a_2)-plane the region of stability of the fixed point of the periodic 2-cycle of the non-autonomous system 29. The black region represents the zone of the attractively of the zero fixed point.

The asymptotically stable 2-cycle is born at the first curve. This curve represents the curve of the periodic 2-cycle.

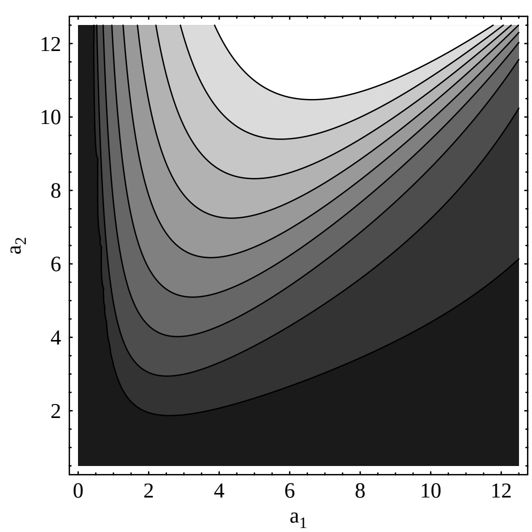

Figure 9. The region of stability of the periodic 2-cycle for the non-autonomous 2-periodic system in the (a_1,a_2)-plane.

4.3 Boom and bust

One of the most interesting problems is to determine whether periodic fluctuations in a dynamical system produces boom or bust in the profit rate. In other words, we would like to know the impact of introducing artificial or natural oscillations in the model and whether this periodical forcing has a deleterious or a booming effect on the system.

This is similar to a question posed earlier by Cushing and Henson [2] regarding certain biological models.

We are now going to give precise definitions for boom and bust in the profit rate. Let $\mathcal{C}_p = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{p-1}\}$ be a periodic p-cycle of equation (26) and let $\mathcal{K}_p = \{K_0, K_1, \ldots, K_{p-1}\}$ be the carrying capacities (rightmost fixed points) of the individual maps f_n, $n = 0, 1, 2, \ldots, p-1$. Let

$$av(\mathcal{C}_p) = \frac{1}{p}\sum_{i=0}^{p-1}\bar{x}_i \text{ and } av(\mathcal{K}_p) = \frac{1}{p}\sum_{i=0}^{p-1}K_i,$$

where $av(\cdot)$ denotes the average of the given set.

Definition 4.6. *Equation* (26) *is said to be*

(1) *bust if* $av(\mathcal{C}_p) < av(\mathcal{K}_p)$
(2) *boom if* $av(\mathcal{C}_p) > av(\mathcal{K}_p)$
(3) *indolent if* $av(\mathcal{C}_p) = av(\mathcal{K}_p)$.

Notice that if the system is bust, then periodic forcing of a system has a deleterious effect on the profit rate, while if the system is boom, periodic forcing leads to a booming effect on the profit rate.

In our model we must compare the average of the periodic orbit and the average of the carrying capacities, that is,

$$av(\mathcal{C}_2) = \frac{\bar{x}_0 + \bar{x}_1}{2} \text{ and } av(\mathcal{K}_2) = \frac{K_0 + K_1}{2},$$

Since the explicit computation of these averages are prohibitively difficult and tedious, we will estimate them numerically.

Using Mathematica software with working precision 10^{-15}, we computed the fixed points of $f_0 = ag$, and $f_1 = (a+\varepsilon)g$ and the fixed points of the two compositions $f_0 \circ f_1$ and $f_1 \circ f_0$. Surprisingly, both boom and bust occur for values of a, $a \in [2.0,3.3]$ and with $b = a + \varepsilon$. In Figure 10 with $\varepsilon = 0.3$ we see the graph of the booming function $B(a)$, as a function of the parameter a

$$B(a) = av(\mathcal{C}_2)(a) - av(\mathcal{K}_2)(a) = \frac{\bar{x}_0(a) + \bar{x}_1(a)}{2} - \frac{K_0(a) + K_1(a)}{2},$$

The function $B(a)$ is positive when we have boom and negative when we have bust. In our system both cases occur. The zeros of the booming function are the indolent points.

In Figure 11 we see the same situation when we vary the parameter ε that measures the difference between a_1 and a_2. The different curves correspond to different values of ε that vary from 0.1 to 0.5.

The presence of both bust and boom in our model has been rarely observed in the literature. It is one of the main characteristics of our model.

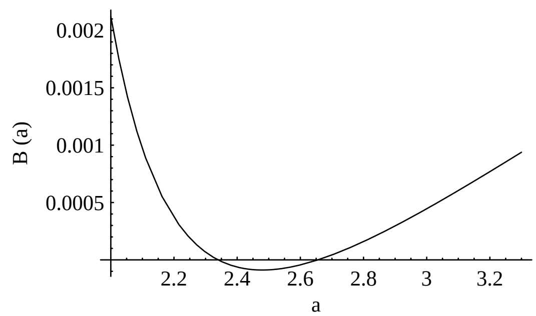

Figure 10. The booming function as a function of the parameter *a*.

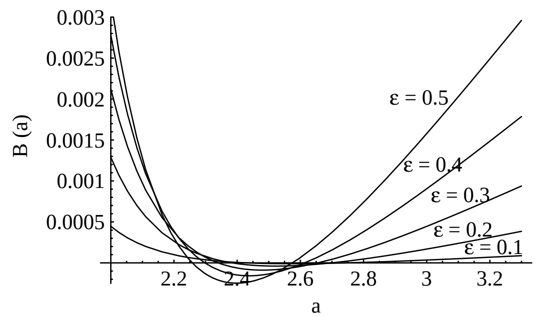

Figure 11. The booming function as a function of the parameter *a* for different values of the parameter ε.

5. Conclusion

The problem of the decrease of the profit rate with time has been the subject of numerous discussions in the last 150 years.

In this study, we use the equation that arises from economics theory, to develop a dynamical system model describing the evolution of the profit rate. This model is a one-dimensional unimodal discrete system which provides a relationship between the profit rate at two consecutive cycles.

When the system does not react desperately to low profits in the previous cycle, increasing the exploitation rate in these circumstances forces the system to approach an equilibrium profit value. This equilibrium value does not vary much, with the variation of the exploitation rate as a response to a null profit. Such conclusion contradicts Marx hypothesis of the decrease of the profit rate in time.

When the exploitation rate with at null profit exceeds a critical value (in our case $a > 12.5$), we have a bifurcation, and the system starts to exhibit a cyclic oscillation between two profit rate values. As the parameter a increases, we obtain a period-doubling bifurcation cascade. For certain values of a, the Lyapunov exponents, computed numerically (equation (21)), become positive, indicating the presence of chaos and unstable orbits. When the exploitation rate with at null profit rate, is too high, the system starts to exhibit, again, stable orbits of odd periods (periods three and five, etc. that implies chaos [7] in the profit rate).

In this economic model we saw that boom in the profit rate is more predominate than bust in the profit rate. From the economic point of view, this means that the periodic fluctuation of the exploitation rate when the profits are low has, in general, a booming effect on the profit rate. Hence, it is generally beneficial for an enterprise micro or macro to vary the exploitation rate.

Notes

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3. This work is part of the first author's Ph.D. dissertation.

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