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Relevance, recovery and recuperation: A prelude to ring withdrawal

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ABSTRACT

In this paper, we introduce *recuperative withdrawals*, belief change operators that satisfy recuperation, a postulate weaker than recovery, all the AGM postulates for contraction except recovery and another postulate which is a slightly stronger condition than conjunctive inclusion. Furthermore, we present a constructive definition for a class of operators —named *ring withdrawals*— which are such that the outcome of a ring withdrawal of a belief set **K** by a sentence α is obtained by adding to the set of most plausible models $\|\mathbf{K}\|$ all the worlds which are as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds. Ring withdrawals satisfy the Lindström and Rabinowicz's interpolation thesis. We show that the classes of recuperative withdrawals and of ring withdrawals are identical. Additionally we show that the class of ring withdrawals is not contained in and does not contain the class of AGM contractions or the class of severe withdrawals. Finally we present methods for defining an operator of ring withdrawal by means of a severe withdrawal operator and by means of an AGM contraction operator, and vice-versa.

1. Introduction

Belief Dynamics is a normative approach to rational alteration in one's body of beliefs. This area has gained traction, and maturity, through persistent research in the area over last half a century since the proposal by Alchourròn, Gärdenforce and Makinson took off in a number of seminal works in 1970s and 80s [1]. This approach may be broadly characterised as *epistemic conservatism* (more aptly *doxastic conservatism*): Any change to one's body of beliefs should be made only if circumstances force it, and that change should be as little as one can get away with. This is often called the *Principle of Minimal Change.*

In the AGM framework there exist three kinds of belief change operators, namely *expansion*, *contraction* and *revision*. Each one of these classes of operators is characterized by a set of postulates that determine the behaviour of the corresponding type of operators, establishing conditions or constraints that they must satisfy. Among the postulates proposed in [1] as the characteristic properties of the contraction operators is *recovery* (in Subsection 2.2 we recall all the AGM postulates for contraction).

In the opinion of the AGM trio, the postulate of recovery guarantees minimal loss of contents in the contraction process. However, several authors have criticized the recovery postulate [4,23,24,26,16,32,31]. Operators that satisfy the other five basic AGM contraction postulates but not recovery were called *withdrawals* [28]. Rott [33] proposed an alternative belief change operator, later

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called *severe withdrawal* [37]. In the same way, severe withdrawal was also criticized [20, p.102]. Lindstrom and Rabinowicz pointed out that a reasonable operation of contraction should be somewhere between AGM contraction and severe withdrawal [26, p.115].

In this paper we present the *ring withdrawals*, operators whose output is situated between the outputs of AGM contractions and of severe withdrawals. These operators were first mentioned by Rott and Pagnucco [37], citing a personal communication from Nayak, but until now there does not seem to exist a standard reference paper concerning these operators. In this paper we present a definition for ring withdrawals, based on Grove's systems of spheres [14] and provide an axiomatic characterization for these operators. Furthermore, we analyse how these operators address the principle of minimal change, and study the relation between these operators and AGM contractions and severe withdrawals.

The rest of the paper is organized as follows: In Section 2 we introduce the notations and recall the main background concepts that will be needed throughout this article. In Section 3 we discuss the postulate of recovery and the notion of minimal change in belief contraction. In Section 4 we introduce the *recuperation* postulate and the *recuperative withdrawal* operators. In Section 5 we present a formal definition for ring withdrawal operators and show that the class of these operators coincides with the class of recuperative withdrawals. In Section 6 we establish some relations between ring withdrawal, severe withdrawal and AGM contraction operators. In Section 7 we present methods for obtaining operators of ring withdrawals from severe withdrawals and AGM contractions and vice-versa. In Section 8 we summarize the main contributions of the paper and briefly discuss their relevance. In the Appendix we provide proofs for all the original results presented, along with some lemmas that are needed in these proofs.

2. Background

2.1. Formal preliminaries

We will assume a propositional language $\mathcal{L}$ that contains the usual truth functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence). We shall make use of a consequence operation Cn that takes sets of sentences to sets of sentences and which satisfies the standard Tarskian properties, namely: (i) $A \subseteq Cn(A)$ (*inclusion*); (ii) If $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$ (*monotony*) and (iii) $Cn(A) = Cn(Cn(A))$ (*iteration*). Furthermore we will assume that Cn satisfies the following three properties: (iv) If α can be derived from A by classical truth-functional logic, then $\alpha \in Cn(A)$ (*supraclassicality*); (v) $\beta \in Cn(A \cup \{\alpha\})$ if and only if $\alpha \rightarrow \beta \in Cn(A)$ (*deduction*) and (vi) If $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite subset A' of A (*compactness*). We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\alpha \vdash \beta$ for $\{\alpha\} \vdash \beta$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\alpha \nvdash \beta$ for $\{\alpha\} \nvdash \beta$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. Greek letters $\alpha, \beta, \ldots$ will be used to denote sentences of $\mathcal{L}$. Lowercase Latin letters such as $p, q, \ldots$ will be used to denote atomic sentences of $\mathcal{L}$. $A, B, \ldots$ shall denote sets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e. $\mathbf{K} = Cn(\mathbf{K})$) — such a set is called a *belief set* or *theory*.

A possible world is a maximal consistent subset of $\mathcal{L}$. The set of all possible worlds will be denoted by $\mathcal{M}_{\mathcal{L}}$. Sets of possible worlds are called propositions. The set of possible worlds that contain $R \subseteq \mathcal{L}$ is denoted by $\|R\|$, i.e., $\|R\| = \{M \in \mathcal{M}_{\mathcal{L}} : R \subseteq M\}$. If R is inconsistent, then $\|R\| = \emptyset$. The elements of R are designated by R-worlds. For any sentence α, $\|\alpha\|$ is an abbreviation of $\|Cn(\{\alpha\}\|$ and its elements are designated by α-worlds.
Unless mentioned otherwise, we will use $\dot{-}$ for AGM contractions, $\ddot{-}$ for severe withdrawals, $\div$ for ring withdrawals and $-$ for generic contraction/withdrawal operators.

2.2. AGM contraction

In the AGM [1,11] account the set of beliefs of a rational agent is represented by a belief set $\mathbf{K}$, which is a set of sentences (of a language $\mathcal{L}$) that is closed under logical consequence Cn.

Given a belief set, three basic epistemic attitudes regarding any given sentence α are assumed: acceptation (when $\alpha \in \mathbf{K}$), rejection (when $\neg\alpha \in \mathbf{K}$) and indetermination (when neither $\alpha \in \mathbf{K}$ nor $\neg\alpha \in \mathbf{K}$). The three basic operations of the AGM model, that correspond with a change of epistemic attitude towards the input sentence α, are: *expansion* ($+$), which incorporates sentences in the original set, without eliminating any sentence from it and is defined as $\mathbf{K} + \alpha = Cn(\mathbf{K} \cup \{\alpha\})$; *contraction* ($\dot{-}$) that eliminates sentences from the original set without incorporating any new ones; and *revision* ($*$), that incorporates a sentence in the original set, but it may eliminate some beliefs in order to preserve consistency of the revised set. Revision can be defined by means of expansion and contraction via the Levi identity ([22]): $\mathbf{K} * \alpha = (\mathbf{K} - \neg\alpha) + \alpha$. The AGM postulates for contraction are the following:

(-1)	$\mathbf{K} - \alpha = Cn(\mathbf{K} - \alpha)$ (*i.e.* $\mathbf{K} - \alpha$ is a belief set).	(Closure)
(-2)	$\mathbf{K} - \alpha \subseteq \mathbf{K}$.	(Inclusion)
(-3)	If $\alpha \notin \mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{K} - \alpha$.	(Vacuity)
(-4)	If $\nvdash \alpha$, then $\alpha \notin \mathbf{K} - \alpha$.	(Success)
(-5)	$\mathbf{K} \subseteq (\mathbf{K} - \alpha) + \alpha$.	(Recovery)
(-6)	If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K} - \alpha = \mathbf{K} - \beta$.	(Extensionality)
(-7)	$(\mathbf{K} - \alpha) \cap (\mathbf{K} - \beta) \subseteq \mathbf{K} - (\alpha \wedge \beta)$.	(Conjunctive overlap)
(-8)	$\mathbf{K} - (\alpha \wedge \beta) \subseteq \mathbf{K} - \alpha$ whenever $\alpha \notin \mathbf{K} - (\alpha \wedge \beta)$.	(Conjunctive inclusion)

Postulates (-1) *to* (-6) *are called* basic AGM postulates for contraction, *and postulates* (-7) and (-8) are designated by *supplementary AGM postulates for contraction.*

An operator $-$ for $\mathbf{K}$ is a *basic AGM contraction* if and only if it satisfies the basic AGM postulates for contraction. An operator $-$ for $\mathbf{K}$ is an *AGM contraction* if and only if it satisfies both the basic and the supplementary AGM postulates for contraction.

In the presence of the basic AGM postulates, it holds that the conjunction of the two supplementary postulates is equivalent to the following postulate:

$(-V)$ $\mathbf{K}-(\alpha\wedge\beta)=\mathbf{K}-\alpha$ or $\mathbf{K}-(\alpha\wedge\beta)=\mathbf{K}-\beta$ or $\mathbf{K}-(\alpha\wedge\beta)=\mathbf{K}-\alpha\cap\mathbf{K}-\beta$. (Conjunctive factoring)

The intuition underlying the conjunctive factoring postulate is a fundamental principle of AGM theory. When contracting a belief set by a conjunction and there is a preference regarding one of the conjuncts, the outcome of that contraction coincides with the outcome of the contraction by the non-preferred conjunct. In cases where there is indifference among the conjuncts, the outcome of contracting by the conjunction is the intersection of the outcomes of the contractions by each one of the conjuncts.

2.3. Withdrawal functions

Functions that satisfy the basic AGM contraction postulates with the exception of recovery were called withdrawals [28]. Formally:

Definition 2.1 *([28]).* Let $\mathbf{K}$ be a belief set. An operator $-$ for $\mathbf{K}$ is a *withdrawal* if and only if it satisfies closure, inclusion, vacuity, success and extensionality.

In [37] an overview of various methods for withdrawal of a belief α from a belief set $\mathbf{K}$ is presented. One of the methods mentioned there was proposed by Nayak and it is the method that underlays the definition of the operators of ring withdrawal that we formally introduce in the present paper.

We now recall the definition of revision equivalent withdrawals, which are withdrawals that give rise to the same revision operator by means of the Levi Identity.

Definition 2.2 *([28,37]).* Let $\mathbf{K}$ be a belief set. Two withdrawals $\div$ and $\div'$ on $\mathbf{K}$ are revision equivalent if and only if for any sentence α it holds that $(\mathbf{K}\div\neg\alpha)+\alpha=(\mathbf{K}\div'\neg\alpha)+\alpha$.

Makinson [28] showed that in any class of revision equivalent withdrawal operators there will be exactly one AGM contraction operator.

2.4. System of sphere-based contractions

Based in the Lewis' semantic of counterfactuals [25], Grove introduced a semantic for AGM operators based on possible worlds models. A proposition (set of possible worlds) can represent either a belief set or an input sentence. The belief set $\mathbf{K}$ can be replaced, as a belief state representation, by $\|\mathbf{K}\|$, which is the set of worlds that contain $\mathbf{K}$. Similarly, each sentence α can be represented by the set $\|\alpha\|=\|Cn(\{\alpha\})\|$. The expansion outcome $\mathbf{K}+\alpha$ will then be represented by the set $\|\mathbf{K}+\alpha\|=\|\mathbf{K}\|\cap\|\alpha\|$, and a contraction outcome $\mathbf{K}-\alpha$ by some superset $\|\mathbf{K}\dot{-}\alpha\|$ of $\|\mathbf{K}\|$ that contains at least one $\neg\alpha$-world.

The semantic for AGM operators proposed by Grove is based on a system of concentric spheres around $\|\mathbf{K}\|$. Intuitively, each sphere represents a degree of closeness or similarity to $\|\mathbf{K}\|$.

Definition 2.3 *([14]).* Let $\mathbf{K}$ be a belief set. A system of spheres centred on $\|\mathbf{K}\|$ is a collection $\mathbb{S}$ of subsets of $\mathcal{M}_{\mathcal{L}}$, i.e., $\mathbb{S}\subseteq\mathcal{P}(\mathcal{M}_{\mathcal{L}})$, that satisfies the following conditions:

($\mathbb{S}$1) $\mathbb{S}$ is totally ordered with respect to set inclusion; that is, if $U,V\in\mathbb{S}$, then $U\subseteq V$ or $V\subseteq U$.
($\mathbb{S}$2) $\|\mathbf{K}\|\in\mathbb{S}$, and if $U\in\mathbb{S}$, then $\|\mathbf{K}\|\subseteq U$ ($\|\mathbf{K}\|$ is the $\subseteq$-minimum of $\mathbb{S}$).
($\mathbb{S}$3) $\mathcal{M}_{\mathcal{L}}\in\mathbb{S}$ ($\mathcal{M}_{\mathcal{L}}$ is the largest element of $\mathbb{S}$).
($\mathbb{S}$4) For every $\alpha\in\mathcal{L}$, if there is any element in $\mathbb{S}$ intersecting $\|\alpha\|$ then there is also a smallest element in $\mathbb{S}$ intersecting $\|\alpha\|$.

The elements of $\mathbb{S}$ are called spheres. For any consistent sentence $\alpha\in\mathcal{L}$, the smallest sphere in $\mathbb{S}$ intersecting $\|\alpha\|$ is denoted by $\mathbb{S}_\alpha$.

Given a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ that contains spheres T,U,V such that $T\subsetneq U\subsetneq V$, intuitively speaking, the worlds in $U\setminus T$ are: (i) better or more likely (in the sense of being more similar to the worlds of $\|\mathbf{K}\|$) than the worlds in $V\setminus U$; and (ii) worse or less likely than the worlds in T. Whenever $\nvdash\alpha$, the set of possible worlds of the outcome of a contraction of $\mathbf{K}$ by α must contain some $\neg\alpha$-worlds. Therefore, using a system of spheres, it is natural to define a contraction operator such that the result of contracting $\mathbf{K}$ by a sentence α is obtained by first adding to $\|\mathbf{K}\|$ the best $\neg\alpha$-worlds (in the sense of the above intuitive interpretation) and then intersecting all the elements of the resulting set of possible worlds.

Definition 2.4 *([14]).* Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$-based contraction on $\|\mathbf{K}\|$ is the operator $\dot{-}_{\mathbb{S}}$ defined, for any $\alpha\in\mathcal{L}$, by:

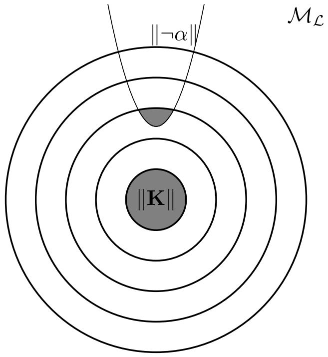

Fig. 1. Representation of a system of spheres centred on $\|\mathbf{K}\|$, where the set $\|\mathbf{K}\| \div_{\mathbb{S}} \|\alpha\|$ is shaded.

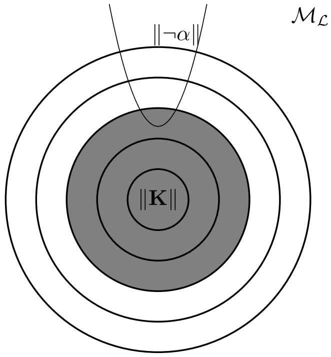

Fig. 2. Representation of a system of spheres centred on $\|\mathbf{K}\|$, where the sphere $S_{\neg\alpha}$ is shaded.

$$\|\mathbf{K}\| \div_{\mathbb{S}} \|\alpha\| = \begin{cases} \|\mathbf{K}\| \cup (S_{\neg\alpha} \cap \|\neg\alpha\|) & \text{if } \|\neg\alpha\| \neq \emptyset \\ \|\mathbf{K}\| & \text{otherwise} \end{cases}$$

An operator $\div$ on $\mathbf{K}$ is a system of spheres-based contraction if and only if there is an $\mathbb{S}$-based contraction $\div_{\mathbb{S}}$ on $\|\mathbf{K}\|$, such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \bigcap(\|\mathbf{K}\| \div_{\mathbb{S}} \|\alpha\|)$.

In Fig. 1 we present a graphical representation of the possible worlds of the outcome of a system of spheres-based contraction.

Proposition 2.5 *([14]). Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

1. $\div$ *is an AGM contraction.*
2. $\div$ *is a system of spheres-based contraction operator on* $\mathbf{K}$*.*

2.5. Severe withdrawals

In [33], Rott proposed an alternative contraction function, later called *severe withdrawal* ([37]). In [37] the authors proposed several methods of constructions for severe withdrawals. One of them was based on Grove's system of spheres.

Definition 2.6 *([37]).* Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$-based severe withdrawal on $\mathbf{K}$ is the operator $\ddot{\div}_{\mathbb{S}}$ defined, for any $\alpha \in \mathcal{L}$, by:

$$\|\mathbf{K}\| \ddot{\div}_{\mathbb{S}} \alpha = \begin{cases} S_{\neg\alpha} & \text{if } \nvdash \alpha \\ \|\mathbf{K}\| & \text{otherwise} \end{cases}$$

An operator $\ddot{\div}$ on $\mathbf{K}$ is a system of spheres-based severe withdrawal (or simply a severe withdrawal) if and only if there is an $\mathbb{S}$-based severe withdrawal $\ddot{\div}_{\mathbb{S}}$ on $\|\mathbf{K}\|$, such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} \ddot{\div} \alpha = \bigcap(\|\mathbf{K}\| \ddot{\div}_{\mathbb{S}} \alpha)$.

In Fig. 2 we present a graphical representation of the possible worlds of the outcome of a severe withdrawal.

Severe withdrawal operators were axiomatized in terms of postulates:

Proposition 2.7 *([37,8,13]). An operator $\ddot{-}$ for* $\mathbf{K}$ *is a severe withdrawal if and only if it satisfies closure, inclusion, vacuity, success, and*

$(-3')$ If $\vdash \alpha$, then $\mathbf{K} = \mathbf{K} - \alpha$ (Failure, [6][1])
(-9) If $\alpha \notin \mathbf{K} - \beta$, then $\mathbf{K} - \beta \subseteq \mathbf{K} - \alpha$ (Strong inclusion)

In [13] it was shown that extensionality follows from the other postulates mentioned in the above proposition. Therefore severe withdrawals are withdrawals in the sense of Definition 2.1. In [37] it was shown that in the above axiomatization strong inclusion can be replaced by conjunctive inclusion and

$(-7a)$ If $\nvdash \alpha$, then $\mathbf{K} - \alpha \subseteq \mathbf{K} - (\alpha \wedge \beta)$

Severe withdrawal operators also satisfy the following postulates:

(-10) If $\nvdash \alpha$ and $\alpha \in \mathbf{K} - \beta$, then $\mathbf{K} - \alpha \subseteq \mathbf{K} - \beta$. ([37])
$(-L)$ Either $\mathbf{K} - \alpha \subseteq \mathbf{K} - \beta$ or $\mathbf{K} - \beta \subseteq \mathbf{K} - \alpha$. (Linearity, [8,37])
$(-D)$ $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$ or $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \beta$. (Decomposition, [8,37])

3. Contraction and minimal change

One of the basic rationality criteria in belief change theories [11,24,36] is that belief changes should take place with minimal loss of previous beliefs. In this paper we will consider two sides to epistemic conservatism:

1. *Something cannot be created out of nothing.* This is the ancient metaphysical dictum *Ex nihilo nihil fit*, applied to epistemology. Information, *ipso facto,* belief and knowledge, is not free. So an epistemic agent must be frugal while acquiring new beliefs. In the belief dynamics literature it is captured by the postulate called *Inclusion*: $\mathbf{K} * \alpha \subseteq \mathbf{K} + \alpha$, i.e., *acceptance of evidence α must not lead to a belief whose veracity cannot be jointly guaranteed by the old corpus of beliefs* $\mathbf{K}$ *and new evidence α.*
2. *Something cannot vanish to nothing. Ad nihil fit nihil.* A rational agent must be frugal in "spending" information: one may not jettison beliefs willy-nilly without compelling reasons. This idea is best captured by the following well known postulate called *Recovery:* $\mathbf{K} \subseteq (\mathbf{K} \dot{-} \alpha) + \alpha$ for all beliefs α: *If a belief β must be discarded, any other belief that is lost along with it for reason of coherence must be such that it can be regained by "purchasing" α back.*

These two principles (1) and (2) above might jointly be termed the *Principle of Epistemic Conservation*: information is not created or destroyed, it simply transforms to different forms such as knowledge, belief and tenable hypotheses. The assumption that information is valuable, which is alluded to as justification for the principle of minimal change, is arguably rooted in this principle of epistemic conservation.

According to the AGM trio, the postulate of recovery guarantees minimal losses of contents in the contraction process [11, p. 65] [27, p. 352] [29, p. 478]. The Recovery postulate is intuitively appealing as it requires that the removal of a belief can be reversed, allowing for its reinstatement. This is based on the notion that "it is reasonable to require that we get all of the beliefs [...] back again after first contracting and then expanding with respect to the same belief" [10]. To illustrate this principle, consider the following example:

Example 3.1 *([20]).* I believed that I had my latchkey on me (α). Then I felt in my left pocket, where I usually keep it, and did not find it. I lost my belief in α (but without starting to believe in $\neg\alpha$ instead). Half a second later, I found the key, and regained my belief in α.

Despite this apparent intuitive appeal, the postulate of recovery has proved to be rather controversial [28,26,23,16,32,31,19,29,24,4,5]. Consider the following examples:

Example 3.2 *([16]).* I believed that Cleopatra had a son (σ). Therefore I also believed that Cleopatra had a child (χ or equivalently $\sigma \vee \delta$ where δ denotes that Cleopatra had a daughter). Then I received information that made me give up my belief in χ, and I contracted my belief set accordingly, forming $\mathbf{K} \dot{-} \chi$. Soon afterwards I learned from a reliable source that Cleopatra had a child. It seems perfectly reasonable for me to then add χ (i.e., $\sigma \vee \delta$) to my set of beliefs without also reintroducing σ.

Example 3.3 *([19]).* I previously entertained the two beliefs "George is a criminal" (α) and "George is a mass murderer" (β). When I received information that induced me to give up the first of these beliefs (α), the second (β) had to go as well (since α would otherwise follow from β). I then received new information that made me accept the belief "George is a shoplifter" (δ). The resulting new belief set is the expansion of $\mathbf{K} \dot{-} \alpha$ by δ, $(\mathbf{K} \dot{-} \alpha) + \delta$. Since α follows from δ, $(\mathbf{K} \dot{-} \alpha) + \alpha$ is a subset of $(\mathbf{K} \dot{-} \alpha) + \delta$. By recovery,

[1] Failure, with the basic five AGM contraction postulates except recovery characterizes the Levi Contraction [21].

$(\mathbf{K} \dot{-} \alpha) + \alpha$ includes β, from which follows that $(\mathbf{K} \dot{-} \alpha) + \delta$ includes β. Thus, since I previously believed George to be a mass murderer, I cannot any longer believe him to be a shoplifter without believing him to be a mass murderer.[2]

This example shows that retaining the sentence $\alpha \to \beta$ after contraction of $\mathbf{K}$ by α gives rise to unintuitive results. There seem to be cases when this sentence has to be removed. Due to recovery, AGM contraction cannot eliminate it.

One way of addressing this issue is to discard recovery as the postulate of minimal change for contraction and propose a different approach to conserve the notion of minimal change. In terms of the possible worlds semantics for AGM contraction proposed by Grove, recovery essentially states that the set of possible worlds of $\mathbf{K} - \alpha$ is the one which results of adding to $\|\mathbf{K}\|$ only $\neg\alpha$-worlds. Grove's constructive definition of AGM contraction, is such that when contracting by α, the worlds that are added to $\|\mathbf{K}\|$ are exactly the $\neg\alpha$-worlds that are closer to $\|\mathbf{K}\|$, where the distance between a possible world and $\|\mathbf{K}\|$ is determined by an associated system of spheres centred on $\|\mathbf{K}\|$. The idea underlying the definition of severe withdrawals (see Subsection 2.5), is that the result of contracting a belief set $\mathbf{K}$ by a sentence α should be the theory corresponding to the set formed by all the possible worlds which are at least as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds. Hansson [20] proved that severe withdrawal operators also satisfy

$(-E)$ If $\nvdash \alpha$ and $\nvdash \beta$, then either $\alpha \notin \mathbf{K} - \beta$ or $\beta \notin \mathbf{K} - \alpha$. (Expulsiveness)

Expulsiveness is an implausible postulate, as recognized by Rott and Pagnucco, since unrelated beliefs affect the contraction of the belief set by each other, not allowing independent beliefs to be retained by each other's contraction [37, p. 515]. However, instead of discarding severe withdrawal, Lindström and Rabinowicz have proposed that it can be used as a lower limit for contraction, while AGM contraction would be the upper limit, and a reasonable operation of contraction should lie somewhere between these two extremes [26, pp. 115]. This has been called Lindström's and Rabinowicz's interpolation thesis [35].

To replace recovery, the following postulates were proposed, in which the focus is the justification for the removal of a sentence β when contracting by α,

$(-Rel)$ If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} - \alpha$, then there is a set $\mathbf{K}'$ such that $\mathbf{K} - \alpha \subseteq \mathbf{K}' \subseteq \mathbf{K}$ and $\mathbf{K}' \nvdash \alpha$ but $\mathbf{K}' \cup \{\beta\} \vdash \alpha$. (Relevance, [15,17])

$(-CR)$ If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} - \alpha$ then there is a set $\mathbf{K}'$ such that $\mathbf{K}' \subseteq \mathbf{K}$ and $\mathbf{K}' \nvdash \alpha$ but $\mathbf{K}' \cup \{\beta\} \vdash \alpha$. (Core-retainment, [16])

$(-DE)$ If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} - \alpha$ then $\mathbf{K} - \alpha \nvdash \alpha \vee \beta$. (Disjunctive Elimination, [7])

However, although the above postulates are more intuitively appealing, they are equivalent to recovery in the presence of the other basic AGM postulates.

In this paper we investigate the possibility of finding an appropriate minimal change postulate alternative to recovery, as the result of fine-tuning the recovery postulate itself. The idea is that, since many beliefs relevant to α are removed when contracting by α, undoing that contraction requires regaining not just α but also many other relevant propositions that were discarded. So the general form of recovery would be: $\mathbf{K} \subseteq Cn((\mathbf{K} - \alpha) \cup f(\alpha))$ for some function f, subject to some possible conditions (e.g. $Cn(f(\alpha)) \subseteq \mathbf{K}$, $f(\alpha) \vdash \alpha$). In this paper we explore this approach that we call *Recuperation*.[3]

We argue that recuperation aptly captures the intuition that underlays the postulate of relevance and based in it, in the next section we will define syntactically a withdrawal operator based on the notion of recuperation.

4. Recuperative withdrawals

In the previous section, we discussed the postulate of recovery and, by means of it, we showed that AGM contractions can give rise to unintuitive results. On the other hand, severe withdrawal provokes the loss of unrelated beliefs in the contraction process. In this subsection, we will introduce recuperative withdrawals, a withdrawal operator based on the notion of recuperation presented in the previous section to capture the notion of minimal change.

In this context, the basic AGM contraction postulates (with the exception of recovery), i.e., closure, inclusion, vacuity, success and extensionality must hold.

We propose the following postulate as the basis of minimal change:

$(-Rec)$ If $\mathbf{K} - \beta \vdash \alpha$, then $\mathbf{K} \subseteq Cn(\mathbf{K} - \alpha \cup \mathbf{K} - \beta)$. (Recuperation)

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[2] It is tempting to draw a parallel between this *Recovery Problem* and the *Gettier's Problem* [12]. Without loss of generality, we can say that in Gettier's Problem the issue is the epistemic status (knowledge or not?) of $\alpha \vee \beta$ which is believed as a consequence of a justified (but false) belief α, and whose veracity is grounded in the hitherto unknown truth β. In the Recovery Problem, $\alpha \vee \beta$ is initially believed as a consequence of belief α; is removed; and finally reintroduced on the basis of β. Both the problems are crucially dependent on $\alpha \vee \beta$ being the common consequence of α and β, in the context of truth, belief, or justification.

[3] A similar approach was studied in [4,9], where the operators of semi-contractions were introduced and axiomatically characterized by a set of postulates which includes "If $\mathbf{K} - \alpha \neq \mathbf{K}$, then there exists some $\beta \in \mathbf{K}$ such that $\mathbf{K} - \alpha \nvdash \beta$ and $\mathbf{K} \subseteq Cn((K - \alpha) \cup \{\beta\})$ (Proxy Recovery)". We note additionally that the mentioned axiomatic characterization for semi-contraction does not contain any of the supplementary AGM postulates for contraction.

Recuperation states that the outcome of any contraction which does not lead to the removal of α contains all the sentences needed to undo the contraction by α. The following proposition highlights that the converse inclusion of the one stated in the recuperation postulate follows directly from the inclusion postulate.

Proposition 4.1. *Let* $\mathbf{K}$ *be a belief set and* $-$ *be an operator on* $\mathbf{K}$*. If* $-$ *satisfies inclusion, then* $Cn(\mathbf{K}-\alpha \cup \mathbf{K}-\beta) \subseteq \mathbf{K}$.

In the presence of closure and inclusion, recuperation implies failure:

Proposition 4.2. *Let* $\mathbf{K}$ *be a belief set and* $-$ *be an operator on* $\mathbf{K}$*. If* $-$ *satisfies closure, inclusion and recuperation then it satisfies failure.*

On the other hand, the following proposition yields that AGM contractions satisfy the *recuperation* postulate.

Proposition 4.3. *Let* $\mathbf{K}$ *be a belief set and* $-$ *be an operator on* $\mathbf{K}$*. If* $-$ *satisfies closure and recovery, then it also satisfies recuperation.*

At the supplementary level, we propose the following postulate:

($-SCI$) If $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$, then $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$. (Strong conjunctive inclusion)

Strong conjunctive inclusion states that if a belief α is removed when contracting by $\alpha \wedge \beta$, then the outcome of the contraction by that conjunction coincides with the outcome of the contraction by α. When giving up $\alpha \wedge \beta$ at least one of the conjuncts must be removed. If α is given up in $\mathbf{K} \div (\alpha \wedge \beta)$, then this set is the same that results of contracting $\mathbf{K}$ simply by α.

Definition 4.4. Let $\mathbf{K}$ be a belief set. An operator $\div$ for $\mathbf{K}$ is a *recuperative withdrawal* if and only if it satisfies closure, inclusion, vacuity, success, extensionality, recuperation and strong conjunctive inclusion.

The following proposition yields that recuperative withdrawals satisfy decomposition and the following postulate:

($-T$) If $\alpha \in \mathbf{K}-(\alpha \wedge \beta)$, then $\alpha \in \mathbf{K}-(\alpha \wedge \beta \wedge \delta)$. (Conjunctive trisection, [34,18])

In the presence of the basic AGM postulates, conjunctive trisection and conjunctive overlap are equivalent ([34]).

Proposition 4.5. *Let* $\mathbf{K}$ *be a belief set and* $-$ *be an operator on* $\mathbf{K}$.

1. *If* $-$ *satisfies closure, success, failure and strong conjunctive inclusion, then it also satisfies decomposition.*
2. *If* $-$ *satisfies extensionality and strong conjunctive inclusion, then it also satisfies conjunctive trisection.*

The following proposition shows that, at the supplementary level, recuperative withdrawals satisfy the supplementary AGM contraction postulates.

Proposition 4.6. *Let* $\mathbf{K}$ *be a belief set and* $-$ *be an operator on* $\mathbf{K}$.

1. *If* $-$ *satisfies strong conjunctive inclusion, then it satisfies conjunctive inclusion.*
2. *If* $-$ *satisfies closure, inclusion, success, recuperation and strong conjunctive inclusion, then it also satisfies conjunctive overlap.*

5. Ring withdrawals

In this section we present a formal definition for ring withdrawal operators and an axiomatic characterization of this kind of operators in terms of postulates. Moreover, we will show that the classes of recuperative withdrawal and of ring withdrawal operators coincide.

In Section 3, we mentioned that, in terms of possible worlds, the AGM contraction of $\mathbf{K}$ by α can be viewed as the process of adding to $\|\mathbf{K}\|$ its closest $\neg\alpha$-worlds, while a severe withdrawal $\dot{-}$ is such that the $\|\mathbf{K}\dot{-}\alpha\|$ is formed by all the possible worlds which are at least as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds. Ring withdrawal takes a middle path, namely by considering that the result of contracting a belief set $\mathbf{K}$ by a sentence α is the theory whose set of possible worlds is the one that results of adding to $\|\mathbf{K}\|$ all the worlds with the same plausibility of the nearest $\neg\alpha$ worlds (i.e. those possible worlds which are as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds).

We will start by presenting a formal definition of ring withdrawals.

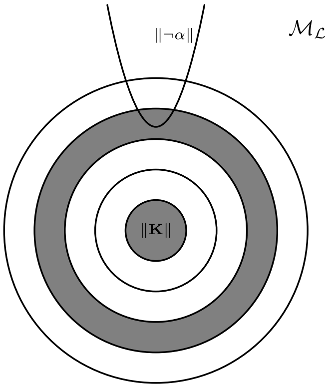

Fig. 3. Representation of a system of spheres centred on $\|\mathbf{K}\|$, where the set $\|\mathbf{K} \div \alpha\|$, formed by the possible worlds of the outcome of the ring withdrawal of $\mathbf{K}$ by α, is shaded.

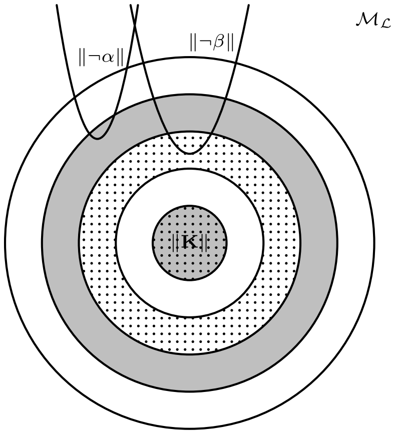

Fig. 4. Graphical representation informally showing that ring withdrawals satisfy recuperation.

Definition 5.1. Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$-based ring withdrawal on $\|\mathbf{K}\|$ is the operator $\div_\mathbb{S}$ defined, for any $\alpha \in \mathcal{L}$, by:

$$\|\mathbf{K}\| \div_\mathbb{S} \|\alpha\| = \begin{cases} \|\mathbf{K}\| \cup (\mathbb{S}_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq \mathbb{S}_{\neg\alpha}\}) & \text{if } \|\neg\alpha\| \neq \emptyset \\ \|\mathbf{K}\| & \text{otherwise} \end{cases}$$

An operator $\div$ on $\mathbf{K}$ is a ring withdrawal if and only if there exists an $\mathbb{S}$-based ring withdrawal $\div_\mathbb{S}$ on $\|\mathbf{K}\|$, such that, for all sentences $\alpha \in \mathcal{L}$, $\|\mathbf{K} \div \alpha\| = \|\mathbf{K}\| \div_\mathbb{S} \|\alpha\|$ and $\mathbf{K} \div \alpha = \bigcap(\|\mathbf{K}\| \div_\mathbb{S} \|\alpha\|)$.[4]

In Fig. 3 we present a graphical representation of the possible worlds of the outcome of a ring withdrawal.

At this point we informally show that ring withdrawals satisfy recuperation but do not satisfy recovery.

For recuperation, if $\vdash \alpha$, it follows by definition that $\mathbf{K} \div \alpha = \mathbf{K}$. Thus in this case, recuperation holds. By symmetry of the case, recuperation also holds if $\vdash \beta$. So, let $\nvdash \alpha$ and $\nvdash \beta$. In Fig. 4 we represent the sets $\|\mathbf{K} \div \alpha\|$ (shadowed) and $\|\mathbf{K} \div \beta\|$ (dotted), where $\div$ is a ring contraction such that $\|\mathbf{K} \div \beta\| \subseteq \|\alpha\|$ (which yields $\mathbf{K} \div \beta \vdash \alpha$). We can easily conclude from this representation that $(\|\mathbf{K} \div \alpha\| \cap \|\mathbf{K} \div \beta\|) = \|\mathbf{K}\|$ (which yields $\mathbf{K} = Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$, cf. Lemma 2). Consequently we can (informally) conclude that recuperation holds.

Regarding recovery, in Fig. 5 we represent the set $\|\mathbf{K} \div \alpha\|$ (shadowed), where $\div$ is a ring contraction, as well as the sets $\|\neg\alpha\|$ and $\|\neg\beta\|$, for two sentences $\alpha, \beta \in \mathbf{K}$. Furthermore, in that figure we highlight a world w which is such that $w \in \|\mathbf{K} \div \alpha\| \cap \|\alpha\|$, but $w \notin \|\mathbf{K}\|$. Therefore, in the situation illustrated, it does not hold that $\|\mathbf{K} \div \alpha\| \cap \|\alpha\| \subseteq \|\mathbf{K}\|$. Therefore $\mathbf{K} \nsubseteq Cn((\mathbf{K} \div \alpha) \cup \{\alpha\})$.[5] Hence, we can conclude that ring withdrawals fail to satisfy recovery.

The following theorem shows the relation between ring withdrawal and recuperative withdrawal operators.

[4] We note that, when the underlying language is defined from a finite set of propositional symbols, the condition $\|\mathbf{K} \div \alpha\| = \|\mathbf{K}\| \div_\mathbb{S} \|\alpha\|$ is redundant in this definition.

[5] See Lemmas 2 and 3.

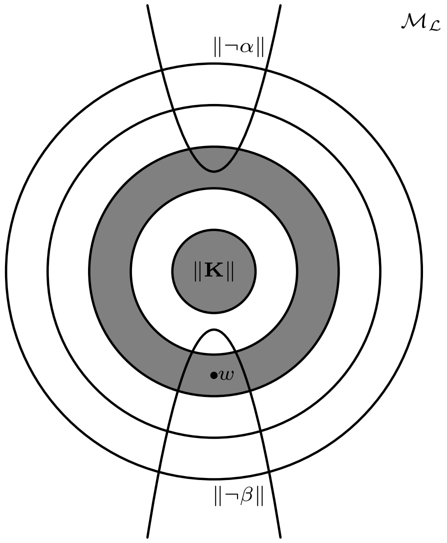

Fig. 5. Graphical representation informally showing that ring withdrawals do not satisfy recovery.

Theorem 5.2. *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

1. $\div$ *is a recuperative withdrawal (i.e.,* $\div$ *satisfies closure, inclusion, vacuity, success, extensionality, recuperation and strong conjunctive inclusion).*
2. $\div$ *is a ring withdrawal operator.*

6. Ring withdrawals versus severe withdrawals and AGM contractions

In this section we compare ring withdrawals (or, equivalently, recuperative withdrawals) with severe withdrawals and AGM contractions. We start by presenting a proposition that illustrates that the outcome of ring withdrawal operator is predetermined whenever it satisfies (-10).

Proposition 6.1. *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a ring withdrawal operator on* $\mathbf{K}$ *that satisfies* (-10)*. It holds for all* α *that:*

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} & \text{if } \vdash \alpha \text{ or } \alpha \notin \mathbf{K} \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

It holds that ring withdrawal operator, in general, do not satisfy the undesirable *expulsiveness* postulate. To see this, consider that $\mathbf{K} = Cn(\{p, q\})$ and let $\div$ be a ring withdrawal operator on $\mathbf{K}$ based on the system of spheres represented in Fig. 6. It holds that $q \in \mathbf{K} \div p$ and $p \in \mathbf{K} \div q$. From which it follows that $\div$ does not satisfy *expulsiveness*.
Having in mind the axiomatic characterizations of the severe withdrawals and of the ring withdrawals as well as Proposition 4.2, it is possible to conclude that ring withdrawal operators satisfy all postulates presented in the axiomatic characterization of severe withdrawals with the exception of strong inclusion (and $(-7a)$).[6]

Conversely, severe withdrawals also satisfy several of the postulates that characterize ring withdrawals. The following proposition states that *strong conjunctive inclusion* follows from two of the postulates satisfied by operators of severe withdrawal.

Proposition 6.2. *Let* $\mathbf{K}$ *be a belief set and* $-$ *an operator that satisfies* $(-7a)$ *and conjunctive inclusion. Then* $-$ *satisfies strong conjunctive inclusion.*

Now we show that, in general, severe withdrawals do not satisfy *recuperation*. Let $\dot{\div}$ be a severe withdrawal on $\mathbf{K} = Cn(\{p, q\})$ based on the system of spheres represented in Fig. 6. It holds that $q \notin \mathbf{K} \dot{\div} p$ and $p \in \mathbf{K} \dot{\div} q$. Thus, by $\dot{\div}$ *strong inclusion*, it holds that $\mathbf{K} \dot{\div} p \subseteq \mathbf{K} \dot{\div} q$. Assume by *reductio* that $\dot{\div}$ satisfies *recuperation*. Then $\mathbf{K} \subseteq Cn(\mathbf{K} \dot{\div} q \cup \mathbf{K} \dot{\div} p) = Cn(\mathbf{K} \dot{\div} q) = \mathbf{K} \dot{\div} q$. This contradicts $\dot{\div}$ *success*. Hence, $\dot{\div}$ does not satisfy *recuperation*. Thus severe withdrawals satisfy all postulates of the axiomatic characterization of ring withdrawals presented in Theorem 5.2 with the exception of *recuperation*.

[6] By Theorem 5.2 and Proposition 4.2, ring withdrawals satisfy all the postulates that characterize severe withdrawals, present in Proposition 2.7 with the exception of strong inclusion. On the other hand, since severe withdrawals satisfy expulsiveness and ring withdrawals don't, we can conclude that ring withdrawals, in general, do not satisfy strong inclusion.

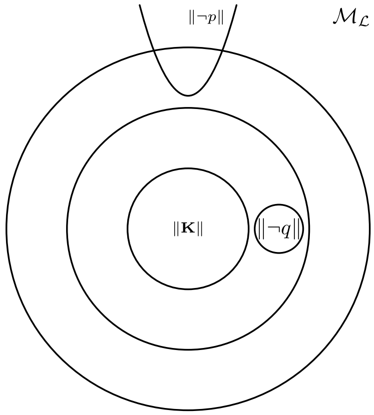

Fig. 6. Graphical representation informally showing that ring withdrawals do not satisfy expulsiveness.

Fig. 6 also illustrates that ring withdrawals fail to satisfy recovery, since $\|\mathbf{K}\ddot{-}p\| \cap \|p\| \nsubseteq \|\mathbf{K}\|$.

From the above, it is possible to conclude that severe withdrawals and ring withdrawals have several features in common as each of them satisfies several of the postulates included in the other one's axiomatic characterization but, they also have some fundamental differences. In fact, ring withdrawals and severe withdrawals based on a system of spheres coincide if and only if that sphere system has at most two spheres. Finally, it is also possible to conclude that the classes of severe withdrawals and of ring withdrawals are such that none of them is a subclass of the other one.

We now compare the postulates presented in the axiomatic characterizations of ring withdrawals and of AGM contractions. From Proposition 4.3 we conclude that AGM contractions satisfy *recuperation*.

At the beginning of Section 5 we saw that ring withdrawals in general do not satisfy *recovery*. From this, Theorem 5.2 and Proposition 4.6 we can conclude that ring withdrawals satisfy all AGM contraction postulates with the exception of *recovery*. Conversely, having in mind Proposition 4.3, AGM contractions satisfy all of the postulates, in the statement of Theorem 5.2, that characterize ring withdrawals with the exception of *strong conjunctive inclusion*. To see that in general AGM contraction operators in general do not satisfy *strong conjunctive inclusion*, consider a system of spheres such that $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta} \neq \|\mathbf{K}\|$ and $\|\neg\alpha\| \cap \|\neg\beta\| = \emptyset$. Thus it holds that $\nvdash \alpha$, $\nvdash \beta$ and the AGM contraction operator $\dot{-}$ based on $\mathbb{S}$ is such that $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \alpha \cap \mathbf{K} \dot{-} \beta$. Hence, $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$, but $\mathbf{K} \dot{-} (\alpha \wedge \beta) \neq \mathbf{K} \dot{-} \alpha$ (since $\beta \in \mathbf{K} \dot{-} \alpha$ and $\beta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta) \subseteq \mathbf{K} \dot{-} \beta$).

We finish this section by showing that, given a belief set $\mathbf{K}$, a fixed system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ that contains more than two spheres, the operators of ring withdrawal, severe withdrawal and AGM contraction on $\mathbf{K}$ that are induced by $\mathbb{S}$, are such that the outcome of the ring withdrawal of $\mathbf{K}$ by α is between the outcomes of the severe withdrawal and of the AGM contraction of $\mathbf{K}$ by α. Let $\mathbb{S}$ be a system of spheres that contains more than two spheres and let $\ddot{-}$, $\div$ and $\dot{-}$ be respectively, the operators of severe withdrawal, ring withdrawal and AGM contraction based on $\mathbb{S}$. If $\|\neg\alpha\| = \emptyset$, then it holds that $\mathbf{K}\ddot{-}\alpha = \mathbf{K} \div \alpha = \mathbf{K} \dot{-} \alpha = \mathbf{K}$. If $\|\neg\alpha\| \neq \emptyset$, then, as Fig. 7 illustrates, it holds that:

$$\|\mathbf{K}\| \cup (S_{\neg\alpha} \cap \|\neg\alpha\|) \subseteq \|\mathbf{K}\| \cup (\mathbb{S}_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq \mathbb{S}_{\neg\alpha}\}) \subseteq \mathbb{S}_{\neg\alpha}$$

Thus, it holds that:

$$\mathbf{K} \dot{-} \alpha \subseteq \mathbf{K} \div \alpha \subseteq \mathbf{K}\ddot{-}\alpha$$

Therefore, we can say that ring withdrawal operators satisfy Lindström and Rabinowicz's interpolation thesis.

7. Inter-defining AGM contractions, ring withdrawals and severe withdrawals

For the sake of completeness we start this section by recalling two conditions, presented in [37] that allow to interdefine severe withdrawal and AGM contraction operators.

Let $\div$ be a withdrawal on $\mathbf{K}$ that satisfies failure, then the following condition allows to define an AGM contraction operator $\dot{-}$ from $\div$.

$$\mathbf{K} \dot{-} \alpha = (\mathbf{K} \div \alpha) + \neg\alpha \cap \mathbf{K} \qquad \text{(Def } \dot{-} \text{ from } \div\text{)}$$

Let $\dot{-}$ be an AGM contraction on $\mathbf{K}$, then the following condition allows to define a severe withdrawal operator $\ddot{-}$ from $\dot{-}$.

$$\mathbf{K}\ddot{-}\alpha = \begin{cases} \{\beta : \beta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases} \qquad \text{(Def } \ddot{-} \text{ from } \dot{-}\text{)}$$

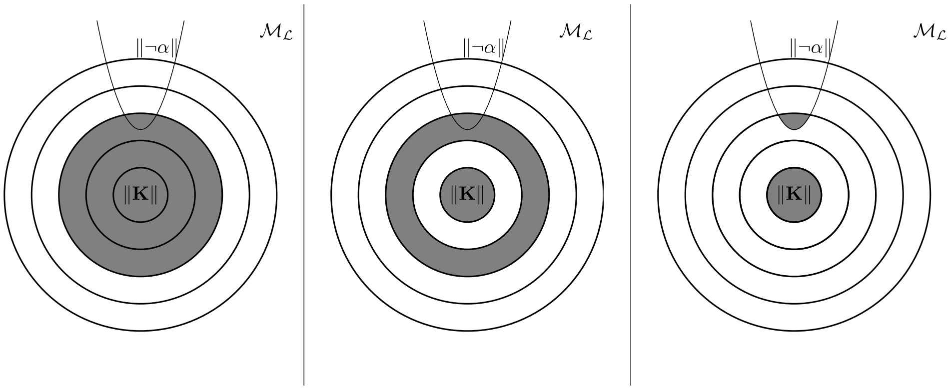

Fig. 7. Sphere semantics for, respectively, severe withdrawal, ring withdrawal and AGM contraction showing $\|\mathbf{K} - \alpha\|$ shaded.

It was proven in [37] that the previous condition is equivalent to[7]:

$$\mathbf{K}\ddot{-}\alpha = \begin{cases} \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \beta) : \beta \in \mathcal{L}\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases} \qquad \text{(Def' } \ddot{-} \text{ from } \dot{-}\text{)}$$

The adequacy of conditions (Def $\ddot{-}$ from $\dot{-}$) and (Def $\dot{-}$ from $\ddot{-}$) is stated in the following propositions.

Proposition 7.1 *([37]). If $\dot{-}$ is an AGM contraction operator, then $\ddot{-}$ as obtained by (Def $\ddot{-}$ from $\dot{-}$) is a severe withdrawal operator revision equivalent to $\dot{-}$, and $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \dot{-} \alpha$, for all $\alpha \in \mathcal{L}$.*

Proposition 7.2 *([37]). If $\ddot{-}$ is a severe withdrawal operator, then $\dot{-}$ as obtained by (Def $\dot{-}$ from $\ddot{-}$) is an AGM contraction operator revision equivalent to $\ddot{-}$, and $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \dot{-} \alpha$, for all $\alpha \in \mathcal{L}$.*

In the remainder of this section we present methods for obtaining operators of ring withdrawals from severe withdrawals and AGM contractions and vice-versa. We start by presenting two conditions which can be used for defining a severe withdrawal operator $\ddot{-}$ from a ring withdrawal operator $\div$.

$$\mathbf{K}\ddot{-}\alpha = \begin{cases} \{\beta : \beta \in \mathbf{K} \div (\alpha \wedge \beta)\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases} \qquad \text{(Def } \ddot{-} \text{ from } \div\text{)}$$

Intuitively, in giving up α, (Def $\ddot{-}$ from $\div$) tells us to retain those beliefs β that are kept when a choice is made between removing α, β or both. An alternative method for obtaining a severe withdrawal operator $\ddot{-}$ from a ring withdrawal operator $\div$ is given by the following condition:

$$\mathbf{K}\ddot{-}\alpha = \begin{cases} \bigcap\{\mathbf{K} \div \beta : \vdash \beta \to \alpha\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases} \qquad \text{(Def' } \ddot{-} \text{ from } \div\text{)}$$

(Def' $\ddot{-}$ from $\div$) states that when trying to remove α we should retain those beliefs that are always kept when removing beliefs that are stronger than α.

It holds that each one of the two conditions presented above is equivalent to the other one.

Proposition 7.3. *Let $\mathbf{K}$ be a belief set and $\div$ be an operator that satisfies success, closure, extensionality and strong conjunctive inclusion. Then (Def $\ddot{-}$ from $\div$) and (Def' $\ddot{-}$ from $\div$) are equivalent.*

The following condition can be used for defining a ring withdrawal $\div$ from a severe withdrawal $\ddot{-}$.

$$\mathbf{K} \div \alpha = \bigcap\{(\mathbf{K}\ddot{-}\alpha + \neg\beta) \cap \mathbf{K} : \mathbf{K}\ddot{-}\alpha = \mathbf{K}\ddot{-}\beta\}. \qquad \text{(Def } \div \text{ from } \ddot{-}\text{)}$$

Intuitively, (Def $\div$ from $\ddot{-}$) states that when contracting by α we should hold those beliefs that are always in the outcome of the set that results from consecutively applying the Levi's and Harper's identities to sentences that have the same plausibility status as α.

We now present two conditions which can be used for defining a ring withdrawal operator $\div$ from an AGM contraction operator $\dot{-}$ and vice-versa. We start by presenting a condition that can be used for defining $\div$ from $\dot{-}$.

[7] According to Lemma 6 this equivalence holds as long as $\dot{-}$ satisfies ($\dot{-}$1), ($\dot{-}$2), ($\dot{-}$5), ($\dot{-}$6), ($\dot{-}$7) and ($\dot{-}$8).

$$\mathbf{K} \div \alpha = \begin{cases} \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \beta) : \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \{\alpha, \beta\} = \emptyset\} & \text{if } \nvdash \alpha \text{ and } \alpha \in \mathbf{K} \\ \mathbf{K} & \text{otherwise} \end{cases} \quad \text{(Def } \div \text{ from } \dot{-}\text{)}$$

In the main case, (Def $\div$ from $\dot{-}$) states that when removing α from $\mathbf{K}$ through a ring withdrawal operator we should keep those beliefs that are always retained when contracting by the conjunction of α with other beliefs that have the same status as α (i.e., when each one of those contractions leads to the removal of both beliefs).

The following condition can be used for defining an AGM contraction operator from a ring withdrawal operator.

$$\mathbf{K} \dot{-} \alpha = Cn(\mathbf{K} \div \alpha \cup \{\neg\alpha\}) \cap \mathbf{K} \quad \text{(Def } \dot{-} \text{ from } \div\text{)}$$

(Def $\dot{-}$ from $\div$) was used in [37] for defining an AGM contraction from a severe withdrawal. It consists in consecutively applying the Levi and Harper identities.

The following propositions express the adequacy of the above definitions.

Proposition 7.4. *Let $\mathbf{K}$ be a belief set. If $\ddot{-}$ is a severe withdrawal operator, then $\div$ as obtained by (Def $\div$ from $\ddot{-}$) is ring withdrawal operator, revision equivalent to $\ddot{-}$, and $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \div \alpha$, for all $\alpha \in \mathcal{L}$.*

Proposition 7.5. *Let $\mathbf{K}$ be a belief set. If $\div$ is ring withdrawal operator, then $\ddot{-}$ as obtained by (Def $\ddot{-}$ from $\div$) is a severe withdrawal operator, revision equivalent to $\div$, and $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \div \alpha$, for all $\alpha \in \mathcal{L}$.*

Proposition 7.6. *Let $\mathbf{K}$ be a belief set. If $\dot{-}$ is an AGM contraction operator on $\mathbf{K}$, then $\div$ as obtained by (Def $\div$ from $\dot{-}$) is a ring withdrawal operator, revision equivalent to $\dot{-}$, and $\mathbf{K} \div \alpha \subseteq \mathbf{K} \dot{-} \alpha$, for all $\alpha \in \mathcal{L}$.*

Proposition 7.7. *Let $\mathbf{K}$ be a belief set. If $\div$ is ring withdrawal operator on $\mathbf{K}$, then $\dot{-}$ as obtained by (Def $\dot{-}$ from $\div$) is an AGM contraction operator, revision equivalent to $\div$, and $\mathbf{K} \div \alpha \subseteq \mathbf{K} \dot{-} \alpha$, for all $\alpha \in \mathcal{L}$.*

8. Conclusion

The AGM model [1] is considered to be the standard model of theory change. In this model three operations are considered, namely expansion, revision and contraction. The latter consists in removing an input sentence from the set of beliefs of a given agent. One of the AGM postulates that characterizes contraction is *recovery*: $\mathbf{K} \subseteq (\mathbf{K} - \alpha) + \alpha$. This postulate characterizes the notion of "minimal change" in the contraction process [11,27,29]. It is based in the intuition that "it is reasonable that we get all of the beliefs (...) back again after first contracting and then expanding with respect to the same belief" [10]. However, there are examples of contractions in which *recovery* seems implausible (see Examples 3.2 and 3.3). Other postulates that capture the notion of minimal change are *relevance* [15,17], *core-retainment*, [16] and *disjunctive elimination* [7]. But, in the presence of the other basic AGM postulates, they are equivalent to *recovery*. Due to the problematic character of the *recovery* postulate, Makinson proposed a wider class of removal operators, that were named *withdrawals*. Makinson defined a *withdrawal* as an operator that satisfies the basic AGM postulates for contraction with the exception of *recovery*. Some withdrawal operators are the Levi contractions [23], the severe withdrawals (or mild contractions or Rott contractions) [33,37], the semi-contractions [4,9] and the systematic withdrawals [30].[8]

In this paper we proposed a postulate, designated by *recuperation* which we have exposed to be an appropriate alternative to the postulate of recovery for formalizing the principle of minimal change. In particular we have shown that this postulate follows from *recovery* and *closure*, so that it can be seen as a weaker version of the former. Furthermore we introduce a new class of withdrawals, which we designate by *recuperative withdrawals*, formed by all the operators which satisfy all the basic AGM postulates, with the exception of recovery, recuperation and strong conjunctive inclusion. Then we presented the class of the, so called, ring withdrawal operators. The ring withdrawal operators were defined by means of a system of spheres and, informally speaking, are such that the outcome of a ring withdrawal of a belief set $\mathbf{K}$ by a sentence α is the theory whose set of possible worlds is the union of $\|\mathbf{K}\|$ with the set of all the worlds which are as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds. These operators were first mentioned in [37, page 543], citing a personal communication from Nayak, however, a formal definition for them had not yet been presented in the literature. The only works that we are aware of which were inspired by the operator proposed in Nayak's communication cited in [37], are those by Booth et al. [2,3]. These papers contain a systematic study of a very general class of operators, there designated by *removal operators* and which have the effect of removing a given sentence from a given belief set. In particular, those papers present axiomatic characterizations for several subfamilies of the class of removal operators. One such removal operator, studied in [3], is called *dichotomous liberation* operator, which, using the same terminology that we have used above, is such that the outcome of a change of a belief set $\mathbf{K}$ by a sentence α is the theory whose set of possible worlds is the set of all the worlds which are as close to $\|\mathbf{K}\|$ as its closest $\neg\alpha$-worlds (graphically speaking, the worlds contained in the first ring containing a $\neg\alpha$-world). We note that this operator differs from the operator of ring withdrawal because the worlds in $\|\mathbf{K}\|$ are not kept (as it is the case in

[8] In [37, pp. 541-543] there is a graphical representation representing most of the above mentioned withdrawal operators by means of Grove's systems of spheres.

our proposal) and, consequently, in general, the dichotomous liberation operators do not satisfy the inclusion postulate. In [3], an axiomatic characterization for these operators has been presented.

It follows immediately that when the three operators are based on the same system of spheres (centred in $\|\mathbf{K}\|$), the output of a ring withdrawal is situated between the outputs of AGM contractions and of severe withdrawals. This yields that ring withdrawals satisfy the interpolation thesis of Lindström and Rabinowicz, which states that a reasonable removal operation should be somewhere between AGM contractions and severe withdrawals [26, p.115].

We have also shown that the class of ring withdrawals is identical to the class of recuperative withdrawals and we have compared, in terms of postulates, the operators of ring withdrawals with the operators of AGM contractions and of severe withdrawals. As a result of that comparison we concluded that:

- ring withdrawal operators satisfy all postulates presented in the axiomatic characterization of severe withdrawals with the exception of strong inclusion (and $(-7a)$);
- ring withdrawal operators satisfy all postulates presented in the axiomatic characterization of AGM contraction with the exception of recovery;
- severe withdrawal operators satisfy all postulates presented in the axiomatic characterization of ring withdrawals with the exception of recuperation;
- AGM contraction operators satisfy all postulates presented in the axiomatic characterization of ring withdrawals with the exception of strong conjunctive inclusion.

At this point we highlight that, in general, ring withdrawals do not satisfy *recovery* and, therefore, most of these operators do not return the undesirable outputs that have been presented in some examples of AGM contractions. When comparing ring withdrawals with severe withdrawals we also showed that the former, contrary to the latter, do not satisfy the undesirable property of *expulsiveness*, according to which independent beliefs cannot be in the outcome of the removal by each other. Another positive aspect regarding ring withdrawals is the fact that there is a one-to-one correspondence between the set of systems of spheres centred in $\|\mathbf{K}\|$ and the set of ring withdrawals on $\mathbf{K}$, since given one such system of spheres $\mathbb{S}$, there is an unique $\mathbb{S}$-based ring withdrawal. Obviously the same happens regarding AGM contractions and severe withdrawals. Nevertheless, there are some operators, such as the Levi contractions [23,21] and the semi-contractions [4,9] for which this does not hold, since for a given system of spheres $\mathbb{S}$ there may exist several different $\mathbb{S}$-based Levi contractions as well as several different $\mathbb{S}$-based semi-contractions.

One additional contribution of the present paper is the presentation of constructive methods for defining operators of ring withdrawals from severe withdrawals and from AGM contractions and vice-versa.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Proofs

Lemma 1 *([1]). Let* $\mathbf{K}$ *be a belief set and* $-$ *a basic AGM contraction on* $\mathbf{K}$*. Then* $-$ *satisfies conjunctive overlap and conjunctive inclusion if and only if* $-$ *satisfies conjunctive factoring.*

Lemma 2 *([14,20]). Let* $\mathbf{K}$ *and* $\mathbf{H}$ *be belief sets and* α *and* β *sentences. Then the following properties hold:*

(a) *If* $w \in \mathcal{L} \perp \perp$*, then* $w \in \|\alpha\|$ *if and only if* $w \notin \|\neg\alpha\|$.
(b) $\|\mathbf{K} \cup \mathbf{H}\| = \|\mathbf{K}\| \cap \|\mathbf{H}\|$.
(c) $\|Cn(\mathbf{K} \cup \mathbf{H})\| = \|\mathbf{K}\| \cap \|\mathbf{H}\|$.
(d) *If* $\mathbf{K} \subseteq \mathbf{H}$*, then* $\|\mathbf{H}\| \subseteq \|\mathbf{K}\|$.
(e) $\|\alpha\| \subseteq \|\beta\|$ *if and only if* $\vdash \alpha \rightarrow \beta$.

Lemma 3 *([14]). Let* $\mathbf{K}$*,* $\mathbf{H}$ *be belief sets and* U*,* V *be sets of possible worlds. Then:*

(a) $\bigcap(\|\mathbf{K}\|) = \mathbf{K}$ *(if the underlying logic is compact).*
(b) *If* $U \subseteq V$*, then* $\bigcap(V) \subseteq \bigcap(U)$.
(c) *For any* $\alpha \in \mathcal{L}$, $\bigcap(V \cap \|\alpha\|) = Cn(\bigcap(V) \cup \{\alpha\})$.

Lemma 4 *([14]). Let* W *be a set of possible worlds. Then* $\bigcap W$ *is a belief set.*

Lemma 5 *([20]). Let* $\mathbb{S}$ *be a system of spheres. Let* S_α *be the minimal sphere that intersects with* $\|\alpha\|$ *and Let* S_β *be the minimal sphere that intersects with* $\|\beta\|$*. If* $\vdash \alpha \to \beta$*, then* $S_\beta \subseteq S_\alpha$.

Lemma 6 *([37]). If* $\dot{-}$ *satisfies* $(\dot{-}1)$, $(\dot{-}2)$, $(\dot{-}5)$, $(\dot{-}6)$, $(\dot{-}7)$ *and* $(\dot{-}8)$*, then (Def* $\ddot{-}$ *from* $\dot{-}$*) and (Def'* $\ddot{-}$ *from* $\dot{-}$*) are equivalent.*

Lemma 7. *Let* $\mathbf{K}$ *be a belief set and* $\dot{-}$ *be an AGM contraction operator on a* $\mathbf{K}$*. It holds that:*

1. $\dot{-}$ *satisfies conjunctive trisection ([34]);*
2. $\dot{-}$ *satisfies the following postulate:*
   If $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$*, then* $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$.

Proof. 2. Let $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$. If $\vdash \beta$, then by $\dot{-}$ *closure* it follows that $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$. Assume now that $\nvdash \beta$. By $\dot{-}$ *conjunctive trisection* (1.) it follows that $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. By $\dot{-}$ *success* it follows that $\alpha \wedge \delta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. From which it follows by $\dot{-}$ *conjunctive inclusion* and *extensionality* that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta) \subseteq \mathbf{K} \dot{-} (\alpha \wedge \delta)$. Hence $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$. □

Lemma 8. *Let* $\mathbf{K}$ *be a belief set. And* $-$ *be an operator on* $\mathbf{K}$ *that satisfies closure. It holds that* $\mathbf{K} \subseteq (\mathbf{K} \cap (\mathbf{K} - \alpha + \neg\alpha)) + \alpha$.

Proof. Suppose toward a contradiction that there exists $\beta \in \mathbf{K}$ such that $\beta \notin (\mathbf{K} \cap (\mathbf{K} - \alpha + \neg\alpha)) + \alpha$. Hence by deduction it follows that $\alpha \to \beta \notin (\mathbf{K} \cap \mathbf{K} - \alpha + \neg\alpha)$. Thus either $\alpha \to \beta \notin \mathbf{K}$ or $\alpha \to \beta \notin \mathbf{K} - \alpha + \neg\alpha$. The former leads to a contradiction, since $\mathbf{K}$ is a belief set and $\beta \in \mathbf{K}$. Assume now that $\alpha \to \beta \notin \mathbf{K} - \alpha + \neg\alpha$. By deduction it follows that $\neg\alpha \to (\alpha \to \beta) \notin \mathbf{K} - \alpha$. This contradicts $-$ *closure*, since $\vdash \neg\alpha \to (\alpha \to \beta)$. □

Proof of Proposition 4.1. By *inclusion* it follows that $\mathbf{K} - \alpha \subseteq \mathbf{K}$ and $\mathbf{K} - \beta \subseteq \mathbf{K}$. Thus $\mathbf{K} - \alpha \cup \mathbf{K} - \beta \subseteq \mathbf{K}$. From which it follows, by the monotony of Cn, that $Cn(\mathbf{K} - \alpha \cup \mathbf{K} - \beta) \subseteq Cn(\mathbf{K}) = \mathbf{K}$. □

Proof of Proposition 4.2. Let $\vdash \alpha$. By *closure*, $\mathbf{K} - \alpha \vdash \alpha$, from which it follows by *recuperation* that $\mathbf{K} \subseteq Cn(\mathbf{K} - \alpha \cup \mathbf{K} - \alpha)$. By closure we obtain $\mathbf{K} \subseteq \mathbf{K} - \alpha$. Hence, by *inclusion* $\mathbf{K} = \mathbf{K} - \alpha$. □

Proof of Proposition 4.3. Let $\mathbf{K}$ be a belief set and $-$ be an operator on $\mathbf{K}$. Assume that $-$ satisfies *recovery* and *closure* and that $\mathbf{K} - \beta \vdash \alpha$. By *recovery* it follows that $\mathbf{K} \subseteq Cn(\mathbf{K} - \alpha \cup \{\alpha\})$. On the other hand it follows by $-$ *closure* and $\mathbf{K} - \beta \vdash \alpha$ that $\mathbf{K} - \alpha \cup \{\alpha\} \subseteq \mathbf{K} - \alpha \cup \mathbf{K} - \beta$. From which it follows by the monotony of Cn that $\mathbf{K} \subseteq Cn(\mathbf{K} - \alpha \cup \mathbf{K} - \beta)$. □

Proof of Proposition 4.5. 1. If $\vdash \alpha \wedge \beta$, then $\vdash \alpha$ and $\vdash \beta$, from which it follows by $-$ *failure* that $\mathbf{K} - (\alpha \wedge \beta) = \mathbf{K} - \alpha = \mathbf{K} - \beta = \mathbf{K}$. If $\nvdash \alpha \wedge \beta$, then it holds, by $-$ *success* and *closure* that $\mathbf{K} - (\alpha \wedge \beta) \nvdash \alpha \wedge \beta$. Hence either $\mathbf{K} - (\alpha \wedge \beta) \nvdash \alpha$ or $\mathbf{K} - (\alpha \wedge \beta) \nvdash \beta$. By $-$ *strong conjunctive inclusion* it follows, in the first case, that $\mathbf{K} - (\alpha \wedge \beta) = \mathbf{K} - \alpha$ and, in the second case, that $\mathbf{K} - (\alpha \wedge \beta) = \mathbf{K} - \beta$.
2. Let $\alpha \notin \mathbf{K} - (\alpha \wedge \beta \wedge \delta) = \mathbf{K} - ((\alpha \wedge \delta) \wedge \beta)$. By $-$ *closure* it follows $\alpha \wedge \delta \notin \mathbf{K} - ((\alpha \wedge \delta) \wedge \beta)$. By $-$ *strong conjunctive inclusion* and *extensionality* it follows that $\mathbf{K} - (\alpha \wedge \beta \wedge \delta) = \mathbf{K} - (\alpha \wedge \delta)$. Thus $\alpha \notin \mathbf{K} - (\alpha \wedge \delta)$. □

Proof of Proposition 4.6.
1. Follows trivially.
2. Due to Proposition 4.2, $-$ satisfies failure. Then by Proposition 4.5 (1), $-$ satisfies decomposition and the rest follows trivially. □

Proof of Theorem 5.2. **Closure:** If $\|\neg\alpha\| = \emptyset$, then $\mathbf{K} \div \alpha = \bigcap \|\mathbf{K}\| = \mathbf{K}$. Thus $\mathbf{K} \div \alpha$ is a belief set. Assume now that $\|\neg\alpha\| \neq \emptyset$. It holds that $\|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$ is a set of possible worlds. Thus by Lemma 4 it follows that $\mathbf{K} \div \alpha = \bigcap(\|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}))$ is a belief set.
Success: Let $\nvdash \alpha$. Hence $\|\neg\alpha\| \neq \emptyset$. Thus by ($\mathbb{S}3$) and ($\mathbb{S}4$) it follows that there exists some minimal sphere $S_{\neg\alpha}$ that intersects with $\|\neg\alpha\|$. Hence for all $S \subsetneq S_{\neg\alpha}$ it holds that $S \cap \|\neg\alpha\| = \emptyset$. Thus $(S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}) \cap \|\neg\alpha\| \neq \emptyset$. Thus $(\|\mathbf{K}\| \cup S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}) \cap \|\neg\alpha\| \neq \emptyset$. Hence $\mathbf{K} \div \alpha = \bigcap(\|\mathbf{K}\| \cup S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})) \nvdash \alpha$.
Inclusion: If $\|\neg\alpha\| = \emptyset$, then $\mathbf{K} \div \alpha = \mathbf{K}$. If $\|\neg\alpha\| \neq \emptyset$, then it holds that $\|\mathbf{K}\| \subseteq \|\mathbf{K}\| \cup S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$. Thus $\bigcap(\|\mathbf{K}\| \cup S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})) \subseteq \bigcap \|\mathbf{K}\| = \mathbf{K}$. From which it follows that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$.
Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. Thus $\|\neg\alpha\| = \|\neg\beta\|$. If $\|\neg\alpha\| = \emptyset$, then $\|\neg\beta\| = \emptyset$. Thus $\mathbf{K} \div \alpha = \mathbf{K} \div \beta = \mathbf{K}$. If $\|\neg\alpha\| \neq \emptyset$, then $\|\neg\beta\| \neq \emptyset$. Thus $\mathbf{K} \div \alpha = \bigcap(\|\mathbf{K}\| \cup S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}))$ and $\mathbf{K} \div \beta = \bigcap(\|\mathbf{K}\| \cup S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\}))$. By Lemma 5 it follows that $S_{\neg\alpha} = S_{\neg\beta}$. From which it follows that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.

Vacuity: Let $\mathbf{K} \nvdash \alpha$. Hence $\|\mathbf{K}\| \cap \|\neg\alpha\| \neq \emptyset$. Thus $S_{\neg\alpha} = \|\mathbf{K}\|$ and $\{S : S \subsetneq S_{\neg\alpha}\} = \emptyset$ (by ($\mathbb{S}$2)). Therefore $\mathbf{K} \div \alpha = \bigcap \|\mathbf{K}\| = \mathbf{K}$.
Strong conjunctive inclusion: Assume that $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. By $\div$ *closure* (proven above) it follows that $\mathbf{K} \div (\alpha \wedge \beta) \nvdash \alpha$. Hence $\nvdash \alpha$ and consequently $\|\neg\alpha\| \neq \emptyset$ and $\|\neg(\alpha \wedge \beta)\| \neq \emptyset$. We will consider two cases:
Case 1) $\mathbf{K} \nvdash \alpha$. Hence $\mathbf{K} \nvdash (\alpha \wedge \beta)$. By *vacuity* and *inclusion* (proven above) it follows that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha = \mathbf{K}$.
Case 2) $\mathbf{K} \vdash \alpha$. Hence $\mathbf{K} \div (\alpha \wedge \beta) \neq \mathbf{K}$. From $\mathbf{K} \vdash \alpha$ it follows that $\|\mathbf{K}\| \cap \|\neg\alpha\| = \emptyset$. From $\mathbf{K} \div (\alpha \wedge \beta) \nvdash \alpha$ it follows that $(\|\mathbf{K}\| \cup S_{\neg(\alpha\wedge\beta)} \setminus \bigcup\{S : S \subsetneq S_{\neg(\alpha\wedge\beta)}\})) \cap \|\neg\alpha\| \neq \emptyset$. Thus $(\|\mathbf{K}\| \cap \|\neg\alpha\|) \cup (S_{\neg(\alpha\wedge\beta)} \setminus \bigcup\{S : S \subsetneq S_{\neg(\alpha\wedge\beta)}\} \cap \|\neg\alpha\|) \neq \emptyset$. Hence $S_{\neg(\alpha\wedge\beta)} \setminus \bigcup\{S : S \subsetneq S_{\neg(\alpha\wedge\beta)}\} \cap \|\neg\alpha\| \neq \emptyset$. Therefore $S_{\neg(\alpha\wedge\beta)} \cap \|\neg\alpha\| \neq \emptyset$. Hence $S_{\neg\alpha} \subseteq S_{\neg(\alpha\wedge\beta)}$. On the other hand it holds that $\vdash \neg\alpha \rightarrow \neg(\alpha \wedge \beta)$. Thus, by Lemma 5, $S_{\neg(\alpha\wedge\beta)} \subseteq S_{\neg\alpha}$. Therefore $S_{\neg(\alpha\wedge\beta)} = S_{\neg\alpha}$. Hence $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$.
Recuperation: Let $\mathbf{K} \div \beta \vdash \alpha$. If $\|\neg\alpha\| = \emptyset$, then $\mathbf{K} \div \alpha = \mathbf{K}$, thus $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$. By symmetry of the case it also follows that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$ if $\|\neg\beta\| = \emptyset$. Assume now that $\|\neg\alpha\| \neq \emptyset$ and $\|\neg\beta\| \neq \emptyset$. Hence $\nvdash \alpha$, from which it follows by $\div$ *success* (proven above) that $\mathbf{K} \div \alpha \neq \mathbf{K} \div \beta$. Assume that $\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta \nvdash \delta$. We intend to prove that $\delta \notin \mathbf{K}$. From $\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta \nvdash \delta$ it follows that $\|\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta\| \cap \|\neg\delta\| \neq \emptyset$. From which it follows by Lemma 2 that $\|\mathbf{K} \div \alpha\| \cap \|\mathbf{K} \div \beta\| \cap \|\neg\delta\| \neq \emptyset$. Thus there exists $w \in (\|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})) \cap (\|\mathbf{K}\| \cup (S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\})) \cap \|\neg\delta\| = (\|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\} \cap S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\})) \cap \|\neg\delta\| = (\|\mathbf{K}\| \cap \|\neg\delta\|) \cup ((S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\} \cap S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\}) \cap \|\neg\delta\|)$. Hence either $w \in \|\mathbf{K}\| \cap \|\neg\delta\|$ or $w \in (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\} \cap S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\}) \cap \|\neg\delta\|$. Assume towards a contradiction that $w \in (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\} \cap S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\}) \cap \|\neg\delta\|$. Hence $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$ and $w \in S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\beta}\}$.
On the other hand, from $\mathbf{K} \div \alpha \neq \mathbf{K} \div \beta$ it follows that $S_{\neg\alpha} \neq S_{\neg\beta}$. By condition ($\mathbb{S}$1) of Definition 2.3 it follows that either $S_{\neg\alpha} \subsetneq S_{\neg\beta}$ or $S_{\neg\beta} \subsetneq S_{\neg\alpha}$.
Assume first that $S_{\neg\alpha} \subsetneq S_{\neg\beta}$. From $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$ it follows that $w \in S_{\neg\alpha}$. Thus $w \in \bigcup\{S : S \subsetneq S_{\neg\beta}\}$. Therefore $w \notin S_{\neg\beta} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$.
Contradiction. By symmetry of the case assuming that $S_{\neg\beta} \subsetneq S_{\neg\beta}$ also leads to a contradiction. Hence $w \in \|\mathbf{K}\| \cap \|\neg\delta\|$. From which it follows that $\mathbf{K} \nvdash \delta$ and consequently that $\delta \notin \mathbf{K}$.

Assume that $\div$ satisfies *closure, inclusion, vacuity, success, extensionality, strong conjunctive inclusion* and *recuperation*. By Propositions 4.2 and 4.5 (1) it follows that $\div$ also satisfies *failure* and *decomposition*.

Let $\mathbb{S}$ be a subset of $\mathcal{M}_{\mathcal{L}}$ such that $S \in \mathbb{S}$ if and only if either:

1. $S = \mathcal{M}_{\mathcal{L}}$;
2. $S = \|\mathbf{K}\|$ or
3. $S \subseteq \{w : w \in \|\mathbf{K} \div \alpha\|$ for some $\alpha \in \mathcal{L}\}$ and $\|\mathbf{K} \div \alpha\| \subseteq S$ for all α such that $S \cap \|\neg\alpha\| \neq \emptyset$ and $\|\mathbf{K} \div \beta\| \cap S = \|\mathbf{K}\|$ for all β such that $S \cap \|\neg\beta\| = \emptyset$.

We will start by showing that $\mathbb{S}$ is a system of spheres. To do so we must verify the four conditions of Definition 2.3.
We first prove ($\mathbb{S}$2) since it is needed in the proof of ($\mathbb{S}$1).
($\mathbb{S}$2) It follows by definition of $\mathbb{S}$ that $\|\mathbf{K}\| \in \mathbb{S}$. It remains to show that $\|\mathbf{K}\| \subseteq S$ for all $S \in \mathbb{S}$. It follows trivially if either $S = \|\mathbf{K}\|$ or $S = \mathcal{M}_{\mathcal{L}}$. Assume now that $S \subseteq \{w : w \in \|\mathbf{K} \div \alpha\|$ for some $\alpha \in \mathcal{L}\}$ and $\|\mathbf{K} \div \alpha\| \subseteq S$ for all α such that $S \cap \|\neg\alpha\| \neq \emptyset$ and $\|\mathbf{K} \div \beta\| \cap S = \|K\|$ for all β such that $S \cap \|\neg\beta\| = \emptyset$.
Let $\alpha \in \mathcal{L}$. If $S \cap \|\neg\alpha\| \neq \emptyset$, then $\|\mathbf{K} \div \alpha\| \subseteq S$. By $\div$ *inclusion* it follows that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. From which it follows, by $\div$ *closure* and Lemma 2, that $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \alpha\|$. Thus $\|\mathbf{K}\| \subseteq S$. Assume now that $S \cap \|\neg\alpha\| = \emptyset$, then $\|\mathbf{K} \div \alpha\| \cap S = \|\mathbf{K}\|$. Hence $\|\mathbf{K}\| \subseteq S$.

($\mathbb{S}$1) We intend to prove that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. It follows trivially if either $S_1 = \mathcal{M}_{\mathcal{L}}$ or $S_2 = \mathcal{M}_{\mathcal{L}}$. By ($\mathbb{S}$2) proven above it also follows trivially if either $S_1 = \|\mathbf{K}\|$ or $S_2 = \|\mathbf{K}\|$. Consider that $S_1 \neq \mathcal{M}_{\mathcal{L}}$, $S_2 \neq \mathcal{M}_{\mathcal{L}}$, $S_1 \neq \|\mathbf{K}\|$ and $S_2 \neq \|\mathbf{K}\|$. Suppose towards a contradiction that $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. From $S_1 \nsubseteq S_2$ it follows that there exists $w_1 \in S_1 \setminus S_2$. Thus $w_1 \in S_1 \subseteq \{w : w \in \|\mathbf{K} \div \alpha\|$ for some $\alpha \in \mathcal{L}\}$. Thus there exists $\alpha \in \mathcal{L}$ such that $w_1 \in \|\mathbf{K} \div \alpha\|$. From $w_1 \notin S_2$ it follows that $\|\mathbf{K} \div \alpha\| \nsubseteq S_2$ and that $w_1 \notin \|\mathbf{K}\|$ (since by ($\mathbb{S}$2), proven above, $\|\mathbf{K}\| \subseteq S_2$). Thus $S_2 \cap \|\neg\alpha\| = \emptyset$. Hence $S_2 \subseteq \|\alpha\|$.
If $S_1 \cap \|\neg\alpha\| = \emptyset$, then $\|\mathbf{K} \div \alpha\| \cap S_1 = \|\mathbf{K}\|$. Contradiction, since $w_1 \in \|\mathbf{K} \div \alpha\| \cap S_1$, but $w_1 \notin \|\mathbf{K}\|$.
Consider now that $S_1 \cap \|\neg\alpha\| \neq \emptyset$. Hence $\|\mathbf{K} \div \alpha\| \subseteq S_1$ and $\nvdash \alpha$.
On the other hand, from $S_2 \nsubseteq S_1$ it follows that there exists $w_2 \in S_2 \setminus S_1$. Reasoning as above, we can conclude that $S_1 \subseteq \|\beta\|$ for some β such that $w_2 \in \mathbf{K} \div \beta$. Assuming that $S_2 \cap \|\neg\beta\| = \emptyset$ leads to a contradiction (reasoning as above). Assume now that $S_2 \cap \|\neg\beta\| \neq \emptyset$. Hence $\|\mathbf{K} \div \beta\| \subseteq S_2$.

From $\|\neg\alpha\| \subseteq \|\neg\alpha \vee \neg\beta)\| = \|\neg(\alpha \wedge \beta)\|$ and $S_1 \cap \|\neg\alpha\| \neq \emptyset$ it follows that $S_1 \cap \|\neg(\alpha \wedge \beta)\| \neq \emptyset$. Thus $\|\mathbf{K} \div (\alpha \wedge \beta)\| \subseteq S_1$. By symmetry of the case it follows from $\|\neg\beta\| \subseteq \|\neg(\alpha \wedge \beta)\|$ and $S_2 \cap \|\neg\beta\| \neq \emptyset$ that $\|\mathbf{K} \div (\alpha \wedge \beta)\| \subseteq S_2$. Hence $\|\mathbf{K} \div (\alpha \wedge \beta)\| \subseteq S_1 \cap S_2 \subseteq \|\alpha\| \cap \|\beta\| = \|\alpha \wedge \beta\|$. Thus $\|\mathbf{K} \div (\alpha \wedge \beta)\| \subseteq \|\alpha \wedge \beta\|$ and consequently $\mathbf{K} \div (\alpha \wedge \beta) \vdash \alpha \wedge \beta$. This contradicts $\div$ *success*.

($\mathbb{S}$3) It follows trivially by definition of $\mathbb{S}$.

($\mathbb{S}$4) Assume that $S \cap \|\alpha\| \neq \emptyset$. Hence $\|\alpha\| \neq \emptyset$. Thus $\nvdash \neg\alpha$. We must show that there exists some minimal sphere S' such that $S' \cap \|\alpha\| \neq \emptyset$.

By ($\mathbb{S}$2) (proven above) if $\|\mathbf{K}\| \cap \|\alpha\| \neq \emptyset$, then $S' = \|\mathbf{K}\|$. Assume now that $\|\mathbf{K}\| \cap \|\alpha\| = \emptyset$. If $\mathcal{M}_{\mathcal{L}}$ is the only sphere that intersects with $\|\alpha\|$, then $S' = \mathcal{M}_{\mathcal{L}}$. Suppose now that there exist other spheres different from $\mathcal{M}_{\mathcal{L}}$ that intersect with $\|\alpha\|$. Let S be one of such spheres. It holds that $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \div \alpha\| \text{ for some } \alpha \in \mathcal{L}\}$. It also holds that $S \cap \|\neg(\neg\alpha)\| \neq \emptyset$. Hence $\|\mathbf{K} \div \neg\alpha\| \subseteq S$. Let S'' be the intersection of all spheres S such that $S \cap \|\alpha\| \neq \emptyset$. Hence $\|\mathbf{K} \div \neg\alpha\| \subseteq S''$. It holds, for all sphere S such that $S \cap \|\alpha\| \neq \emptyset$, that $S'' \subseteq S$. Hence $S'' \subseteq S \subseteq \{w : w \in \|\mathbf{K} \div \alpha\| \text{ for some } \alpha \in \mathcal{L}\}$.
Let β be such that $S'' \cap \|\neg\beta\| \neq \emptyset$. We must prove that $\|\mathbf{K} \div \beta\| \subseteq S''$. Let S_i be such that $S_i \cap \|\alpha\| \neq \emptyset$. By definition of S'' it follows that $S'' \subseteq S_i$, thus $S_i \cap \|\neg\beta\| \neq \emptyset$. Thus $\|\mathbf{K} \div \beta\| \subseteq S_i$. Hence S'' is the intersection of spheres that contain $\|\mathbf{K} \div \beta\|$. Thus $\|\mathbf{K} \div \beta\| \subseteq S''$.
Let δ be such that $S'' \cap \|\neg\delta\| = \emptyset$. We must prove that $\|\mathbf{K} \div \delta\| \cap S'' = \|\mathbf{K}\|$.
For all spheres S it holds that $\|\mathbf{K}\| \subseteq S$. S'' is the intersection of some spheres, thus $\|\mathbf{K}\| \subseteq S''$. On the other hand it holds, by $\div$ *inclusion*, that $\mathbf{K} \div \delta \subseteq \mathbf{K}$. From which it follows by Lemma 2 and $\div$ *closure* that $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \delta\|$. Hence $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \delta\| \cap S''$. To prove the other inclusion we will consider two cases:
Case 1) $S_i \cap \|\neg\delta\| = \emptyset$ for some sphere S_i such that $S_i \cap \|\alpha\| \neq \emptyset$. Hence $\|\mathbf{K} \div \delta\| \cap S'' \subseteq \|\mathbf{K} \div \delta\| \cap S_i = \|\mathbf{K}\|$.
Case 2) $S_i \cap \|\neg\delta\| \neq \emptyset$ for all spheres S_i such that $S_i \cap \|\alpha\| \neq \emptyset$. Thus, for all S_i it holds that $\|\mathbf{K} \div \delta\| \subseteq S_i$. Hence $\|\mathbf{K} \div \delta\| \subseteq S''$. Thus $\|\mathbf{K} \div \delta\| \cap S'' = \|\mathbf{K} \div \delta\|$. On the other hand, from $S'' \cap \|\neg\delta\| = \emptyset$ it follows that $\|\mathbf{K} \div \delta\| \cap \|\neg\delta\| = \emptyset$. Thus $\mathbf{K} \div \delta \vdash \delta$. By $\div$ *success* it follows that $\vdash \delta$. Thus, by $\div$ *failure*, it follows that $\|\mathbf{K} \div \delta\| = \|\mathbf{K}\|$. Hence $\|\mathbf{K} \div \delta\| \cap S'' = \|\mathbf{K} \div \delta\| = \|\mathbf{K}\|$.
Therefore $S'' \in \mathbb{S}$ and $S'' \subseteq S$ for all $S \in \mathbb{S}$ such that $S \cap \|\alpha\| \neq \emptyset$. It remains to prove that $S'' \cap \|\alpha\| \neq \emptyset$. As seen above $\|\mathbf{K} \div \neg\alpha\| \subseteq S''$. By success it holds that $\mathbf{K} \div \neg\alpha \nvdash \neg\alpha$. Thus $\|\mathbf{K} \div \neg\alpha\| \cap \|\alpha\| \neq \emptyset$. From which it follows that $S'' \cap \|\alpha\| \neq \emptyset$.

Verification of coincidence with $\div$. If $\vdash \alpha$, then $\|\neg\alpha\| = \emptyset$. On the other hand, by $\div$ *failure*, it follows that $\mathbf{K} \div \alpha = \mathbf{K}$. Assume now that $\nvdash \alpha$. Hence $\|\neg\alpha\| \neq \emptyset$
Let

$$S_j = \bigcup\{\|\mathbf{K} \div \delta\| : \|\neg\alpha\| \subseteq \|\neg\delta\|\}.$$

We must show that 1) $S_j \in \mathbb{S}$; 2) S_j is the minimal sphere that intersects with $\|\neg\alpha\|$ (*i.e.* $S_j = S_{\neg\alpha}$); 3) $\|\mathbf{K} \div \alpha\| = \|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$ and $\mathbf{K} \div \alpha = \bigcap \|\mathbf{K} \div \alpha\|$.

1) Let $w_i \in S_j$. Hence there exists β such that $w_i \in \|\mathbf{K} \div \beta\|$ for some β such that $\|\neg\alpha\| \subseteq \|\neg\beta\|$. Thus $w_i \in \{w : w \in \|\mathbf{K} \div \alpha\| \text{ for some } \alpha \in \mathcal{L}\}$. Hence $S_j \subseteq \{w : w \in \|\mathbf{K} \div \alpha\| \text{ for some } \alpha \in \mathcal{L}\}$. Assume now that $S_j \cap \|\neg\delta\| \neq \emptyset$. We intend to prove that $\|\mathbf{K} \div \delta\| \subseteq S_j$.
Let $S_j \cap \|\neg\delta\| \neq \emptyset$. It holds that $\nvdash \delta$. By definition of S_j it follows that $\|\mathbf{K} \div \beta\| \cap \|\neg\delta\| \neq \emptyset$ for some β such that $\|\neg\alpha\| \subseteq \|\neg\beta\|$. It holds that $\|\neg\alpha\| \subseteq \|\neg\beta\| \subseteq \|\neg(\beta \wedge \delta)\|$. By definition of S_j it follows that $\|\mathbf{K} \div (\beta \wedge \delta)\| \subseteq S_j$.
Suppose that $\delta \in \mathbf{K} \div (\beta \wedge \delta)$. Hence by $\div$ *decomposition* and *success* $\mathbf{K} \div (\beta \wedge \delta) = \mathbf{K} \div \beta$. From which it follows that $\delta \in \mathbf{K} \div \beta$. Hence $\|\mathbf{K} \div \beta\| \cap \|\neg\delta\| = \emptyset$. Contradiction. Therefore, $\delta \notin \mathbf{K} \div (\beta \wedge \delta)$. By *strong conjunctive inclusion* it follows that $\mathbf{K} \div (\beta \wedge \delta) = \mathbf{K} \div \delta$. Therefore $\|\mathbf{K} \div \delta\| = \|\mathbf{K} \div (\beta \wedge \delta)\| \subseteq S_j$.
Assume now that $S_j \cap \|\neg\delta\| = \emptyset$. We intend to show that $\|\mathbf{K} \div \delta\| \cap S_j = \|\mathbf{K}\|$.
Case 1) $\|\neg\delta\| = \emptyset$. Hence $\vdash \delta$. From which it follows by $\div$ *failure* that $\mathbf{K} \div \delta = \mathbf{K}$. On the other hand it holds that $\|\mathbf{K}\| \subseteq S_j$ (since $\|\mathbf{K} \div \alpha\| \subseteq S_j$ and $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \alpha\|$). Thus $\|\mathbf{K} \div \delta\| \cap S_j = \|\mathbf{K}\|$.
Case 2) $\|\neg\delta\| \neq \emptyset$. By $\div$ *success* it follows that $\|\mathbf{K} \div \delta\| \cap \|\neg\delta\| \neq \emptyset$. Let $w \in \|\mathbf{K} \div \delta\| \cap S_j$. Hence $w \in \|\mathbf{K} \div \delta\|$ and $w \in S_j$. Hence there exists β such that $\|\neg\alpha\| \subseteq \|\neg\beta\|$ and $w \in \|\mathbf{K} \div \beta\| \subseteq S_j$. Hence $w \in \|\mathbf{K} \div \beta\| \cap \|\mathbf{K} \div \delta\|$. From $\|\mathbf{K} \div \beta\| \subseteq S_j$ it follows that $\|\mathbf{K} \div \beta\| \cap \|\neg\delta\| = \emptyset$ (since $S_j \cap \|\neg\delta\| = \emptyset$). Thus $\mathbf{K} \div \beta \vdash \delta$. By *recuperation* it follows that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \delta \cup \mathbf{K} \div \beta)$. By Lemma 2, $\|Cn(\mathbf{K} \div \delta \cup \mathbf{K} \div \beta)\| \subseteq \|\mathbf{K}\|$. By Lemma 2 and $\div$ *closure* it follows that $\|\mathbf{K} \div \delta\| \cap \|\mathbf{K} \div \beta\| = \|Cn(\mathbf{K} \div \delta \cup \mathbf{K} \div \beta)\| \subseteq \|\mathbf{K}\|$. Hence $w \in \|\mathbf{K}\|$. Thus $\|\mathbf{K} \div \delta\| \cap S_j \subseteq \|\mathbf{K}\|$. On the other hand it holds that $\|\mathbf{K}\| \subseteq S_j$ (since by $\div$ *inclusion* and *closure* $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \alpha\|$ and $\|\mathbf{K} \div \alpha\| \subseteq S_j$, by definition of S_j, letting $\delta = \alpha$). By $\div$ *inclusion* and Lemma 2 it follows that $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \delta\|$. Thus $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \delta\| \cap S_j$. Therefore $\|\mathbf{K} \div \delta\| \cap S_j = \|\mathbf{K}\|$.
Therefore $S_j \in \mathbb{S}$.

2) It follows by definition of S_j that $\|\mathbf{K} \div \alpha\| \subseteq S_j$ (by letting $\delta = \alpha$). By $\div$ *success* it follows that $\mathbf{K} \div \alpha \nvdash \alpha$. Thus $\|\mathbf{K} \div \alpha\| \cap \|\neg\alpha\| \neq \emptyset$. Hence $S_j \cap \|\neg\alpha\| \neq \emptyset$.
Let $S' \subsetneq S_j$. Therefore there exists $w \in S_j \setminus S'$. Thus there exists β such that $\|\neg\alpha\| \subseteq \|\neg\beta\|$, $w \in \|\mathbf{K} \div \beta\|$ and $\|\mathbf{K} \div \beta\| \nsubseteq S'$. It follows by definition of $\mathbb{S}$ that $S' \cap \|\neg\beta\| = \emptyset$. Therefore, $S' \cap \|\neg\alpha\| = \emptyset$. Hence S_j is the minimal sphere that intersects with $\|\neg\alpha\|$. Thus $S_{\neg\alpha} = S_j$.

3) Let $w \in \|\mathbf{K} \div \alpha\|$. If $w \in \|\mathbf{K}\|$, then $w \in \|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$. Assume now that $w \notin \|\mathbf{K}\|$. It holds that $S_{\neg\alpha} \cap \|\neg\alpha\| \neq \emptyset$. From which it follows, by definition of $\mathbb{S}$, that $\|\mathbf{K} \div \alpha\| \subseteq S_{\neg\alpha}$. Thus $w \in S_{\neg\alpha}$.
Let $S \subsetneq S_{\neg\alpha}$. Hence $S \cap \|\neg\alpha\| = \emptyset$ (since $S_{\neg\alpha}$ is the minimal sphere that intersects with $\|\neg\alpha\|$). Thus, by definition of $\mathbb{S}$, it follows that $\|\mathbf{K} \div \alpha\| \cap S = \|\mathbf{K}\|$. From $w \in \|\mathbf{K} \div \alpha\|$ and $w \notin \|\mathbf{K}\|$ it follows that $w \notin S$. Thus it holds that $w \notin S$ for all spheres S such that $S \subsetneq S_{\neg\alpha}$. Thus $w \notin \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$. Therefore $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$. Hence $\|\mathbf{K} \div \alpha\| \subseteq \|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$.

Let $w \in \|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$. Hence $w \in \|\mathbf{K}\|$ or $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$.
Case 1) $w \in \|\mathbf{K}\|$. By $\div$ *inclusion* it follows that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. From which it follows, by $\div$ *closure* and Lemma 2, that $\|\mathbf{K}\| \subseteq \|\mathbf{K} \div \alpha\|$.

Hence $w \in \|\mathbf{K} \div \alpha\|$.
Case 2) $w \notin \|\mathbf{K}\|$. Hence $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$. Thus $w \in S_{\neg\alpha}$ and $w \notin \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$. Hence there exists δ such that $w \in \|\mathbf{K} \div \delta\| \subseteq S_{\neg\alpha}$ and $\|\neg\alpha\| \subseteq \|\neg\delta\|$. From $\nvdash \alpha$ and $\|\neg\alpha\| \subseteq \|\neg\delta\|$ it follows that $\nvdash \delta$.

We will now show that $\|\mathbf{K} \div \delta\| \cap \|\neg\alpha\| \neq \emptyset$.
Let $S_{\neg\delta} = \bigcup\{\|\mathbf{K} \div \beta\| : \|\neg\delta\| \subseteq \|\neg\beta\|\}$. It holds that $S_{\neg\delta} \in \mathbb{S}$ and that $S_{\neg\delta} \subseteq S_{\neg\alpha}$ (since $\|\neg\alpha\| \subseteq \|\neg\delta\|$). It also holds that $\|\mathbf{K} \div \delta\| \subseteq S_{\neg\delta}$. Thus $S_{\neg\delta} = S_{\neg\alpha}$ (since $w \in S_{\neg\delta}$ and $w \in S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}$). Therefore $S_{\neg\delta} \cap \|\neg\alpha\| \neq \emptyset$. Hence there exists β such that $\|\neg\delta\| \subseteq \|\neg\beta\|$ and $\|\mathbf{K} \div \beta\| \cap \|\neg\alpha\| \neq \emptyset$. From $\|\neg\alpha\| \subseteq \|\neg\delta\|$ it follows that $\|\mathbf{K} \div \beta\| \cap \|\neg\delta\| \neq \emptyset$. Thus $\mathbf{K} \div \beta \nvdash \delta$. From which it follows by $\div$ *decomposition* and *success* that $\mathbf{K} \div (\beta \wedge \delta) \nvdash \delta$. So, by $\div$ *strong conjunctive inclusion* it follows that $\mathbf{K} \div (\beta \wedge \delta) = \mathbf{K} \div \delta$. On the other hand it holds that $\|\neg\delta\| \subseteq \|\neg\beta\|$. Hence $\vdash \neg\delta \rightarrow \neg\beta$. Thus $\vdash \beta \rightarrow \delta$. Hence $\vdash (\beta \wedge \delta) \leftrightarrow \beta$. From which it follows, by $\div$ *extensionality* that $\mathbf{K} \div (\beta \wedge \delta) = \mathbf{K} \div \beta$. Thus $\mathbf{K} \div \beta = \mathbf{K} \div \delta$. Hence $\|\mathbf{K} \div \delta\| \cap \|\neg\alpha\| \neq \emptyset$. Thus $\mathbf{K} \div \delta \nvdash \alpha$. Thus by $\div$ *decomposition* and *success* it follows that $\mathbf{K} \div (\alpha \wedge \delta) \nvdash \alpha$. Thus by $\div$ *strong conjunctive inclusion* it follows that $\mathbf{K} \div (\alpha \wedge \delta) = \mathbf{K} \div \alpha$. On the other hand it holds that $\|\neg\alpha\| \subseteq \|\neg\delta\|$. Hence $\vdash \neg\alpha \rightarrow \neg\delta$. Thus $\vdash \delta \rightarrow \alpha$. Hence $\vdash (\alpha \wedge \delta) \leftrightarrow \delta$. From which it follows, by $\div$ *extensionality* that $\mathbf{K} \div (\alpha \wedge \delta) = \mathbf{K} \div \delta$. Thus $\mathbf{K} \div \alpha = \mathbf{K} \div \delta$. Hence $w \in \|\mathbf{K} \div \alpha\|$. Therefore, $\|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\}) \subseteq \|\mathbf{K} \div \alpha\|$.
Hence $\|\mathbf{K} \div \alpha\| = \|\mathbf{K}\| \cup (S_{\neg\alpha} \setminus \bigcup\{S : S \subsetneq S_{\neg\alpha}\})$.
On the other hand, by $\div$ *closure* and Lemma 3, it follows that $\mathbf{K} \div \alpha = \bigcap \|\mathbf{K} \div \alpha\|$. $\square$

Proof of Proposition 6.1. Let $\mathbf{K}$ be a belief set and $\div$ be a ring withdrawal on $\mathbf{K}$ that satisfies ($\dot{-}$10). By Theorem 5.2 it holds that $\div$ satisfies *closure, inclusion, vacuity, success, extensionality, strong conjunctive inclusion* and *recuperation*. By Proposition 4.2 it follows that $\div$ also satisfies *failure*. Let $\alpha \in \mathcal{L}$. If $\mathbf{K} = Cn(\emptyset)$, then by $\div$ *inclusion* and *closure* it follows that $\mathbf{K} \div \alpha = Cn(\emptyset) = \mathbf{K}$. Assume now that $\mathbf{K} \neq Cn(\emptyset)$. If $\vdash \alpha$, then by $\div$ *failure* it follows that $\mathbf{K} \div \alpha = \mathbf{K}$. If $\alpha \notin \mathbf{K}$, then by $\div$ *vacuity* and *inclusion* it follows that $\mathbf{K} \div \alpha = \mathbf{K}$. Consider now that $\nvdash \alpha$ and $\alpha \in \mathbf{K}$. Assume towards a contradiction that $\mathbf{K} \div \alpha \neq Cn(\emptyset)$. By $\div$ *closure* it holds that $Cn(\emptyset) \subsetneq \mathbf{K} \div \alpha$. Hence there exists $\beta \in \mathbf{K} \div \alpha$ such that $\nvdash \beta$. By ($\dot{-}$10) it follows that $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$. On the other hand, by *recuperation* it follows that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$. Thus $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha) = \mathbf{K} \div \alpha$. This contradicts $\div$ *success*, since $\alpha \in \mathbf{K}$. $\square$

Proof of Proposition 7.3. It is sufficient to show that it holds that $\beta \in \mathbf{K} \div (\alpha \wedge \beta)$ iff $\beta \in \bigcap\{\mathbf{K} \div \delta : \vdash \delta \rightarrow \alpha\}$, whenever $\nvdash \alpha$. Let $\beta \in \bigcap\{\mathbf{K} \div \delta : \vdash \delta \rightarrow \alpha\}$. It holds that $\vdash (\alpha \wedge \beta) \rightarrow \alpha$, from which it follows that $\beta \in \mathbf{K} \div (\alpha \wedge \beta)$. For the other direction, assume that $\beta \in \mathbf{K} \div (\alpha \wedge \beta)$ and let δ be such that $\vdash \delta \rightarrow \alpha$. From the former it follows by *conjunctive trisection* (Proposition 4.5 (2)) that $\beta \in \mathbf{K} \div (\alpha \wedge \beta \wedge \delta)$. By $\div$ *success* it follows that $\alpha \wedge \delta \notin \mathbf{K} \div (\alpha \wedge \beta \wedge \delta)$. Hence, by $\div$ *closure* it follows that $\delta \notin \mathbf{K} \div (\alpha \wedge \beta \wedge \delta)$ (since $\vdash \delta \rightarrow \alpha$). Thus, by $\div$ *strong disjunctive inclusion* it follows that $\mathbf{K} \div (\alpha \wedge \beta \wedge \delta) = \mathbf{K} \div \delta$. Hence $\beta \in \mathbf{K} \div \delta$. From the arbitrariness of δ it follows that $\beta \in \bigcap\{\mathbf{K} \div \delta : \vdash \delta \rightarrow \alpha\}$. $\square$

Proof of Proposition 7.4. Let $\dot{-}$ be a severe withdrawal operator on $\mathbf{K}$ and $\div$ be obtained from $\dot{-}$ by (Def $\div$ from $\dot{-}$). We first show that $\div$ is a ring withdrawal. To do that (according to Theorem 5.2) it is sufficient to show that $\div$ satisfies *closure, inclusion, vacuity, success, extensionality, strong conjunctive inclusion* and *recuperation*.
Closure: Follows by (Def $\div$ from $\dot{-}$) and the fact that the intersection of belief sets is a belief set.
Inclusion: Follows by (Def $\div$ from $\dot{-}$).
Vacuity: Let $\alpha \notin \mathbf{K}$. By $\dot{-}$ *vacuity* and *inclusion* it follows that $\mathbf{K} \dot{-} \alpha = \mathbf{K}$. Thus, for all β it holds that $\mathbf{K} \subseteq (\mathbf{K} \dot{-} \alpha + \neg\beta) \cap \mathbf{K}$. Thus by (Def $\div$ from $\dot{-}$) $\mathbf{K} \subseteq \mathbf{K} \div \alpha$.
Success: Let $\nvdash \alpha$. Suppose towards a contradiction that $\alpha \in \mathbf{K} \div \alpha$. Hence by (Def $\div$ from $\dot{-}$) it follows that $\alpha \in \mathbf{K} \dot{-} \alpha + \neg\alpha$ (by letting $\delta = \alpha$). From which it follows by deduction and $\dot{-}$ *closure* that $\neg\alpha \rightarrow \alpha \in \mathbf{K} \dot{-} \alpha$. Thus, by $\dot{-}$ *closure*, $\alpha \in \mathbf{K} \dot{-} \alpha$. This contradicts, $\dot{-}$ *success*.
Extensionality: Follows by (Def $\div$ from $\dot{-}$) and $\dot{-}$ *extensionality*.
Strong conjunctive inclusion: Let $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. We will consider two cases:
Case 1) $\alpha \notin \mathbf{K}$. Hence $\alpha \wedge \beta \notin \mathbf{K}$. From which it follows by $\div$ *inclusion* and *vacuity* that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K}$.
Case 2) $\alpha \in \mathbf{K}$. By (Def $\div$ from $\dot{-}$) it follows that there exists δ such that $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \delta$ and $\alpha \notin (\mathbf{K} \dot{-} (\alpha \wedge \beta) + \neg\delta) \cap \mathbf{K}$. Thus $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta) + \neg\delta$. Therefore $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$. It holds, by Proposition 6.2, that severe withdrawals satisfy *strong conjunctive inclusion*, thus $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \alpha$. From which it follows by (Def $\div$ from $\dot{-}$) that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$.
Recuperation: Let $\mathbf{K} \div \beta \vdash \alpha$. As proven above $\div$ satisfies *closure, failure, vacuity* and *success*.
If $\vdash \alpha$, then by $\dot{-}$ *failure* it follows that $\mathbf{K} \dot{-} \alpha = \mathbf{K}$. Thus $(\mathbf{K} \dot{-} \alpha + \neg\delta) \cap \mathbf{K} = \mathbf{K}$, for all δ. From which it follows by (Def $\div$ from $\dot{-}$) that $\mathbf{K} \div \alpha = \mathbf{K}$. By symmetry of the case it holds that $\mathbf{K} \div \beta = \mathbf{K}$ if $\vdash \beta$. Hence it holds that if $\vdash \alpha$ or $\vdash \beta$ that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$.
It also holds that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$ if either $\alpha \notin \mathbf{K}$ or $\beta \notin \mathbf{K}$ (by $\div$ *vacuity*). Assume now that $\nvdash \alpha$, $\nvdash \beta$, $\alpha \in \mathbf{K}$ and $\beta \in \mathbf{K}$.
By $\dot{-}$ *linearity* it follows that either $\mathbf{K} \dot{-} \alpha \subseteq \mathbf{K} \dot{-} \beta$ or $\mathbf{K} \dot{-} \beta \subseteq \mathbf{K} \dot{-} \alpha$.
Assume towards a contradiction that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \beta$. Hence, by (Def $\div$ from $\dot{-}$) it follows that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. Hence, by $\div$ *closure*, $\alpha \in \mathbf{K} \div \alpha$. This contradicts $\div$ *success*. Hence $\mathbf{K} \dot{-} \alpha \neq \mathbf{K} \dot{-} \beta$. We will consider two cases:
Case 1) $\mathbf{K} \dot{-} \alpha \subsetneq \mathbf{K} \dot{-} \beta$. Let δ be such that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \delta$. Hence $\mathbf{K} \dot{-} \delta \subsetneq \mathbf{K} \dot{-} \beta$. From which it follows that $\mathbf{K} \dot{-} \beta \nsubseteq \mathbf{K} \dot{-} \delta$. By ($\dot{-}$9) it follows that $\delta \in \mathbf{K} \dot{-} \beta$. By $\dot{-}$ *inclusion* it follows that $\delta \in \mathbf{K}$. Thus $\delta \in \bigcap\{(\mathbf{K} \dot{-} \beta + \neg\xi) \cap \mathbf{K} : \mathbf{K} \dot{-} \beta = \mathbf{K} \dot{-} \xi\}$. Thus $\delta \in \mathbf{K} \div \beta$.
Hence $\{\delta : \mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \delta\} \subseteq \mathbf{K} \div \beta$.
Let η be such that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \eta$.
Let $\theta \in (\mathbf{K} \cap (\mathbf{K} \dot{-} \eta + \neg\eta)) \cup \{\eta\}$. Hence $\theta \in \mathbf{K} \cap (\mathbf{K} \dot{-} \eta + \neg\eta)$ or $\theta = \eta$. If for all η such that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \eta$, it holds that $\theta \in \mathbf{K} \cap (\mathbf{K} \dot{-} \eta + \neg\eta)$ then $\theta \in \bigcap\{(\mathbf{K} \dot{-} \alpha + \neg\beta) \cap \mathbf{K} : \mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \beta\}$. If for some η such that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \eta$, it holds that $\theta \notin \mathbf{K} \cap (\mathbf{K} \dot{-} \eta + \neg\eta)$, then $\theta = \eta \in \{\delta : \mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \delta\}$. Hence, for all η such that $\mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \eta$ it holds that $(\mathbf{K} \cap (\mathbf{K} \dot{-} \eta + \neg\eta)) \cup \{\eta\} \subseteq \bigcap\{(\mathbf{K} \dot{-} \alpha + \neg\xi) \cap \mathbf{K} : \mathbf{K} \dot{-} \alpha = \mathbf{K} \dot{-} \xi\} \cup \{\delta :$

$\mathbf{K}\ddot{-}\alpha = \mathbf{K}\ddot{-}\delta\} \subseteq \mathbf{K} \div \alpha \cup \mathbf{K} \div \beta$.
On the other hand, by Lemma 8 it follows that $\mathbf{K} \subseteq Cn((\mathbf{K} \cap (\mathbf{K}\ddot{-}\eta + \neg\eta)) \cup \{\eta\})$. Therefore, $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$.
Case 2) $\mathbf{K}\ddot{-}\beta \subsetneq \mathbf{K}\ddot{-}\alpha$. This case is symmetric with the previous one.

Let β be such that $\mathbf{K}\ddot{-}\alpha = \mathbf{K}\ddot{-}\beta$. It holds that $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K}\ddot{-}\alpha + \neg\beta$. By $\ddot{-}$ *inclusion*, $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K}$. Thus $\mathbf{K}\ddot{-}\alpha \subseteq (\mathbf{K}\ddot{-}\alpha + \neg\beta) \cap \mathbf{K}$. From the arbitrariness of β it follows that $\mathbf{K}\ddot{-}\alpha \subseteq \bigcap\{(\mathbf{K}\ddot{-}\alpha + \neg\beta) \cap \mathbf{K} : \mathbf{K}\ddot{-}\alpha = \mathbf{K}\ddot{-}\beta\} = \mathbf{K} \div \alpha$.

It remains to prove that $\ddot{-}$ and $\div$ are revision equivalent, i.e. that $Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\}) = Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. We will prove by double inclusion. From $\mathbf{K}\ddot{-}\neg\alpha \subseteq \mathbf{K} \div \neg\alpha$ it follows by the monotony of Cn that $Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. For the other inclusion, let $\beta \in Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. By deduction it follows that $\alpha \to \beta \in \mathbf{K} \div \neg\alpha$. By (Def $\div$ from $\ddot{-}$) it follows that $\alpha \to \beta \in Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\neg\neg\alpha\})$. Thus $\beta \in Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\})$. □

Proof of Proposition 7.5. Let $\div$ be a ring withdrawal operator on $\mathbf{K}$ and $\ddot{-}$ be obtained from $\div$ by (Def $\ddot{-}$ from $\div$). We first show that $\ddot{-}$ is a severe withdrawal. To do that it is sufficient to show that $\ddot{-}$ satisfies *closure, inclusion, vacuity, success, failure, conjunctive inclusion* and ($\ddot{-}7a$).
It holds that (Def $\ddot{-}$ from $\div$) and (Def' $\ddot{-}$ from $\div$) are equivalent, by Proposition 7.3, so we can make use of these two conditions to simplify the proof.
Closure: It holds that the intersection of belief sets is a belief set, thus by (Def' $\ddot{-}$ from $\div$) it follows that $\ddot{-}$ satisfies *closure*.
Success: Let $\nvdash \alpha$. It holds that $\vdash \alpha \to \alpha$. Thus, by (Def' $\ddot{-}$ from $\div$) it follows that $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \div \alpha$. By $\div$ *success* it follows that $\alpha \notin \mathbf{K} \div \alpha$, from which it follows that $\alpha \notin \mathbf{K}\ddot{-}\alpha$.
Vacuity: Assume that $\alpha \notin \mathbf{K}$. Let $\beta \in \mathbf{K}$. Hence $\alpha \wedge \beta \notin \mathbf{K}$. Hence, by $\div$ *inclusion* and *vacuity* $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K}$. From which it follows by (Def $\ddot{-}$ from $\div$) that $\beta \in \mathbf{K}\ddot{-}\alpha$. Thus $\mathbf{K} \subseteq \mathbf{K}\ddot{-}\alpha$.
Inclusion: Follows by (Def $\ddot{-}$ from $\div$) and $\div$ *inclusion*.
Failure: Follows by (Def $\ddot{-}$ from $\div$).
Conjunctive inclusion: Assume that $\alpha \notin \mathbf{K}\ddot{-}(\alpha \wedge \beta)$. Hence $\nvdash \alpha$. By (Def $\ddot{-}$ from $\div$) it follows $\alpha \notin \mathbf{K} \div ((\alpha \wedge \beta) \wedge \alpha)$. Thus, by $\div$ *extensionality* $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. Let $\delta \in \mathbf{K}\ddot{-}(\alpha \wedge \beta)$. By (Def $\ddot{-}$ from $\div$) it follows that $\delta \in \mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta)$. By $\div$ *success* it follows that $\alpha \wedge \beta \notin \mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta)$. By $\div$ *strong conjunctive inclusion* it follows that $\mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta) = \mathbf{K} \div (\alpha \wedge \beta)$. Thus $\alpha \notin \mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta)$. By $\div$ *closure* it follows that $\alpha \wedge \delta \notin \mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta)$. So, by $\div$ *extensionality* and *strong conjunctive inclusion* it follows $\mathbf{K} \div ((\alpha \wedge \beta) \wedge \delta) = \mathbf{K} \div (\alpha \wedge \delta)$. Thus $\delta \in \mathbf{K} \div (\alpha \wedge \delta)$. Therefore, $\delta \in \mathbf{K}\ddot{-}\alpha$, by (Def $\ddot{-}$ from $\div$).
($\ddot{-}$7a): Let $\nvdash \alpha$. Let $\delta \in \mathbf{K}\ddot{-}\alpha$. By (Def $\ddot{-}$ from $\div$) it follows that $\delta \in \mathbf{K} \div (\alpha \wedge \delta)$. Hence, by $\div$ *conjunctive trisection* (Proposition 4.5 (2)) it follows that $\delta \in \mathbf{K} \div (\alpha \wedge \delta \wedge \beta)$. From which it follows, by $\div$ *extensionality* and (Def $\ddot{-}$ from $\div$), that $\delta \in \mathbf{K}\ddot{-}(\alpha \wedge \beta)$.

If $\vdash \alpha$, then by $\ddot{-}$ and $\div$ *failure* (Proposition 4.2) it follows that $\mathbf{K}\ddot{-}\alpha = \mathbf{K} \div \alpha = \mathbf{K}$. Consider now that $\nvdash \alpha$. Hence $\mathbf{K}\ddot{-}\alpha = \bigcap\{\mathbf{K} \div \beta : \vdash \beta \to \alpha\}$. It holds that $\mathbf{K} \div \alpha \in \{\mathbf{K} \div \beta : \ \vdash \beta \to \alpha\}$. Thus $\mathbf{K}\ddot{-}\alpha = \bigcap\{\mathbf{K} \div \beta : \ \vdash \beta \to \alpha\} \subseteq \mathbf{K} \div \alpha$.

It remains to prove that $\ddot{-}$ and $\div$ are revision equivalent, i.e. that $Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\}) = Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\})$. We will prove by double inclusion. From $\mathbf{K}\ddot{-}\alpha \subseteq \mathbf{K} \div \alpha$ it follows by monotony that $Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. For the other inclusion, let $\beta \in Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. By deduction and $\div$ *closure* it follows that $\alpha \to \beta \in \mathbf{K} \div \neg\alpha$. Thus by $\div$ *extensionality* $\alpha \to \beta \in \mathbf{K} \div (\neg\alpha \wedge (\alpha \to \beta))$. Hence, by (Def $\ddot{-}$ from $\div$), $\alpha \to \beta \in \mathbf{K}\ddot{-}\neg\alpha$. From which it follows by deduction, that $\beta \in Cn((\mathbf{K}\ddot{-}\neg\alpha) \cup \{\alpha\})$. □

Proof of Proposition 7.6. Let $\dot{-}$ be an AGM contraction operator on $\mathbf{K}$ and $\div$ be obtained from $\dot{-}$ by (Def $\div$ from $\dot{-}$). We first show that $\div$ is a ring withdrawal. According to Theorem 5.2 it is sufficient to show that $\div$ satisfies *closure, inclusion, vacuity, success, extensionality, strong conjunctive inclusion* and *recuperation*.
Closure: Follows by (Def $\div$ from $\dot{-}$) and the fact that the intersection of belief set is a belief set ($\dot{-}$ satisfies *closure*).
Inclusion: Follows by (Def $\div$ from $\dot{-}$) and $\dot{-}$ *inclusion*.
Success: Let $\nvdash \alpha$. If $\alpha \notin \mathbf{K}$, then by (Def $\div$ from $\dot{-}$) it follows that $\mathbf{K} \div \alpha = \mathbf{K} \nvdash \alpha$. Assume now that $\alpha \in \mathbf{K}$. It holds, by $\dot{-}$ *success* and *extensionality* that $\mathbf{K} \dot{-} (\alpha \wedge \alpha) \cap \{\alpha, \alpha\} = \emptyset$. Thus $\alpha \notin \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \beta) : \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \{\alpha, \beta\} = \emptyset\}$. From which it follows, by (Def $\div$ from $\dot{-}$), that $\alpha \notin \mathbf{K} \div \alpha$.
Vacuity: Let $\alpha \notin \mathbf{K}$. It follows by (Def $\div$ from $\dot{-}$) that $\mathbf{K} \div \alpha = \mathbf{K}$.
Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. It follows that if $\vdash \alpha$ or $\alpha \notin \mathbf{K}$ that $\mathbf{K} \div \alpha = \mathbf{K}$. On the other hand, in the former case it follows that $\vdash \beta$ and in the latter that $\beta \notin \mathbf{K}$. In both those cases it holds that $\mathbf{K} \div \beta = \mathbf{K}$. Thus $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.
Assume now that $\nvdash \alpha$ and $\alpha \in \mathbf{K}$. Hence $\nvdash \beta$ and $\beta \in \mathbf{K}$.
Let $\delta \in \mathbf{K} \div \alpha$. Hence, by (Def $\div$ from $\dot{-}$) it follows that $\delta \in \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \eta) : \mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset\}$.
Let ξ be such that $\mathbf{K} \dot{-} (\beta \wedge \xi) \cap \{\beta, \xi\} = \emptyset$. By $\dot{-}$ *extensionality* and *closure* it follows that $\mathbf{K} \dot{-} (\alpha \wedge \xi) \cap \{\alpha, \xi\} = \emptyset$. Thus $\delta \in \mathbf{K} \dot{-} (\alpha \wedge \xi)$. Thus, by $\dot{-}$ *extensionality*, $\delta \in \mathbf{K} \dot{-} (\beta \wedge \xi)$. From the arbitrariness of ξ it follows that $\delta \in \bigcap\{\mathbf{K} \dot{-} (\beta \wedge \xi) : \mathbf{K} \dot{-} (\alpha \wedge \xi) \cap \{\beta, \xi\} = \emptyset\}$. Thus $\delta \in \mathbf{K} \div \beta$. Therefore $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \beta$. By symmetry of the case it follows that $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$. Hence $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.
Strong conjunctive inclusion: Let $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. Hence $\nvdash \alpha$. If $\alpha \notin \mathbf{K}$, then $\alpha \wedge \beta \notin \mathbf{K}$. From which it follows, by $\div$ *inclusion* and *vacuity* (proven above) that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} = \mathbf{K} \div \alpha$.
It also holds that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$, by $\div$ *extensionality*, if $\vdash \beta$. Assume now that $\alpha \in \mathbf{K}$ and $\nvdash \beta$.
From $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$, by (Def $\div$ from $\dot{-}$), that there exists ξ such that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \xi) \cap \{\alpha \wedge \beta, \xi\} = \emptyset$ and $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \xi)$.

By $\dot{-}$ *conjunctive trisection* (Lemma 7) it follows that $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$ and $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \xi)$. Let δ be such that $\mathbf{K} \dot{-} (\alpha \wedge \delta) \cap \{\alpha, \delta\} = \emptyset$. By $\dot{-}$ *conjunctive factoring* (Lemma 1) and *extensionality* it follows that $\mathbf{K} \dot{-} (\alpha \wedge \delta \wedge \xi)$ is either $\mathbf{K} \dot{-} (\alpha \wedge \delta)$, $\mathbf{K} \dot{-} (\alpha \wedge \xi)$ or $\mathbf{K} \dot{-} (\alpha \wedge \delta) \cap \mathbf{K} \dot{-} (\alpha \wedge \xi)$. Thus $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \delta \wedge \xi)$. From which it follows, by $\dot{-}$ *closure* that $\alpha \wedge \delta \notin \mathbf{K} \dot{-} (\alpha \wedge \delta \wedge \xi)$. Thus, by $\dot{-}$ *conjunctive inclusion*, $\mathbf{K} \dot{-} (\alpha \wedge \delta \wedge \xi) \subseteq \mathbf{K} \dot{-} (\alpha \wedge \delta)$.
Let $\theta \in \mathbf{K} \div (\alpha \wedge \beta)$. Hence, by (Def $\div$ from $\dot{-}$) it follows that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \xi)$. We will consider two cases:
Case 1) $\delta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. From $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$ and $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \delta)$ it follows by $\dot{-}$ *conjunctive factoring* that $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. Thus by $\dot{-}$ *closure* $\alpha \wedge \beta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. From $\theta \in \mathbf{K} \div (\alpha \wedge \beta)$ it follows by (Def $\div$ from $\dot{-}$) that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. From $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$ it follows by $\dot{-}$ *closure* that $\alpha \wedge \delta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. From which it follows, by $\dot{-}$ *conjunctive inclusion* that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta) \subseteq \mathbf{K} \dot{-} (\alpha \wedge \delta)$. Thus $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$.
Case 2) $\delta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta)$. By $\dot{-}$ *conjunctive trisection* it follows that $\delta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi)$. From which it follows by $\dot{-}$ *success* that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi) \nsubseteq \mathbf{K} \dot{-} \delta$, thus by *conjunctive factoring* and *extensionality* $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi) = \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \xi)$. Thus $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi)$. On the other hand, from $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$ and $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \delta \wedge \xi)$ it follows by $\dot{-}$ *conjunctive factoring* and *extensionality* that $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi)$. Thus by $\dot{-}$ *closure* $\alpha \wedge \delta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi)$. From which it follows by $\dot{-}$ *conjunctive inclusion* that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \delta \wedge \xi) \subseteq \mathbf{K} \dot{-} (\alpha \wedge \delta)$. Therefore $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$.
Therefore, in both cases it holds that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \delta)$. From the arbitrariness of δ it follows that $\theta \in \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \delta) : \mathbf{K} \dot{-} (\alpha \wedge \delta) \cap \{\alpha, \delta\} = \emptyset\} = \mathbf{K} \div \alpha$. Thus $\mathbf{K} \div (\alpha \wedge \beta) \subseteq \mathbf{K} \div \alpha$.
For the other inclusion, let $\theta \in \mathbf{K} \div \alpha$. It holds for all ϵ such that $\mathbf{K} \dot{-} (\alpha \wedge \epsilon) \cap \{\alpha, \epsilon\} = \emptyset$ that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \epsilon)$. Let η be such that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta) \cap \{\alpha \wedge \beta, \eta\} = \emptyset$. We intend to prove that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$.
By $\dot{-}$ *conjunctive inclusion* and *conjunctive overlap* it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta) = \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \mathbf{K} \dot{-} \eta$. From $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$ it follows that $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$. On the other hand, it also holds that $\eta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$. By $\dot{-}$ *conjunctive trisection* it follows from the latter that $\eta \notin \mathbf{K} \dot{-} (\alpha \wedge \eta)$ and from the former that $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \eta)$. Thus by hypothesis $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \eta)$. On the other hand, from $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \eta)$ it follows by $\dot{-}$ *conjunctive inclusion* that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \subseteq \mathbf{K} \dot{-} \alpha$. Thus $\theta \in \mathbf{K} \dot{-} \alpha$. From $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$ it follows by *closure* that $\alpha \wedge \beta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$ and $\alpha \wedge \eta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$. Thus by $\dot{-}$ *conjunctive inclusion* and *conjunctive overlap* it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta) = \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \mathbf{K} \dot{-} (\alpha \wedge \eta)$. We will consider two cases:
Case 1) $\beta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$. By hypothesis it follows that $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$. Thus $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$.
Case 2) $\beta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$. By $\dot{-}$ *conjunctive factoring* and *success* it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \alpha$. Thus $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$. Thus $\theta \in \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta)$.
From the arbitrariness of η it follows that $\theta \in \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta) : \mathbf{K} \dot{-} (\alpha \wedge \beta \wedge \eta) \cap \{\alpha \wedge \beta, \eta\} = \emptyset\} = \mathbf{K} \div (\alpha \wedge \beta)$. Thus $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div (\alpha \wedge \beta)$.
Recuperation: Assume that $\mathbf{K} \div \beta \vdash \alpha$. If $\vdash \alpha$ or $\alpha \notin \mathbf{K}$, then by (Def $\div$ from $\dot{-}$) it follows that $\mathbf{K} \div \alpha = \mathbf{K}$. Thus $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$. By symmetry of the case, the same also holds if either $\vdash \beta$ or $\beta \notin \mathbf{K}$. Assume now that $\nvdash \alpha$, $\alpha \in \mathbf{K}$, $\nvdash \beta$ and $\beta \in \mathbf{K}$. By $\dot{-}$ *success* it follows that $\alpha \wedge \beta \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$. By (Def $\div$ from $\dot{-}$) it follows that $\mathbf{K} \div \alpha = \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \xi) : \mathbf{K} \dot{-} (\alpha \wedge \xi) \cap \{\alpha, \xi\} = \emptyset\}$ and $\mathbf{K} \div \beta = \bigcap\{\mathbf{K} \dot{-} (\beta \wedge \delta) : \mathbf{K} \dot{-} (\beta \wedge \delta) \cap \{\beta, \delta\} = \emptyset\}$.
If it holds that $\eta \in \mathbf{K} \div \beta$ for all η such that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$, then by $\div$ *closure* $\alpha \wedge \eta \in \mathbf{K} \div \beta$, for all η such that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$.
By $\dot{-}$ *recovery* it follows that $\mathbf{K} \subseteq Cn(\mathbf{K} \dot{-} (\alpha \wedge \eta) \cup \{\alpha \wedge \eta\})$, for all η such that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$. Thus $\mathbf{K} \subseteq \bigcap\{Cn(\mathbf{K} \dot{-} (\alpha \wedge \eta) \cup \{\alpha \wedge \eta\}) : \mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset\}$.
Let $\theta \in \mathbf{K}$. Hence it holds that $\theta \in \bigcap\{Cn(\mathbf{K} \dot{-} (\alpha \wedge \eta) \cup \{\alpha \wedge \eta\}) : \mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset\}$. Hence $(\alpha \wedge \eta) \to \theta \in \mathbf{K} \dot{-} (\alpha \wedge \eta)$, for all η such that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$. Thus $(\alpha \wedge \eta) \to \theta \in \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \eta) : \mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset\}$. Thus $(\alpha \wedge \eta) \to \theta \in \mathbf{K} \div \alpha$. Hence $(\alpha \wedge \eta) \to \theta \in \mathbf{K} \div \alpha \cup \mathbf{K} \div \beta$. Therefore, $\theta \in Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$.
Assume now that $\eta \notin \mathbf{K} \div \beta$ for some η such that $\mathbf{K} \dot{-} (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$. Hence $\mathbf{K} \div (\alpha \wedge \eta) \cap \{\alpha, \eta\} = \emptyset$. From which it follows by $\div$ *strong conjunctive inclusion* (proven above) that $\mathbf{K} \div (\alpha \wedge \eta) = \mathbf{K} \div \alpha = \mathbf{K} \div \eta$. On the other hand, from $\eta \notin \mathbf{K} \div \beta$ it follows by $\div$ *success* and *strong conjunctive inclusion* that $\mathbf{K} \div (\beta \wedge \eta) = \mathbf{K} \div \eta$.
If $\beta \notin \mathbf{K} \div (\beta \wedge \eta)$, then by $\div$ *strong conjunctive inclusion* $\mathbf{K} \div (\beta \wedge \eta) = \mathbf{K} \div \beta = \mathbf{K} \div \eta = \mathbf{K} \div \alpha$. This contradicts $\div$ *success*. Thus $\beta \in \mathbf{K} \div (\beta \wedge \eta) = \mathbf{K} \div \alpha$.
Let δ be such that $\mathbf{K} \dot{-} (\beta \wedge \delta) \cap \{\beta, \delta\} = \emptyset$. Hence $\mathbf{K} \div (\beta \wedge \delta) \cap \{\beta, \delta\} = \emptyset$. Thus, by $\div$ *strong conjunctive inclusion* and *success* $\mathbf{K} \div \beta = \mathbf{K} \div \delta$. If for all such δ it holds that $\delta \in \mathbf{K} \div \alpha$, then reasoning as above we can conclude that $\mathbf{K} \subseteq Cn(\mathbf{K} \div \alpha \cup \mathbf{K} \div \beta)$. Assume towards a contradiction that it is not the case. Hence there exists δ such that $\mathbf{K} \dot{-} (\beta \wedge \delta) \cap \{\beta, \delta\} = \emptyset$ and $\delta \notin \mathbf{K} \div \alpha$. Thus by $\div$ *success* and *strong conjunctive inclusion* (proven above) $\mathbf{K} \div (\alpha \wedge \delta) = \mathbf{K} \div \delta$. If $\alpha \notin \mathbf{K} \div (\alpha \wedge \delta)$, then by $\div$ *strong conjunctive inclusion* it follows that $\mathbf{K} \div (\alpha \wedge \delta) = \mathbf{K} \div \alpha$. Thus $\mathbf{K} \div \beta = \mathbf{K} \div \alpha$. This contradicts $\div$ *success*. Assume now that $\alpha \in \mathbf{K} \div (\alpha \wedge \delta)$. Then by $\div$ *conjunctive trisection* (Proposition 4.5 (2)) it follows from the latter and $\beta \in \mathbf{K} \div (\beta \wedge \eta)$ that $\{\alpha, \beta\} \subseteq \mathbf{K} \div (\alpha \wedge \beta \wedge \delta \wedge \eta)$. Thus by $\div$ *closure* $\alpha \wedge \beta \in \mathbf{K} \div (\alpha \wedge \beta \wedge \delta \wedge \eta)$. By $\div$ *success* it follows that $\delta \wedge \eta \notin \mathbf{K} \div (\alpha \wedge \beta \wedge \delta \wedge \eta)$. From which it follows by $\div$ *strong conjunctive inclusion* that $\mathbf{K} \div (\alpha \wedge \beta \wedge \delta \wedge \eta) = \mathbf{K} \div (\delta \wedge \eta)$. By $\div$ *decomposition* (Proposition 4.5 (1)) it follows that either $\mathbf{K} \div (\delta \wedge \eta) = \mathbf{K} \div \delta$ or $\mathbf{K} \div (\delta \wedge \eta) = \mathbf{K} \div \eta$. Hence either $\alpha \wedge \beta \in \mathbf{K} \div \delta = \mathbf{K} \div \beta$ or $\alpha \wedge \beta \in \mathbf{K} \div \eta = \mathbf{K} \div \alpha$. Both cases contradict $\div$ *success*.

If $\vdash \alpha$ or $\alpha \notin \mathbf{K}$, then by $\dot{-}$ and *failure* (Propositions 4.2 and 4.3), *inclusion* and *vacuity* it follows that $\mathbf{K} \dot{-} \alpha = \mathbf{K}$. Thus $\mathbf{K} \dot{-} \alpha = \mathbf{K} \div \alpha$. Assume now that $\nvdash \alpha$ and $\alpha \in \mathbf{K}$. It holds that $\mathbf{K} \dot{-} \alpha \in \{\mathbf{K} \dot{-} (\alpha \wedge \beta) : \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \{\alpha, \beta\} = \emptyset\}$ (choosing $\beta = \alpha$ by *extensionality* and *success*). Thus $\mathbf{K} \div \alpha = \bigcap\{\mathbf{K} \dot{-} (\alpha \wedge \beta) : \mathbf{K} \dot{-} (\alpha \wedge \beta) \cap \{\alpha, \beta\} = \emptyset\} \subseteq \mathbf{K} \dot{-} \alpha$.

It remains to prove that $\dot{-}$ and $\div$ are revision equivalent, i.e. that $Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\}) = Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. We will prove by double inclusion.
From $\mathbf{K} \div \neg\alpha \subseteq \mathbf{K} \dot{-} \neg\alpha$ it follows by monotony that $Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$. For the other inclusion, let $\beta \in$

$Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$. If $\vdash \neg\alpha$ or $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} \div \neg\alpha = \mathbf{K}$. Thus by $\dot{-}$ *inclusion* $\mathbf{K} \dot{-} \neg\alpha \subseteq \mathbf{K} \div \neg\alpha$. Thus by monotony $Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. Assume now that $\nvdash \neg\alpha$ and $\neg\alpha \not\in \mathbf{K}$.
By deduction and $\dot{-}$ *closure* it follows from $\beta \in Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$ that $\alpha \to \beta \in \mathbf{K} \dot{-} \neg\alpha$. It holds that $\vdash (\neg\alpha \wedge (\alpha \to \beta)) \leftrightarrow \neg\alpha$. Thus, by $\dot{-}$ *extensionality* it follows that $\alpha \to \beta \in \mathbf{K} \dot{-} (\neg\alpha \wedge (\alpha \to \beta))$. By Lemma 7 (2) it follows that $\alpha \to \beta \in \bigcap\{\mathbf{K} \dot{-} (\neg\alpha \wedge \delta) : \delta \in \mathcal{L}\}$. Thus $\alpha \to \beta \in \bigcap\{\mathbf{K} \dot{-} (\neg\alpha \wedge \delta) : \mathbf{K} \dot{-} (\neg\alpha \wedge \delta) \cap \{\neg\alpha, \delta\} = \emptyset\}$. Hence, by (Def $\dot{-}$ from $\div$), $\alpha \to \beta \in \mathbf{K} \div \neg\alpha$. From which it follows by deduction that $\beta \in Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. □

Proof of Proposition 7.7. Let $\div$ be a ring withdrawal operator on $\mathbf{K}$ and $\dot{-}$ be obtained from $\div$ by (Def $\dot{-}$ from $\div$). We first show that $\dot{-}$ is an AGM contraction. To do so we will show that $\dot{-}$ satisfies the eight AGM postulates for contraction.
Closure: It holds that the intersection of belief sets is a belief set. Thus $\mathbf{K} \dot{-} \alpha = (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$ is a belief set.
Success: Let $\nvdash \alpha$. Assume towards a contradiction that $\mathbf{K} \dot{-} \alpha \vdash \alpha$. By $\dot{-}$ *closure* (proven above) it follows that $\alpha \in \mathbf{K} \dot{-} \alpha$. Hence, by (Def $\dot{-}$ from $\div$), $\alpha \in (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$. Thus $\alpha \in \mathbf{K} \div \alpha + \neg\alpha$, from which it follows by deduction and $\div$ *closure* that $\neg\alpha \to \alpha \in \mathbf{K} \div \alpha$. Thus, by $\div$ *closure*, $\alpha \in \mathbf{K} \div \alpha$. This contradicts $\div$ *success*.
Vacuity: Let $\alpha \notin \mathbf{K}$. By (Def $\dot{-}$ from $\div$), $\mathbf{K} \dot{-} \alpha = (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$. By $\div$ *vacuity* and *inclusion* it follows that $\mathbf{K} \div \alpha = \mathbf{K}$, from which it follows that $\mathbf{K} \dot{-} \alpha = \mathbf{K}$.
Inclusion: Follows by (Def $\dot{-}$ from $\div$).
Recovery: Suppose towards a contradiction that there exists $\beta \in \mathbf{K}$ such that $\beta \notin \mathbf{K} \dot{-} \alpha + \alpha$. Hence, by deduction and $\dot{-}$ *closure* (proven above), $\alpha \to \beta \notin \mathbf{K} \dot{-} \alpha$. From which it follows by (Def $\dot{-}$ from $\div$) that $\alpha \to \beta \notin (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$. From $\beta \in \mathbf{K}$ it follows that $\alpha \to \beta \in \mathbf{K}$. Hence $\alpha \to \beta \notin \mathbf{K} \div \alpha + \neg\alpha$. Thus, by deduction and $\div$ *closure*, $\neg\alpha \to (\alpha \to \beta) \notin \mathbf{K} \div \alpha$. This contradicts $\div$ *closure* since $\vdash \neg\alpha \to (\alpha \to \beta)$.
Extensionality: Follows by (Def $\dot{-}$ from $\div$) and $\div$ *extensionality*.
Conjunctive overlap: Let $\delta \in \mathbf{K} \dot{-} \alpha \cap \mathbf{K} \dot{-} \beta$. If $\vdash \alpha$, then by $\dot{-}$ *extensionality* (proven above) it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \alpha$. By symmetry of the case it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta) = \mathbf{K} \dot{-} \beta$, if $\vdash \beta$. Thus, in both cases it follows that $\delta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$. Assume now that $\nvdash \alpha$ and $\nvdash \beta$. From $\delta \in \mathbf{K} \dot{-} \alpha \cap \mathbf{K} \dot{-} \beta$ it follows that $\delta \in \mathbf{K} \dot{-} \alpha$ and $\delta \in \mathbf{K} \dot{-} \beta$. By (Def $\dot{-}$ from $\div$) it follows that $\delta \in \mathbf{K} \div \alpha + \neg\alpha$, $\delta \in \mathbf{K} \dot{-} \beta + \neg\beta$ and $\delta \in \mathbf{K}$. Hence, by deduction and $\div$ *closure*, $\neg\alpha \to \delta \in \mathbf{K} \div \alpha$ and $\neg\beta \to \delta \in \mathbf{K} \div \beta$.
We intend to show that $\delta \in \mathbf{K} \dot{-} (\alpha \wedge \beta)$, i.e., by (Def $\dot{-}$ from $\div$) that $\delta \in (\mathbf{K} \div (\alpha \wedge \beta) + \neg(\alpha \wedge \beta)) \cap \mathbf{K}$. To do so, by deduction and $\div$ *closure*, it is enough to show that $\neg(\alpha \wedge \beta) \to \delta \in \mathbf{K} \div (\alpha \wedge \beta)$ (since $\delta \in \mathbf{K}$). We will prove by cases:
Case 1) $\alpha \in \mathbf{K} \div (\alpha \wedge \beta)$. Thus, by $\div$ *success* it follows that $\beta \notin \mathbf{K} \div (\alpha \wedge \beta)$. Thus, by $\div$ *strong conjunctive inclusion* $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \beta$. Thus $\{\alpha, \neg\beta \to \delta\} \subseteq \mathbf{K} \div (\alpha \wedge \beta)$. It holds that $\{\alpha, \neg\beta \to \delta\} \vdash \neg(\alpha \wedge \beta) \to \delta$. From which it follows by monotony and $\div$ *closure* that $\neg(\alpha \wedge \beta) \to \delta \in \mathbf{K} \div (\alpha \wedge \beta)$.
Case 2) $\beta \in \mathbf{K} \div (\alpha \wedge \beta)$. Reasoning as in the previous case it is possible to conclude that $\neg(\alpha \wedge \beta) \to \delta \in \mathbf{K} \div (\alpha \wedge \beta)$.
Case 3) $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$ and $\beta \notin \mathbf{K} \div (\alpha \wedge \beta)$. Hence, by $\div$ *strong conjunctive inclusion* $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha = \mathbf{K} \div \beta$. Thus $\{\neg\alpha \to \delta, \neg\beta \to \delta\} \subseteq \mathbf{K} \div (\alpha \wedge \beta)$. It holds that $\{\neg\alpha \to \delta, \neg\beta \to \delta\} \vdash \neg(\alpha \wedge \beta) \to \delta$. Thus by monotony and $\div$ *closure* $\neg(\alpha \wedge \beta) \to \delta \in \mathbf{K} \div (\alpha \wedge \beta)$.
Conjunctive inclusion: Let $\alpha \notin \mathbf{K} \dot{-} (\alpha \wedge \beta)$. Thus by (Def $\dot{-}$ from $\div$) it follows that $\alpha \notin (\mathbf{K} \div (\alpha \wedge \beta) + \neg(\alpha \wedge \beta)) \cap \mathbf{K}$. Thus either $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$ or $\alpha \notin \mathbf{K}$.

Case 1) $\alpha \notin \mathbf{K}$. Hence by $\dot{-}$ *vacuity* and *inclusion* (proven above) it follows that $\mathbf{K} \dot{-} \alpha = \mathbf{K}$. Thus, by $\dot{-}$ *inclusion*, $\mathbf{K} \dot{-} (\alpha \wedge \beta) \subseteq \mathbf{K} \dot{-} \alpha$.
Case 2) $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. Thus by $\div$ *strong conjunctive inclusion* it follows that $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$. Thus $\mathbf{K} \div (\alpha \wedge \beta) + \neg(\alpha \wedge \beta) = \mathbf{K} \div \alpha + \neg(\alpha \wedge \beta) \subseteq \mathbf{K} \div \alpha + \neg\alpha$. Thus $(\mathbf{K} \div (\alpha \wedge \beta) + \neg(\alpha \wedge \beta)) \cap \mathbf{K} \subseteq (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$. Therefore, by (Def $\dot{-}$ from $\div$), it follows that $\mathbf{K} \dot{-} (\alpha \wedge \beta) \subseteq \mathbf{K} \dot{-} \alpha$.

By $\div$ *inclusion* it follows that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. Thus $\mathbf{K} \div \alpha \subseteq (\mathbf{K} \div \alpha + \neg\alpha) \cap \mathbf{K}$. From which it follows, by (Def $\dot{-}$ from $\div$) that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \dot{-} \alpha$.

It remains to prove that $\dot{-}$ and $\div$ are revision equivalent, i.e. that $Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\}) = Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. We will prove by double inclusion. From $\mathbf{K} \div \neg\alpha \subseteq \mathbf{K} \dot{-} \neg\alpha$ it follows by the monotony of Cn that $Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$. For the other inclusion, let $\beta \in Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$. If $\vdash \neg\alpha$ or $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} \div \neg\alpha = \mathbf{K}$. Thus by $\dot{-}$ *inclusion* (proven above) $\mathbf{K} \dot{-} \neg\alpha \subseteq \mathbf{K} \div \neg\alpha$. Thus by monotony $Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\}) \subseteq Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. Assume now that $\nvdash \neg\alpha$ and $\neg\alpha \in \mathbf{K}$.
By deduction and $\dot{-}$ *closure* (proven above) it follows from $\beta \in Cn((\mathbf{K} \dot{-} \neg\alpha) \cup \{\alpha\})$ that $\alpha \to \beta \in \mathbf{K} \dot{-} \neg\alpha$. Hence, by (Def $\dot{-}$ from $\div$), it holds that $\alpha \to \beta \in (\mathbf{K} \div \neg\alpha + \neg\neg\alpha) \cap \mathbf{K}$. From which it follows that $\beta \in Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$. □

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