Maps between Different Kinds of Two-Level Credibility-Limited Revision Operators

MARCO GARAPA, Universidade da Madeira, Funchal, Portugal and Centro de Investigação em Matemática e Aplicações (CIMA), Évora, Portugal
EDUARDO FERMÉ, Universidade da Madeira, Funchal, Portugal and NOVA Laboratory for Computer Science and Informatics (NOVA LINCS), Caparica, Portugal
MAURÍCIO D. L. REIS, Universidade da Madeira, Funchal, Portugal and Centro de Investigação em Matemática e Aplicações (CIMA), Évora, Portugal

Two-level credibility-limited revision is a non-prioritized revision operation. When revising through a two-level credibility-limited revision, two levels of credibility and one level of incredibility are considered. When revising by a sentence at the highest level of credibility, the operator behaves like a standard revision, if the sentence is at the second level of credibility, the revision process results in a standard contraction by the negation of that sentence. If the sentence is not credible, then the original belief set remains unchanged. In this article, we introduce a novel constructive method for two-level credibility-limited revision operators, based on a modified version of entrenchment relations. Additionally, we propose a semantics for this type of operators, based on Grove's systems of spheres. Furthermore, we present axiomatic characterizations for the newly proposed operators.

CCS Concepts: • **Computing methodologies → Nonmonotonic, default reasoning and belief revision**; • **Theory of computation → Logic**;

Additional Key Words and Phrases: Two Level Credibility-limited Revisions, System of Spheres, Epistemic Entrenchment

ACM Reference format:
Marco Garapa, Eduardo Fermé, and Maurício D. L. Reis. 2025. Maps between Different Kinds of Two-Level Credibility-Limited Revision Operators. *ACM Trans. Comput. Logic* 26, 3, Article 16 (July 2025), 33 pages.
https://doi.org/10.1145/3733830

1 Introduction

The research area that studies the dynamics of knowledge is known as *belief change* (also referred to as *belief revision* or *belief dynamics*). One of the main goals within this area is to model how

This article was partially supported by FCT-Fundação para a Ciência e a Tecnologia, Portugal through project PTDC/CCI-COM/4464/2020. E.F. was partially supported by UID/04516/NOVA Laboratory for Computer Science and Informatics (NOVA LINCS) with the financial support of FCT.IP. M.G. and M.R. were partially supported by the Centro de Investigação em Matemática e Aplicações (CIMA), through the grant UIDB/04674/2020 of FCT (DOI 10.54499/UIDB/04674/2020).
Authors' Contact Information: Marco Garapa (corresponding author), Universidade da Madeira, Funchal, Portugal and Centro de Investigação em Matemática e Aplicações (CIMA), Évora, Portugal; e-mail: mgarapa@staff.uma.pt; Eduardo Fermé, Universidade da Madeira, Funchal, Portugal and NOVA Laboratory for Computer Science and Informatics (NOVA LINCS), Caparica, Portugal; e-mail: ferme@uma.pt; Maurício D. L. Reis, Universidade da Madeira, Funchal, Portugal and Centro de Investigação em Matemática e Aplicações (CIMA), Évora, Portugal; e-mail: m_reis@staff.uma.pt.

ACM 1557-945X/2025/7-ART16
https://doi.org/10.1145/3733830

rational agents modify their set of beliefs when confronted with some new information. The main model of belief change is the AGM model [1]. In this conceptual framework, an agent's *belief* is represented by a sentence, and the agent's *belief state* is represented by a logically closed set of (belief-representing) sentences. These sets are called *belief sets*. The AGM model considers three kinds of belief change operators, namely *expansion, contraction*, and *revision*. An expansion occurs when new information is added to the set of beliefs of an agent, and none of the previous beliefs is removed. The expansion of a belief set $\mathbf{K}$ by a sentence α (denoted by $\mathbf{K} + \alpha$) is the logical closure of $\mathbf{K} \cup \{\alpha\}$. Contraction, on the other hand, involves the removal of information from an agent's belief set. Revision is the process wherein new information is added to an agent's belief set while ensuring consistency, provided that the new information itself is consistent. Of these operations, expansion is the only one that can be uniquely defined. The remaining two operations are governed by a set of postulates that determine the behavior of each function, thereby establishing conditions or constraints that must be satisfied.

Although the AGM model has acquired the status of a standard model of belief change, several researchers (for an overview see [7] and [8]) have highlighted its limitations in various contexts. Consequently, several extensions and generalizations to the framework have been proposed. One criticism of the AGM model that appears in the belief change literature is the total acceptance of the new information, which is characterized by the *success* postulate for revision. "The AGM model always accepts the new information. This feature appears, in general, to be unrealistic, since rational agents, when confronted with information that contradicts previous beliefs, often reject it altogether or accept only parts of it" [9]. This can happen due to various reasons. For instance, the new information may lack credibility, or it may contradict previously held highly entrenched beliefs.

Models in which the belief change operators considered do not always satisfy the *success* postulate (contrary to what is the case regarding the AGM model) are designated by *non-prioritized belief change operators* [23]. **Credibility-limited (CL)** revision operators [25] are *non-prioritized belief revision operators*. They rely on the credibility of the sentences by which the revision is performed. If that sentence is considered to be credible, then the credibility revision operator works as a standard AGM revision; otherwise, it leaves the original belief set unchanged. **Two-level credibility-limited (2LCL)** revision operators are non-prioritized revision operators that were proposed (independently) in [11] and [2]. In the definition of these operators, an additional level of credibility is taken into account. When revising by means of a 2LCL revision operator two levels of credibility and one level of incredibility are considered. When revising by a sentence at the highest level of credibility, the operator behaves as a standard revision. In this case the new information is incorporated in the agent's belief set. If the sentence is at the second level of credibility, then the outcome of the revision process coincides with a standard contraction by the negation of that sentence. In this case, the new information is not accepted but all the beliefs that are inconsistent with it are removed. This behavior is motivated by the intuition that the new information lacks sufficient credibility to be to incorporated in the agent's belief set, but creates some doubt in the agent's mind making her lose all the beliefs that are inconsistent with it. If the sentence is considered to be not credible, then the original belief set remains unaltered.

To illustrate the different attitudes towards a particular item of information depending on the level of (in)credibility assigned to it, Bonanno presented the following example:

Example 1.1 ([3]). Consider three husbands, A, B, and C, who all believe their wives are faithful. A trusted friend informs each that his wife is having an affair:

- Husband A accepts the claim, believes his wife is unfaithful, and files for divorce.
- Husband B dismisses the claim as not credible and remains firm in his belief in his wife's faithfulness.

—Husband C reaction is somewhere in the middle: he abandons his belief in his wife's faithfulness without fully accepting her unfaithfulness, remaining open to the possibility of her being unfaithful (and, perhaps, hires a private investigator to resolve the uncertainty).

In real-world scenarios, new information varies in its degree of credibility. By using information categorized into three distinct and mutually exclusive categories, 2LCL revision operators ensure that:

—Highly credible information is incorporated effectively, in accordance with AGM revision principles. This ensures that the belief set evolves to accommodate trustworthy new information.
—Moderately credible information prompts a cautious approach. Rather than fully accepting it, the agent removes conflicting beliefs while not incorporating the new information directly. This corresponds to a standard contraction and is useful for dealing with uncertain or partially reliable sources.
—Non-credible information is ignored, preserving the integrity of the belief set. This avoids polluting the belief set with unreliable data.

Therefore, 2LCL revision operators enable a more context-aware approach to belief revision compared to standard revisions. This can be useful in designing more effective artificial rational agents for applications such as decision-making, knowledge management, and AI systems.

In this article, we propose a construction for 2LCL revision operators based on entrenchment relations. Then we present a semantic characterization for this operator based on Grove's systems of spheres. Additionally, we present axiomatic characterizations for such operators. This article subsumes and significantly extends the study presented in [12], where a construction for 2LCL revision operators was introduced, based on Grove's systems of spheres. However, no proofs were provided in that paper.

This article is organized as follows: In Section 2, we introduce the notations and recall the main background concepts and results that will be needed throughout this article. In Section 3, we recall the definition of 2LCL revisions and provide some results concerning these operators. In Section 4, we introduce the **entrenchment-based two-level credibility-limited (E2LCL)** revision operators and present a representation theorem for such operators. In Section 5, we present the definition of **system of spheres-based two-level credibility-limited (SS2LCL)** revision operators, thus providing the semantic counterpart for the 2LCL revision operators and we axiomatically characterize these operators. In Section 6, we present a brief survey of related works. In Section 7, we summarize the main contributions of the article and briefly discuss their relevance. In the Appendix, we provide the proofs for Theorems 4.5 and 5.4.

2 Background

2.1 Formal Preliminaries

We will assume a propositional language $\mathcal{L}$ that contains the usual truth functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), and $\leftrightarrow$ (equivalence). We will also use $\mathcal{L}$ to denote the set of all formulas of the language. We denote by $Cn(A)$ the set of all logical consequences of A. We assume that Cn satisfies the standard Tarskian properties, namely *inclusion, monotony*, and *iteration*. Furthermore, we will assume that Cn also satisfies *supraclassicality, compactness*, and *deduction*. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, and $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. The letters $\alpha, \beta, \ldots$ will be used to denote sentences of $\mathcal{L}$. Lowercase Latin letters such as $p, q, \ldots$ will be used to denote atomic sentences of $\mathcal{L}$. $A, B, \ldots$ shall denote sets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e., $\mathbf{K} = Cn(\mathbf{K})$)—such a set is called a *belief set* or *theory.* $\mathbf{K}_\perp$ denotes the inconsistent belief set formed by all sentences (i.e., $\mathbf{K}_\perp = \mathcal{L}$). The expansion of a belief set $\mathbf{K}$ by α is $Cn(\mathbf{K} \cup \{\alpha\})$, which we will denote by $\mathbf{K} + \alpha$. We will use the symbol $\top$ to represent an arbitrary tautology and the symbol $\perp$ to represent an arbitrary contradiction. A possible world is a maximal consistent subset of $\mathcal{L}$. The set of all possible worlds will be denoted by $\mathcal{M}_\mathcal{L}$. Sets of possible worlds are called propositions. The set of possible worlds that contain $R \subseteq \mathcal{L}$ is denoted by $\|R\|$, i.e., $\|R\| = \{M \in \mathcal{M}_\mathcal{L} : R \subseteq M\}$. If R is inconsistent, then $\|R\| = \emptyset$. It holds that $\bigcap \emptyset = \mathcal{L}$. The elements of R are designated by R–worlds. For any sentence α, $\|\alpha\|$ is an abbreviation of $\|Cn(\{\alpha\}\|)$ and its elements are designated by α-worlds. Given a binary relation $\preceq$ on $\mathcal{L}$ we shall write $\alpha \prec \beta$ to denote that $\alpha \preceq \beta$ and $\beta \npreceq \alpha$. Additionally, we use $\alpha =_\preceq \beta$ as an abbreviation for $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

2.2 AGM Revisions

The revision of a belief set consists of incorporating new beliefs into that set. In a revision process, some prior beliefs may be retracted in order to produce a consistent belief set as the output. The following postulates, which were originally presented in [17–19], are commonly known as *AGM postulates for revision*:[1]

- **($\star$1)** $\mathbf{K}\star\alpha = Cn(\mathbf{K}\star\alpha)$ (*i.e.,* $\mathbf{K}\star\alpha$ is a belief set). (Closure)
- **($\star$2)** $\alpha \in \mathbf{K}\star\alpha$. (Success)
- **($\star$3)** $\mathbf{K}\star\alpha \subseteq \mathbf{K} + \alpha$. (Inclusion)
- **($\star$4)** If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} + \alpha \subseteq \mathbf{K}\star\alpha$. (Vacuity)
- **($\star$5)** If α is consistent, then $\mathbf{K}\star\alpha$ is consistent. (Consistency)
- **($\star$6)** If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}\star\alpha = \mathbf{K}\star\beta$. (Extensionality)
- **($\star$7)** $\mathbf{K}\star\alpha \cap \mathbf{K}\star\beta \subseteq \mathbf{K}\star(\alpha \vee \beta)$. (Disjunctive overlap)
- **($\star$8)** If $\neg\alpha \notin \mathbf{K}\star(\alpha \vee \beta)$, then $\mathbf{K}\star(\alpha \vee \beta) \subseteq \mathbf{K}\star\alpha$. (Disjunctive inclusion)

Closure states that the revised belief set is deductively closed. *Success* asserts that the newly received information must be accepted. *Inclusion* states that at most elements of $\mathbf{K} + \alpha$ are incorporated as a result of revising the belief set $\mathbf{K}$ by the sentence α. *Vacuity* states that if the incoming information is consistent with the initial belief set, then any belief deducible from the incoming information taken together with the initial belief set should be included in the resulting belief set. *Consistency* assures that the outcome of the revision of a belief set by a consistent sentence is itself consistent. *Extensionality* states that the revisions of a belief set by logical equivalent sentences produce the same output. *Disjunctive overlap* and *disjunctive inclusion* specify the principles for revising by disjunctions. When taken together with the other six AGM revision postulates, they are equivalent to the more intuitive postulate:

($\star V$) $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha$ or $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\beta$ or $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha \cap \mathbf{K}\star\beta$. (Disjunctive factoring)

According to *disjunctive factoring*, the outcome of a revision by a disjunction is either identical to the revision by one of the disjuncts or to the intersection of the outcomes of the revisions by each of the disjuncts. For a more detailed discussion of the rationale behind the AGM postulates, see [19, 21, 24].

Definition 2.1 ([1]). An operator $\star$ for a belief set $\mathbf{K}$ is a basic AGM revision if and only if it satisfies postulates ($\star$1) to ($\star$6). It is an AGM revision if and only if it satisfies postulates ($\star$1) to ($\star$8).

[1]The postulates were already presented in [1] but with slightly different formulations.

2.3 AGM Contractions

The contraction of a belief set consists of removing some beliefs from that set, without adding any new beliefs. The following postulates, which were presented in [1] (following [17, 18]), are commonly known as *AGM postulates for contraction*:

(÷1) $\mathbf{K} \div \alpha = Cn(\mathbf{K} \div \alpha)$ (i.e., $\mathbf{K} \div \alpha$ is a belief set). (Closure)
(÷2) $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. (Inclusion)
(÷3) If $\alpha \notin \mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. (Vacuity)
(÷4) If $\nvdash \alpha$, then $\alpha \notin \mathbf{K} \div \alpha$. (Success)
(÷5) $\mathbf{K} \subseteq (\mathbf{K} \div \alpha) + \alpha$. (Recovery)
(÷6) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. (Extensionality)
(÷7) $(\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta) \subseteq \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive overlap)
(÷8) $\mathbf{K} \div (\alpha \wedge \beta) \subseteq \mathbf{K} \div \alpha$ whenever $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive inclusion)

Closure states that the outcome of a contraction is a belief set. *Inclusion* asserts that no new beliefs are incorporated as a result of a contraction operation. *Vacuity* states that nothing is removed when contracting by a sentence that is not in the original belief set. *Success* asserts that contracting by a non-tautological sentence results in its removal. *Recovery* states that contracting and then expanding by any given sentence allows to recover (at least) the original belief set. *Conjunctive overlap* and *conjunctive inclusion* determine how to contract conjunctions. When combined with the other six AGM contraction postulates, they are equivalent to the following postulate, which states that the outcome of a contraction by a conjunction is identical to either the contraction by one of the conjuncts or to the intersection of the outcomes of the contractions by each of the conjuncts.

(÷V) $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$ or $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \beta$ or $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha \cap \mathbf{K} \div \beta$. (Conjunctive factoring)

For a deeper discussion of the rationale behind these postulates, see [19, 24].

Definition 2.2 ([1]). An operator $\div$ for a belief set $\mathbf{K}$ is a basic AGM contraction if and only if it satisfies postulates (÷1) to (÷6). It is an AGM contraction if and only if it satisfies postulates (÷1) to (÷8).

The Levi and Harper identities make contraction and revision interchangeable.
Levi Identity ([28]): $\mathbf{K}\star\alpha = (\mathbf{K} \div \neg\alpha) + \alpha$.
Harper Identity ([26]): $\mathbf{K} \div \alpha = (\mathbf{K}\star\neg\alpha) \cap \mathbf{K}$.

These identities allow us to define the revision and the contraction operators in terms of each other. The Levi (respectively Harper) identity enables the use of contraction (resp. revision) as primitive function and treat revision (resp. contraction) as defined in terms of contraction (resp. revision).

2.4 Entrenchment-Based Contractions and Revisions

Roughly speaking, an entrenchment is an ordering of the sentences of the language according to their epistemic importance. The idea underlying these structures is that when compelled to remove one of two beliefs, an agent is more willing to abandon the belief that occupies a lower position in the entrenchment. In the following definition we recall, from [19, 20], the set of properties that an epistemic entrenchment order (with respect to a belief set $\mathbf{K}$) is assumed to satisfy.

Definition 2.3 ([19, 20]). An ordering of *epistemic entrenchment* with respect to a belief set $\mathbf{K}$ is a binary relation $\leq$ on $\mathcal{L}$ which satisfies the following postulates:

(EE1) For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \delta$ then $\alpha \leq \delta$. (Transitivity)
(EE2) For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \vdash \beta$ then $\alpha \leq \beta$. (Dominance)
(EE3) For all $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$. (Conjunctiveness)
(EE4) When $\mathbf{K} \neq \mathbf{K}_\perp$, $\alpha \notin \mathbf{K}$ iff $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$. (Minimality)
(EE5) If $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$, then $\vdash \alpha$. (Maximality)

(EE1) states that an epistemic entrenchment order is transitive. (EE2) asserts that if α entails β, then α is at most as entrenched as β, since it is impossible to give up β without also giving up α. (EE3) relies on the fact that to remove a conjunction, one must give up at least one of its conjuncts; hence, a conjunction must be at least as entrenched as one of its conjuncts. (EE4) states that beliefs with lower entrenchment are those which are not included in the belief set, while (EE5) asserts that tautologies are (exactly) the beliefs with the highest degree of entrenchment.

In the following proposition we recall some properties satisfied by epistemic entrenchments.

Proposition 2.4 ([20, 21]). *Let $\leq$ be a relation.*

- *If $\leq$ satisfies (EE1) and (EE2), then it also satisfies:*
  Intersubstitutivity: *If $\vdash \alpha \leftrightarrow \alpha'$ and $\vdash \beta \leftrightarrow \beta'$, then $\alpha \leq \beta$ if and only if $\alpha' \leq \beta'$.*
- *If $\leq$ satisfies (EE1), (EE2), and (EE3), then it also satisfies:*
  Connectedness: *Either $\alpha \leq \beta$ or $\beta \leq \alpha$.*

It is possible to define an entrenchment-based contraction $\div$ and an entrenchment-based revision $\star$ from an epistemic entrenchment ordering $\leq$, by means of the following conditions [4, 19, 20, 25, 30, 34]:

$$\mathbf{K} \div \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{if } \vdash \alpha \end{cases}$$

$$\mathbf{K} \star \alpha = \begin{cases} \{\beta : \neg\alpha < \alpha \rightarrow \beta\} & \text{if } \nvdash \neg\alpha \\ \mathcal{L} & \text{if } \vdash \neg\alpha \end{cases}$$

For an explanation of why the conditions $\alpha < \alpha \vee \beta$ and $\alpha < \alpha \rightarrow \beta$ appear in the definitions of entrenchment-based contraction and revision operators, respectively, see [24, p. 100].

We now present an example to illustrate the operation of epistemic entrenchment-based contraction, which can be easily adapted for epistemic entrenchment-based revision.

Example 2.5. Consider a language $\mathcal{L}$ built from the finite set of propositional symbols $\{p, q\}$ and the Boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\{p \wedge q\})$, and let $\div$ be the epistemic entrenchment-based contraction on $\mathbf{K}$, based on an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$, such that $p < q$. As $(p \wedge q) \vee p$ and $(p \wedge q) \vee q$ are logically equivalent to p and q, respectively, it follows by intersubstitutivity that $(p \wedge q) \vee p < (p \wedge q) \vee q$. By properties (EE1), (EE2), and (EE3), it follows that $p \wedge q =_\leq p < q$. Therefore, $p \wedge q =_\leq (p \wedge q) \vee p$ and $p \wedge q < (p \wedge q) \vee q$. Hence, $q \in \mathbf{K} \div (p \wedge q)$, but $p \notin \mathbf{K} \div (p \wedge q)$. Since $\neg p$ and $\neg q$ are not in $\mathbf{K}$, it follows that they are also not elements of $\mathbf{K} \div (p \wedge q)$. Furthermore, by (EE1) and (EE2), it also follows that if α is a logical consequence of q, then $p \wedge q < (p \wedge q) \vee \alpha$. Thus, $Cn(\{q\}) \subseteq \mathbf{K} \div (p \wedge q)$. Among the distinct belief sets of the language under consideration, the only ones that contain $Cn(\{q\})$ are $\mathcal{L}, Cn(\{\neg p \wedge q\}), Cn(\{p \wedge q\})$ and $Cn(\{q\})$. Of these, the only one which is disjoint from $\{p, \neg p, \neg q\}$ is $Cn(\{q\})$. Therefore, since the outcome of an epistemic entrenchment-based contraction is a belief set, we can conclude that $\mathbf{K} \div (p \wedge q) = Cn(\{q\})$.

It holds that entrenchment-based revision and contraction operators are characterized by the (eight) AGM postulates for revision and contraction, respectively [19, 20].

2.5 System of Spheres-Based Operations of Belief Change

Grove [22], inspired by the semantics for counterfactuals [29], proposed a structure called *system of spheres* and a way of using this structure for defining revision functions. Figuratively, the distance between a possible world and the innermost sphere reflects its plausibility towards $\|\mathbf{K}\|$. The closer a possible world is to $\|\mathbf{K}\|$, the more plausible it is. A system of spheres can also be viewed as a special type of total pre-order on possible worlds. In this pre-order, the worlds in $\|\mathbf{K}\|$ occupy the lowest position, while higher positions correspond to worlds that are considered increasingly less plausible.

Definition 2.6 ([22, 24]). Let $\mathbf{K}$ be a belief set. A system of spheres, or spheres' system, centered on $\|\mathbf{K}\|$ is a collection $\mathbb{S}$ of subsets of $\mathcal{M}_{\mathcal{L}}$, i.e., $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, that satisfies the following conditions:
($\mathbb{S}$1) $\mathbb{S}$ is totally ordered with respect to set inclusion; that is, if $U, V \in \mathbb{S}$, then $U \subseteq V$ or $V \subseteq U$.
($\mathbb{S}$2) $\|\mathbf{K}\| \in \mathbb{S}$, and if $U \in \mathbb{S}$, then $\|\mathbf{K}\| \subseteq U$ ($\|\mathbf{K}\|$ is the $\subseteq$-minimum of $\mathbb{S}$).
($\mathbb{S}$3) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ ($\mathcal{M}_{\mathcal{L}}$ is the largest element of $\mathbb{S}$).
($\mathbb{S}$4) For every $\alpha \in \mathcal{L}$, if there is any element in $\mathbb{S}$ intersecting $\|\alpha\|$ then there is also a smallest element in $\mathbb{S}$ intersecting $\|\alpha\|$.
The elements of $\mathbb{S}$ are called spheres. For any consistent sentence $\alpha \in \mathcal{L}$, the smallest sphere in $\mathbb{S}$ intersecting $\|\alpha\|$ is denoted by $\mathbb{S}_{\alpha}$.

($\mathbb{S}$1) states that spheres are totally ordered by set inclusion, i.e., that they are concentric. ($\mathbb{S}$2) and ($\mathbb{S}$3) specify, respectively, that $\|\mathbf{K}\|$ and $\mathcal{M}_{\mathcal{L}}$ (the set of all possible worlds) are the smallest and largest spheres. ($\mathbb{S}$4) asserts that for any consistent sentence α, there exists a smallest sphere that intersects $\|\alpha\|$.[2]

Given a system of spheres $\mathbb{S}$ centered on $\|\mathbf{K}\|$, it is possible to define expansion, revision, and contraction operators based on $\mathbb{S}$. To expand a belief set $\mathbf{K}$ by a sentence α, we intersect the set of all α-worlds that are contained in $\|\mathbf{K}\|$. In the principal case where $\|\mathbf{K}\| \cap \|\alpha\| \neq \emptyset$ (i.e., α is consistent with $\mathbf{K}$), the most plausible α-worlds reside within the innermost sphere $\|\mathbf{K}\|$. Intersecting all these α-worlds produces a set that, on one hand, implies α, and on the other hand, because fewer worlds are intersected than those lying in $\|\mathbf{K}\|$, results in a superset of $\mathbf{K}$. If $\|\mathbf{K}\| \cap \|\alpha\| = \emptyset$, then $\mathbf{K} + \alpha = \bigcap \emptyset = \mathcal{L}$. To revise $\mathbf{K}$ by a non-contradictory sentence α, we intersect the α-worlds that lie in the narrowest sphere around $\|\mathbf{K}\|$ which contains α-worlds. To contract $\mathbf{K}$ by a non-tautological sentence α, we start by adding the most plausible $\neg\alpha$-worlds to the set of $\|\mathbf{K}\|$-worlds. In this way, we obtain a superset of $\|\mathbf{K}\|$ that has some $\neg\alpha$-worlds. Afterwards, by intersecting all these worlds, we obtain a belief set that is contained in $\mathbf{K}$ and does not imply α. We note that by "most plausible $\neg\alpha$-worlds" we mean the worlds lying in the smallest sphere around $\|\mathbf{K}\|$ that contains some $\neg\alpha$-worlds. The following definition formalizes the ideas exposed in this paragraph.

Definition 2.7 ([22]). Let $\mathbf{K}$ be a belief set.

(a) An operation $+$ on $\mathbf{K}$ is a system of spheres-based expansion operator if and only if there exists a system of spheres $\mathbb{S}$ centered on $\|\mathbf{K}\|$ such that for all α it holds that:

$$\mathbf{K} + \alpha = \bigcap(\|\mathbf{K}\| \cap \|\alpha\|).$$

(b) An operation $\div$ on $\mathbf{K}$ is a system of spheres-based contraction operator if and only if there exists a system of spheres $\mathbb{S}$ cenetred on $\|\mathbf{K}\|$ such that for all α it holds that:

$$\mathbf{K} \div \alpha = \begin{cases} \bigcap((\mathbb{S}_{\neg\alpha} \cap \|\neg\alpha\|) \cup \|\mathbf{K}\|) & \text{if } \|\neg\alpha\| \neq \emptyset \\ \mathbf{K} & \text{otherwise} \end{cases}$$

[2]If α is inconsistent, then $\|\alpha\| = \emptyset$.

(c) An operation $\star$ on $\mathbf{K}$ is a system of spheres-based revision operator if and only if there exists a system of spheres $\mathbb{S}$ centered on $\|\mathbf{K}\|$ such that for all α it holds that:

$$\mathbf{K}\star\alpha = \begin{cases} \bigcap(\mathbb{S}_\alpha \cap \|\alpha\|) & \text{if } \|\alpha\| \neq \emptyset \\ \mathcal{L} & \text{otherwise} \end{cases}$$

We now present an example illustrating the operations of system of spheres-based contraction and system of spheres-based revision.

Example 2.8. Consider a language $\mathcal{L}$ built from the finite set of propositional symbols $\{p, q\}$ and the Boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\{p, q\})$. In the language under consideration, there are only four possible worlds: $w_1 = Cn(\{p, q\}), w_2 = Cn(\{\neg p, q\}), w_3 = Cn(\{p, \neg q\})$, and $w_4 = Cn(\{\neg p, \neg q\})$. Therefore, $\|\mathbf{K}\| = \{w_1\}$ and $\|\neg p\| = \{w_2, w_4\}$. Let $\mathbb{S}$ be a system of spheres centered in $\|\mathbf{K}\|$, such that $\mathbb{S} = \{\{w_1\}, \{w_1, w_2\}, \{w_1, w_2, w_3, w_4\}\}$. Hence, $\mathbb{S}_{\neg p} = \{w_1, w_2\}$. Let $\div$ and $\star$ be, respectively, the system of spheres-based contraction and revision operators based on $\mathbb{S}$. Then, $\mathbf{K} \div p = \bigcap((\mathbb{S}_{\neg p} \cap \|\neg p\|) \cup \|\mathbf{K}\|) = \bigcap(\{w_2\} \cup \{w_1\}) = \bigcap(\{w_1, w_2\}) = w_1 \cap w_2 = Cn(\{p, q\}) \cap Cn(\{\neg p, q\}) = Cn(\{q\})$ and $\mathbf{K}\star\neg p = \bigcap(\mathbb{S}_{\neg p} \cap \|\neg p\|) = \bigcap\{w_2\} = w_2 = Cn(\{\neg p, q\})$.

It holds that system of spheres-based revision and contraction operators are characterized, by the (eight) AGM postulates for revision and contraction, respectively, as stated in the following two propositions.

Proposition 2.9 ([19, 20, 22]). *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$. *The following statements are equivalent:*

(a) $\div$ *is an AGM contraction operator.*
(b) $\div$ *is an entrenchment-based contraction operator.*
(c) $\div$ *is a system of spheres-based contraction operator.*

Proposition 2.10 ([19, 20, 22]). *Let* $\mathbf{K}$ *be a belief set and* $\star$ *be an operator on* $\mathbf{K}$. *The following statements are equivalent:*

(a) $\star$ *is an AGM revision operator.*
(b) $\star$ *is an entrenchment-based revision operator.*
(c) $\star$ *is a system of spheres-based revision operator.*

The last two propositions state that the classes of entrenchment-based contraction (respectively, revision) and system of spheres-based contraction (respectively, revision) operators coincide. In fact, in [33] it was proven in a direct way that an epistemic entrenchment relation with respect to $\mathbf{K}$ can be constructed from a system of spheres centered on $\|\mathbf{K}\|$, and *vice versa*.

3 2LCL Revisions

The 2LCL revisions are operators of non-prioritized revision. When revising a belief set by a sentence α, the first step is to analyze the degree of credibility of that sentence. If the original status of a sentence α is rejected (i.e., the agent believes that $\neg\alpha$ is true), then after performing a 2LCL revision by α, the resulting status of α is: (i) accepted if α is at the first level of credibility; (ii) indeterminate (i.e., neither α nor $\neg\alpha$ belong to the revised belief set), if α is at the second level of credibility; (iii) rejected, if it is not credible. When performing a 2LCL revision by a sentence that is considered to be at the highest level of credibility, the 2LCL operator works as a standard revision operator. If the sentence is considered to be at the second level of credibility, then that sentence is not incorporated in the revision process, but its negation is removed. When revising by a non-credible sentence, the original belief set remains unchanged. The following definition formalizes this concept.

Definition 3.1 ([2, 11]). Let $\mathbf{K}$ be a belief set, $\star$ be a basic AGM revision operator on $\mathbf{K}$ and C_H and C_L be subsets of $\mathcal{L}$. Then $\odot$ is a 2LCL revision operator induced by $\star$, C_H and C_L if and only if:

$$\mathbf{K} \odot \alpha = \begin{cases} \mathbf{K}\star\alpha & \text{if } \alpha \in C_H \\ (\mathbf{K}\star\alpha) \cap \mathbf{K} & \text{if } \alpha \in C_L \\ \mathbf{K} & \text{if } \alpha \notin (C_L \cup C_H) \end{cases}$$

In the previous definition $C_H \cup C_L$ represent the sentences that are considered to have some degree of credibility. C_H and C_L represent, respectively, the set of sentences that are considered to be at the first (highest) and at the second level of credibility. Note that if $\star$ is a basic AGM revision (respectively, AGM revision), then the operator $\div$ defined from $\star$ by means of the Harper identity $(\mathbf{K} \div \beta) = (\mathbf{K}\star\neg\beta) \cap \mathbf{K}$ is a basic AGM contraction (respectively, AGM contraction) [1, 17, 18, 31]. Thus, the second condition of the previous definition states that if α is at the second level of credibility, then the outcome of the 2LCL revision by α coincides with the outcome of a standard contraction of $\mathbf{K}$ by $\neg\alpha$. The revision of the belief set by a non-credible sentence (i.e., by a sentence that is not in $C_H \cup C_L$) leaves the original belief set unchanged.

This construction can be further specified by adding constraints to the structure of two the sets of credible sentences. In [13, 15, 25], the following properties for a set of credible sentences C were proposed:

Credibility of Logical Equivalents: If $\vdash \alpha \leftrightarrow \beta$, then $\alpha \in C$ if and only if $\beta \in C$.[3]
Single Sentence Closure: If $\alpha \in C$, then $Cn(\alpha) \subseteq C$.
Element Consistency: If $\alpha \in C$, then $\alpha \nvdash \perp$.
Credibility Lower Bounding: If $\mathbf{K}$ is consistent, then $\mathbf{K} \subseteq C$.[4]

Additionally, in [11] the following condition that relates a set of credible sentences C and a revision function $\star$ was introduced. This condition, designated by *condition* (**C - $\star$**), states that if a sentence α is not credible, then any possible outcome of revising the belief set $\mathbf{K}$ through $\star$ by a credible sentence contains $\neg\alpha$. The intuition underlying this property is that if α is not credible then its negation cannot be removed. Thus, its negation should still be in the outcome of the revision by any credible sentence.

$$\text{If } \alpha \notin C \text{ and } \beta \in C, \text{ then } \neg\alpha \in \mathbf{K} \star \beta. \tag{C - $\star$}$$

3.1 2LCL Revision Postulates

We now recall some of the postulates proposed in [11] to express properties of the 2LCL revision operators. The first postulate was originally proposed in [32], the second in [27], the third in [25], and the remaining ones in [11].

(Consistency Preservation) If $\mathbf{K}$ is consistent, then $\mathbf{K} \odot \alpha$ is consistent.
(Confirmation) If $\alpha \in \mathbf{K}$, then $\mathbf{K} \odot \alpha = \mathbf{K}$.
(Strict Improvement) If $\alpha \in \mathbf{K} \odot \alpha$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in \mathbf{K} \odot \beta$.

[3] In [25], this property was designated by *closure under logical equivalence* and was formulated as follows: If $\vdash \alpha \leftrightarrow \beta$, and $\alpha \in C$, then $\beta \in C$.
[4] We note that, more rigorously, the expression "with respect to $\mathbf{K}$" should be included in the designation of this property, as it relates C and $\mathbf{K}$. However, this will be omitted throughout the text, as there is no risk of ambiguity when referring to this property.

(N-Recovery) $\mathbf{K} \subseteq \mathbf{K} \odot \alpha + \neg\alpha$.
(N-Relative Success) If $\neg\alpha \in \mathbf{K} \odot \alpha$, then $\mathbf{K} \odot \alpha = \mathbf{K}$.
(N-Persistence) If $\neg\beta \in \mathbf{K} \odot \beta$, then $\neg\beta \in \mathbf{K} \odot \alpha$.
(N-Success Propagation) If $\neg\alpha \in \mathbf{K} \odot \alpha$ and $\beta \vdash \alpha$, then $\neg\beta \in \mathbf{K} \odot \beta$.
(Weak Relative Success) $\alpha \in \mathbf{K} \odot \alpha$ or $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$.
(Weak Vacuity) If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$.
(Weak Disjunctive Inclusion) If $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$, then $\mathbf{K} \odot (\alpha \vee \beta) + (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha + \alpha$.
(Containment) If $\mathbf{K}$ is consistent, then $\mathbf{K} \cap ((\mathbf{K} \odot \alpha) + \alpha) \subseteq \mathbf{K} \odot \alpha$.

Consistency preservation states that the outcome of a revision of a consistent belief set is consistent. *Confirmation* ensures that the outcome of a revision by a sentence that is contained in the belief set leaves that belief set unchanged. *Strict improvement* states that if a sentence is in the outcome of a revision by it, then the same thing happens regarding every logical consequence of that sentence. *N-recovery* ensures that if the outcome of revising a belief set by a certain sentence is (subsequently) expanded by the negation of that sentence then all the initial sentences are recovered. *N-relative success* ensures that when revising by any given sentence, either the belief set is left unchanged or the negation of that sentence is removed. *N-persistence* intuitively states that if a negation of a sentence is kept when revising a belief set $\mathbf{K}$ by that sentence, then it should also be kept when revising $\mathbf{K}$ by any other sentence. *N-success propagation* informally states that if the negation of a sentence is removed when revising a belief set by that sentence, then the same thing happens regarding every logical consequence of that sentence. *Weak relative success* states that either α is incorporated when revising by it or nothing is added. *Weak vacuity* states that a revision by a sentence that is consistent with the original belief set does not cause the loss of any original belief. *Weak disjunctive inclusion* ensures that if α is consistent with the outcome of the revision of $\mathbf{K}$ by $\alpha \vee \beta$, then the outcome of the expansion of $\mathbf{K} \odot (\alpha \vee \beta)$ by $\alpha \vee \beta$ is a subset of the outcome of the expansion of $\mathbf{K} \odot \alpha$ by α. *Containment* informally states that whenever $\mathbf{K}$ is consistent the sentences that are added to $\mathbf{K} \odot \alpha$ as a result of an expansion by α are not contained in $\mathbf{K}$.

The following propositions present relations among some of the postulates presented above.

Proposition 3.2 ([11]). *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$. *It holds that:*

1. *If* $\odot$ *satisfies closure, consistency preservation, weak relative success, and N-recovery, then it satisfies N-relative success.*
2. *If* $\odot$ *satisfies weak vacuity and inclusion, then it satisfies confirmation.*

Proposition 3.3. *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$ *that satisfies N-recovery and closure. Then* $\odot$ *satisfies containment.*

Proof. Let $\mathbf{K}$ be a consistent and logically closed set and $\odot$ be an operator on $\mathbf{K}$ that satisfies N-recovery and closure. Assume that $\beta \in \mathbf{K} \cap ((\mathbf{K} \odot \alpha) + \alpha)$. Hence, $\beta \in \mathbf{K}$ and $\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$ (by deduction and $\odot$ closure). On the other hand, by $\odot$ N-recovery it holds that $\beta \in (\mathbf{K} \odot \alpha) + \neg\alpha$. Thus, by deduction and $\odot$ closure, it yields that $\neg\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. Therefore, from $\{\alpha \rightarrow \beta, \neg\alpha \rightarrow \beta\} \vdash \beta$ it follows, by $\odot$ closure, that $\beta \in \mathbf{K} \odot \alpha$. Thus, $\mathbf{K} \cap ((\mathbf{K} \odot \alpha) + \alpha) \subseteq \mathbf{K} \odot \alpha$ □

Proposition 3.4. *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$. *If* $\odot$ *satisfies closure, vacuity, consistency preservation, N-recovery, strict improvement, and disjunctive inclusion, then it also satisfies:*

If $\alpha \in \mathbf{K} \odot \alpha$ *and* $\neg\beta \notin \mathbf{K} \odot \alpha$, *then* $\beta \in \mathbf{K} \odot \beta$.

Proof. Assume that $\alpha \in \mathbf{K} \odot \alpha$ and $\neg\beta \notin \mathbf{K} \odot \alpha$. We will prove by cases:

Case (1) $\neg\alpha \notin \mathbf{K}$. Hence by $\odot$ vacuity it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Hence $\neg\beta \notin \mathbf{K}$, from which it follows by $\odot$ vacuity it follows that $\beta \in \mathbf{K} \odot \beta$.

Case (2) $\neg\alpha \in \mathbf{K}$. By $\odot$ N-recovery it follows that $\neg\alpha \in \mathbf{K} \odot \beta + \neg\beta$. Thus, by deduction and $\odot$ closure, it yields that $\alpha \rightarrow \beta \in \mathbf{K} \odot \beta$.

On the other hand, from $\alpha \in \mathbf{K} \odot \alpha$ it follows by $\odot$ strict improvement that $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$. Thus by $\odot$ consistency preservation it holds that either $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ or $\neg\beta \notin \mathbf{K} \odot (\alpha \vee \beta)$. Assume towards a contradiction that $\neg\beta \in \mathbf{K} \odot (\alpha \vee \beta)$. Hence $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$, from which it follows by $\odot$ disjunctive inclusion that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$. Hence $\neg\beta \in \mathbf{K} \odot \alpha$. Contradiction. Therefore $\neg\beta \notin \mathbf{K} \odot (\alpha \vee \beta)$. From which it follows by $\odot$ disjunctive inclusion that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \beta$. Hence $\{\alpha \rightarrow \beta, \alpha \vee \beta\} \subseteq \mathbf{K} \odot \beta$. It yields that $\{\alpha \rightarrow \beta, \alpha \vee \beta\} \vdash \beta$. From which it follows by $\odot$ closure that $\beta \in \mathbf{K} \odot \beta$.

Hence in both cases considered it yields that $\beta \in \mathbf{K} \odot \beta$. □

In the following proposition we recall from [11] an axiomatic characterization for a 2LCL revision operator induced by an AGM revision and sets C_H and C_L satisfying certain properties.[5]

Proposition 3.5 ([11]). *Let $\mathbf{K}$ be a consistent and logically closed set and $\odot$ be an operator on $\mathbf{K}$. The following conditions are equivalent:*

(1) *$\odot$ satisfies weak relative success, closure, inclusion, consistency preservation, weak vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and weak disjunctive inclusion.*
(2) *$\odot$ is a 2LCL revision operator induced by an AGM revision operator $\star$ for $\mathbf{K}$ and some sets $C_H, C_L \subseteq \mathcal{L}$ such that: C_L satisfies credibility of logical equivalents and element consistency, $C_H \cap C_L = \emptyset$, C_H satisfies element consistency, credibility lower bounding and single sentence closure and condition ($C_H \cup C_L$ - $\star$) holds.*

4 E2LCL Revisions

In this section, we present the definition of E2LCL revision operators. We start by presenting the notion of extended epistemic entrenchment with respect to a belief set $\mathbf{K}$.

Definition 4.1. An ordering of an *extended epistemic entrenchment* with respect to a belief set $\mathbf{K}$ is a binary relation $\preceq$ on $\mathcal{L}$ which satisfies the following postulates:

(EE1) For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\alpha \preceq \beta$ and $\beta \preceq \delta$ then $\alpha \preceq \delta$. (Transitivity)
(EE2) For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \vdash \beta$ then $\alpha \preceq \beta$. (Dominance)
(wEE3) If $\bot \prec \alpha \wedge \beta$, then $\alpha \preceq \alpha \wedge \beta$ or $\beta \preceq \alpha \wedge \beta$. (Weak conjunctiveness)
(wEE4) If $\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}$, then $\alpha \prec \beta$. (Weak minimality)
(EEc) For all α and β it holds that either $\alpha \preceq \beta$ or $\beta \preceq \alpha$. (Connectedness)
(EEv) If $\neg\alpha \notin \mathbf{K}$, then $\bot \prec \alpha$. (Vacuity)
(EE6) If $\alpha \preceq \bot$ and $\bot \prec \beta$, then $\neg\beta \prec \neg\alpha$. (Negation ordering)

An extended epistemic entrenchment with respect to a belief set $\mathbf{K}$ is a binary relation $\preceq$ on $\mathcal{L}$. Contrary to standard epistemic entrenchments, sentences that are not in $\mathbf{K}$ may not be equally entrenched, as condition (EE4) of standard epistemic entrenchments is replaced by (wEE4) in extended epistemic entrenchments. This allows to place sentences, that are not in $\mathbf{K}$, in different

[5]The list of postulates of the representation theorem presented in [11] included additionally the postulate *containment*; however, as Proposition 3.3 illustrates, containment follows from closure and N-recovery.

levels of the entrenchment. This assumption appears to be perfectly reasonable. For example, one might not believe that there is any kind of life form on Mars, nor that the Moon is made of cheese. But it is natural to attribute a greater degree of credibility to the former than to the latter. Another crucial difference between extended epistemic entrenchments and standard ones is that (EE5) does not hold in the former, allowing the possibility of non-tautological sentences to be maximally entrenched.[6] (EE1) and (EE2) are properties of standard entrenchments. (wEE3) and (wEE4) are weakened versions of (EE3) and (EE4), respectively. (wEE3) imposes a condition on conjunctions. It states that conjunctiveness can be applied as long as the considered conjunction is not minimally entrenched. Note that, as we permit the possibility that sentences not in $\mathbf{K}$ may not be equally entrenched, the precondition $\perp \prec \alpha \wedge \beta$ of (wEE3) is required. Let us consider the following example: one can attribute a certain level of credibility to the assertion that a particular planet hosts some form of life (α) and at the same time assign a certain degree of credibility to the absence of life in that same planet ($\neg\alpha$). However, the conjunction of both claims—that there is both some form of life and no form of life in that planet $\alpha \wedge \neg\alpha$ is not credible at all. Thus $\alpha \wedge \neg\alpha \preceq \perp$, but $\perp \prec \alpha$ and $\perp \prec \neg\alpha$. (wEE4) states that sentences in $\mathbf{K}$ are more entrenched that those that are not. (EEc) is a property of standard entrenchments, which follows from (EE1), (EE2), and (EE3) [20]. Since in this definition a weaker version of (EE3) is used, it is necessary to include (EEc) in the list of properties that characterize extended epistemic entrenchments. According to (EE2) contradictions are in the lowest level of the entrenchment. Having this in mind, (EEv) states that if $\neg\alpha \notin \mathbf{K}$, then α is more entrenched than the contradictions. On the other hand, having in mind that formulas with lower level of entrenchment are more easily abandoned, (EE6) intuitively states that if α is at the lowest level of the entrenchment and β is not, then $\neg\beta$ should be easier to abandon than $\neg\alpha$.

The following propositions introduce other properties of extended epistemic entrenchments: (i) if a formula is not minimally entrenched, then its negation is not maximally entrenched; (ii) if α and β are maximally entrenched, then the same happens regarding $\alpha \wedge \beta$.

Proposition 4.2. *Let $\preceq$ be a relation that satisfies (EE1), (EE2), and (EE6). Then it also satisfies:*
(EE7) *If $\perp \prec \alpha$, then $\neg\alpha \prec \top$.* *(Boundary)*

Proof. Let $\preceq$ be a relation that satisfies (EE1), (EE2), and (EE6). Assume that $\perp \prec \alpha$. It holds (by (EE2)) that $\perp \preceq \perp$. Thus by (EE6) $\neg\alpha \prec \neg\perp$. From which it follows, by Proposition 2.4 that $\neg\alpha \prec \top$. □

Proposition 4.3. *Let $\mathbf{K}$ be a consistent belief set and $\preceq$ be a relation on $\mathbf{K}$ that satisfies (EE1), (wEE3), and (wEE4). Then it also satisfies:*
(EE8) *If $\top \preceq \alpha$ and $\top \preceq \beta$, then $\top \preceq \alpha \wedge \beta$.* *(Weak maximality propagation)*

Proof. Assume that $\top \preceq \alpha$ and $\top \preceq \beta$. By (wEE4) it follows that $\alpha \in \mathbf{K}$ and $\beta \in \mathbf{K}$. Thus $\alpha \wedge \beta \in \mathbf{K}$. Hence, by (wEE4), $\perp \prec \alpha \wedge \beta$. Therefore, by (wEE3), $\alpha \preceq \alpha \wedge \beta$ or $\beta \preceq \alpha \wedge \beta$. In both cases it yields, by (EE1), that $\top \preceq \alpha \wedge \beta$. □

We now define the E2LCL revision operators. Three clusters in the entrenchment are considered. If α is not minimally entrenched, then the operator works as a standard entrenchment-based revision

[6]Note that, since (EE2) still holds, tautological sentences are still maximally entrenched. The modified version of the entrenchment for credibility limited revision operators proposed in [25] also does not satisfy (EE5).

by α.[7] If α is minimally entrenched (i.e., $\alpha \preceq \bot$), the position of $\neg\alpha$ in the extended epistemic entrenchment must be taken into account. Specifically, if $\neg\alpha$ is maximally entrenched (i.e., $\top \preceq \neg\alpha$), then the revision by α leaves the belief set unchanged. Otherwise, the revision by α works as a standard entrenchment-based contraction by $\neg\alpha$.[8]

Definition 4.4. Let $\mathbf{K}$ be a belief set and $\preceq$ be an extended epistemic entrenchment with respect to $\mathbf{K}$. Then $\odot_{\preceq}$ is the *E2LCL revision operator based on* $\preceq$ if and only if:

$$\mathbf{K} \odot_{\preceq} \alpha = \begin{cases} \{\beta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \to \beta\} & \text{if } \bot \prec \alpha \\ \{\beta \in \mathbf{K} : \neg\alpha \prec \alpha \to \beta\} & \text{if } \alpha \preceq \bot \text{ and } \neg\alpha \prec \top \\ \mathbf{K} & \top \preceq \neg\alpha \end{cases}$$

An operator $\odot$ on $\mathbf{K}$ is an E2LCL *revision operator* if and only if there exists an extended epistemic entrenchment $\preceq$ with respect to $\mathbf{K}$ such that $\mathbf{K} \odot \alpha = \mathbf{K} \odot_{\preceq} \alpha$ holds for all α.

We now present a representation theorem for E2LCL revision operators.

Theorem 4.5. *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

(1) $\odot$ *satisfies weak relative success, closure, inclusion, consistency preservation, vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and disjunctive inclusion.*
(2) $\odot$ *is an E2LCL revision operator.*

Sketch Proof. The complete proof is provided in the Appendix.
(2) to (1): Let $\odot$ be an E2LCL revision operator based on an extended epistemic entrenchment $\preceq$. It can be shown that $\odot$ satisfies all the postulates present in statement (1).
(1) to (2): Let $\odot$ be an operator satisfying the postulates listed in (1), and let $\preceq$ be defined as follows:

$$\alpha \preceq \beta \text{ iff either } \alpha \notin \mathbf{K} \odot \alpha \text{ or } \beta \in \mathbf{K} \odot \beta \text{ and if } \alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta), \text{ then } \beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta).$$

It can be shown that $\preceq$ is an extended epistemic entrenchment and $\odot$ is the E2LCL revision operator induced by $\preceq$. □

The following theorem relates E2LCL revision operators and the 2LCL revision operators induced by AGM revision operators and sets $C_H, C_L \subseteq \mathcal{L}$ satisfying certain properties. Considering the axiomatic characterization for the latter, recalled in Proposition 3.5, we note that we only need to ensure (additionally) that the Condition **(**C_H **-** $\star$**)** holds, to guarantee that the class of these operators coincide with the class of E2LCL revision operators.

Theorem 4.6. *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

(1) $\odot$ *is an E2LCL revision operator.*
(2) $\odot$ *is a 2LCL revision operator induced by an AGM revision operator* $\star$ *for* $\mathbf{K}$ *and some sets* $C_H, C_L \subseteq \mathcal{L}$ *such that:* C_L *satisfies credibility of logical equivalents and element consistency,* $C_H \cap C_L = \emptyset$*,* C_H *satisfies element consistency, credibility lower bounding and single sentence closure and conditions* **(**$C_H \cup C_L$ **-** $\star$**)** *and* **(**C_H **-** $\star$**)** *hold.*

[7]We note that, in the definition of entrenchment-based revision, replacing the set $\{\beta : \neg\alpha < \alpha \to \beta\}$ with $\{\beta \in K + \alpha : \neg\alpha < \alpha \to \beta\}$ yields an equivalent definition. This holds because, by deduction, if $\beta \notin \mathbf{K} + \alpha$, then $\alpha \to \beta \notin \mathbf{K}$. From this, it follows by (EE4) that $\neg\alpha \not< \alpha \to \beta$.
[8]Note that according to Proposition 4.2, if $\preceq$ satisfies (EE1), (EE2), (EEc), and (EE6), then it holds that if $\top \preceq \neg\alpha$, then $\alpha \preceq \bot$.

Proof. From Theorem 4.5, it is only necessary to prove that the operator $\odot$ mentioned in the second statement of the theorem is exactly characterized by the following set of postulates: weak relative success, closure, inclusion, consistency preservation, vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and disjunctive inclusion. To do so, we start by proving that if $\odot$ is a 2LCL revision operator, then it satisfies these postulates.

By Proposition 3.5 it only remains to prove that $\odot$ also satisfies vacuity and disjunctive inclusion. It also holds that $\odot$ satisfies weak vacuity.

Vacuity: Let $\neg\alpha \notin \mathbf{K}$. By $\odot$ weak vacuity it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Thus, by the monotony of the consequence operator Cn, it only remains to prove that $\alpha \in \mathbf{K} \odot \alpha$. Assume towards a contradiction that this is not the case. Hence, by definition of $\odot$, it follows that $\alpha \notin C_H$ (by $\star$ success) and $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$. Thus $\mathbf{K} \odot \alpha = \mathbf{K}$. It holds that $\top \in C_H$ (since C_H satisfies credibility lower boundary). Thus, by condition **(C_H - $\star$)**, it follows that $\neg\alpha \in \mathbf{K} \star \top = \mathbf{K} \odot \top$. Therefore, by $\odot$ confirmation (Proposition 3.2), it yields that $\neg\alpha \in \mathbf{K} \odot \top = \mathbf{K}$. Contradiction.

Disjunctive inclusion: Assume that $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. We will prove by cases:

Case (1) $\alpha \in C_H$. It holds that C_H satisfies single sentence closure. Thus $\alpha \vee \beta \in C_H$. Thus $\mathbf{K} \odot \alpha = \mathbf{K} \star \alpha$ and $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \star (\alpha \vee \beta)$. It holds that $\neg\alpha \notin \mathbf{K} \star (\alpha \vee \beta)$. From which it follows by $\star$ *disjunctive inclusion* that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Case (2) $\alpha \notin C_H$. It holds that $C_H \cap C_L = \emptyset$. Assume towards a contradiction that $\alpha \vee \beta \in C_H$. By condition **($C_H$ - $\star$)**, it follows that $\neg\alpha \in \mathbf{K} \odot (\alpha \vee \beta)$. Contradiction. Hence $\alpha \vee \beta \notin C_H$.

Case (2.1) $\alpha \vee \beta \in C_L$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \cap \mathbf{K} \star (\alpha \vee \beta)$.

From $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ it follows that either $\neg\alpha \notin \mathbf{K}$ or $\neg\alpha \notin \mathbf{K} \star (\alpha \vee \beta)$. Hence there are two sub-cases to consider:

Case (2.1.1) $\neg\alpha \notin \mathbf{K}$. By $\odot$ vacuity it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Hence $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Case (2.1.2) $\neg\alpha \in \mathbf{K}$. Hence $\neg\alpha \notin \mathbf{K} \star (\alpha \vee \beta)$. From which it follows by $\star$ disjunctive inclusion that $\mathbf{K} \star (\alpha \vee \beta) \subseteq \mathbf{K} \star \alpha$. We will consider two more sub-cases:

Case (2.1.2.1) $\alpha \in C_L$. Hence $\mathbf{K} \odot \alpha = \mathbf{K} \cap \mathbf{K} \star \alpha$. Thus $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \cap \mathbf{K} \star (\alpha \vee \beta) \subseteq \mathbf{K} \cap \mathbf{K} \star \alpha = \mathbf{K} \odot \alpha$.

Case (2.1.2.2) $\alpha \notin C_H \cup C_L$. Thus $\mathbf{K} \odot \alpha = \mathbf{K}$. Thus $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} = \mathbf{K} \odot \alpha$.

Case (2.2) $\alpha \vee \beta \notin C_H \cup C_L$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K}$. Thus $\neg\alpha \notin \mathbf{K}$. From which it follows by $\odot$ vacuity that $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \subseteq \mathbf{K} \odot \alpha$.

Hence, in all cases considered, it holds that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Let $\odot$ be an operator satisfying weak relative success, closure, inclusion, consistency preservation, vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and disjunctive inclusion. Let $\star$ be the operation such that:

(i) If $\neg\alpha \notin \mathbf{K} \odot \alpha$, then $\mathbf{K} \star \alpha = \mathbf{K} \odot \alpha + \alpha$;

(ii) If $\neg\alpha \in \mathbf{K} \odot \alpha$, then $\mathbf{K} \star \alpha = Cn(\alpha)$.

Furthermore let $C_H = \{\alpha : \alpha \in \mathbf{K} \odot \alpha\}$ and $C_L = \{\alpha : \neg\alpha \notin \mathbf{K} \odot \alpha\} \setminus C_H$.

These are the same constructions used in the corresponding part of the proof of Proposition 3.5. Then, regarding that proof, it remains only to show that condition **(C_H - $\star$)** holds. Let $\alpha \notin C_H$ and $\beta \in C_H$, we intend to prove that $\neg\alpha \in \mathbf{K} \star \beta$. From $\alpha \notin C_H$ and $\beta \in C_H$ it follows respectively that $\alpha \notin \mathbf{K} \odot \alpha$ and $\beta \in \mathbf{K} \odot \beta$. From which it follows, by Proposition 3.4, that $\neg\alpha \in \mathbf{K} \odot \beta$. By $\odot$ consistency preservation it yields that $\neg\beta \notin \mathbf{K} \odot \beta$. Therefore, by definition of $\star$, it yields that $\neg\alpha \in \mathbf{K} \star \beta$. □

5 SS2LCL Revisions

In this section we present the definition of a SS2LCL revision operator, which is the semantical counterpart of the 2LCL revision operators. We start by presenting the notion of two-level system of spheres, centered on $\|\mathbf{K}\|$.

Definition 5.1. Let $\mathbf{K}$ be a belief set. A two-level system of spheres centered on $\|\mathbf{K}\|$ is a pair $(\mathbb{S}_{in}, \mathbb{S})$ whose elements are subsets of $\mathcal{M}_{\mathcal{L}}$, i.e., $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ and $\mathbb{S}_{in} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, such that:
(a) $\mathbb{S}$ and $\mathbb{S}_{in}$ satisfy conditions ($\mathbb{S}1$), ($\mathbb{S}2$) and ($\mathbb{S}4$) of Definition 2.6;
(b) $\mathbb{S}_{in} \subseteq \mathbb{S}$;
(c) If $X \in \mathbb{S}_{in}$, then $X \subseteq Y$ for all $Y \in \mathbb{S} \setminus \mathbb{S}_{in}$.

Intuitively, a two-level system of spheres $(\mathbb{S}_{in}, \mathbb{S})$, centered on $\|\mathbf{K}\|$ is a system composed by two systems of spheres $\mathbb{S}_{in}$ and $\mathbb{S}$, both centered on $\|\mathbf{K}\|$, where $\mathbb{S}_{in} \subseteq \mathbb{S}$ and in which the condition ($\mathbb{S}3$) of Definition 2.6 is relaxed for $\mathbb{S}_{in}$ and $\mathbb{S}$, allowing the existence of possible worlds outside the union of all spheres of $\mathbb{S}_{in}$ and of $\mathbb{S}$.[9] Conditions (b) and (c) impose that the spheres of $\mathbb{S}_{in}$ are the innermost ones of $\mathbb{S}$.

The following proposition is a direct consequence of condition (c). It states that all spheres of $\mathbb{S}$ contained in a given sphere of $\mathbb{S}_{in}$ belong to $\mathbb{S}_{in}$.

Proposition 5.2. *If $\mathbb{S}_{in}$ and $\mathbb{S}$ satisfy condition (c) of* Definition 5.1, *then it holds that: If $X \in \mathbb{S}$ and $Y \in \mathbb{S}_{in}$ are such that $X \subseteq Y$, then $X \in \mathbb{S}_{in}$.*

Proof. Suppose towards a contradiction that $X \in \mathbb{S}$ and $Y \in \mathbb{S}_{in}$, $X \subseteq Y$, and $X \notin \mathbb{S}_{in}$. Hence $X \in \mathbb{S} \setminus \mathbb{S}_{in}$. Thus, by hypothesis, it follows that $Y \subseteq X$. Hence $X = Y$. Contradiction since $X \notin \mathbb{S}_{in}$ and $Y \in \mathbb{S}_{in}$. □

In a system of spheres centered on $\|\mathbf{K}\|$, the closer a possible world is to the center, the more plausible it is considered to be. Similarly, the worlds lying in the spheres of $\mathbb{S}_{in}$ have a higher degree of plausibility than those in the spheres of $\mathbb{S} \setminus \mathbb{S}_{in}$. Intuitively, a two-level system of spheres $(\mathbb{S}_{in}, \mathbb{S})$, centered on $\|\mathbf{K}\|$ defines three clusters. The first cluster is formed by the worlds in the spheres of $\mathbb{S}_{in}$. These worlds are the ones to which a higher degree of plausibility is attributed (relatively to those outside the spheres of $\mathbb{S}_{in}$). The second cluster is formed by the worlds in the spheres of $\mathbb{S} \setminus \mathbb{S}_{in}$, which are assigned some (lower) degree of plausibility. Finally, the third cluster is formed by the worlds outside the spheres of $\mathbb{S}$, which are considered to be implausible.

By employing a two-level system of spheres, it is possible to define a revision operator, as presented in the following definition.

Definition 5.3. Let $\mathbf{K}$ be a belief set and $(\mathbb{S}_{in}, \mathbb{S})$ be a two-level system of spheres centered on $\|\mathbf{K}\|$. The *SS2LCL revision operator induced by* $(\mathbb{S}_{in}, \mathbb{S})$ is the operator $\odot_{(\mathbb{S}_{in},\mathbb{S})}$ such that, for all α:

$$\mathbf{K} \odot_{(\mathbb{S}_{in},\mathbb{S})} \alpha = \begin{cases} \bigcap(S_\alpha \cap \|\alpha\|) & \text{if } S_\alpha \in \mathbb{S}_{in} \\ \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) & \text{if } S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in} \\ \mathbf{K} & \text{if } X \cap \|\alpha\| = \emptyset, \text{for all } X \in \mathbb{S} \end{cases}$$

An operator $\odot$ on $\mathbf{K}$ is a *SS2LCL revision operator* if and only if there exists a two-level system of spheres $(\mathbb{S}_{in}, \mathbb{S})$ centered on $\|\mathbf{K}\|$ such that $\mathbf{K} \odot \alpha = \mathbf{K} \odot_{(\mathbb{S}_{in},\mathbb{S})} \alpha$ holds for all α.

As Figure 1 illustrates, the outcome of the revision by means of a SS2LCL revision operator of a belief set $\mathbf{K}$ by a sentence α is:

- the intersection of the most plausible α-worlds, if these worlds belong to the cluster of the most plausible worlds.[10]

[9]Condition ($\mathbb{S}3$) of Definition 2.6 was also relaxed in [25] when constructing a (modified) system of spheres for credibility-limited revision operators.
[10]Note that if X is a set of possible worlds, then $\bigcap X$ is a belief set.

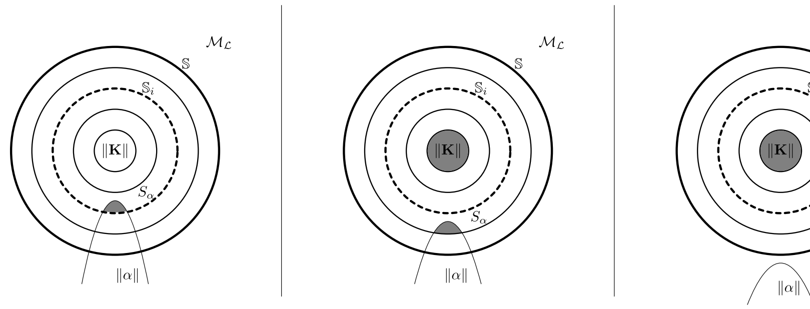

Fig. 1. Schematic representation of the worlds in the outcome of the SS2LCL revision operator induced by a two-level system of spheres $(\mathbb{S}_{in}, \mathbb{S})$ centered on $\|\mathbf{K}\|$ by a sentence α. In the first case, it holds that $S_\alpha \in \mathbb{S}_{in}$, in the second that $S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$, and in the third case, all the α-worlds are outside the spheres of $\mathbb{S}$. The color-filled area represents the worlds that are selected in each case. Thus, the SS2LCL revision operator functions as a revision by α in the first case, as a contraction by $\neg\alpha$ in the second, and leaves the belief set unchanged in the third.

— the intersection of all the worlds contained in the union of the set of $\mathbf{K}$-worlds with the set of the most plausible α-worlds, if those α-worlds are considered to be plausible, but are not in the cluster of the most plausible ones.
— $\mathbf{K}$ if the α-worlds are not plausible, i.e., in this case the belief set remains unchanged.

We now present a representation theorem for the SS2LCL revision operators.

THEOREM 5.4. *Let $\mathbf{K}$ be a consistent and logically closed set and $\odot$ be an operator on $\mathbf{K}$. The following conditions are equivalent:*

(1) *$\odot$ satisfies weak relative success, closure, inclusion, consistency preservation, vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and disjunctive inclusion.*
(2) *$\odot$ is a SS2LCL revision operator.*

SKETCH PROOF. The complete proof is provided in the Appendix.
(2) to (1): Let $\odot$ be a system of spheres-based two-level CL revision operator induced by a two-level system of spheres $(\mathbb{S}_{in}, \mathbb{S})$. It can be shown that $\odot$ satisfies all the postulates present in statement (1).
(1) to (2): Assume that $\odot$ satisfies all the postulates listed in statement (1). Consider the following constructions for $\mathbb{S}$ and $\mathbb{S}_{in}$:

$S \in \mathbb{S}_{in}$ iff (i) $S = \|\mathbf{K}\|$ or (ii) $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|\}$ and $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$.

$S \in \mathbb{S}$ iff (i) $S = \|\mathbf{K}\|$ or (ii) $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset\}$, $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$ and if $S \cap \|\alpha\| = \emptyset$ and $S \notin \mathbb{S}_{in}$, then $\|\mathbf{K} \odot \alpha\| \cap S = \|\mathbf{K}\|$.

It can be shown to show that:

(1) $(\mathbb{S}_{in}, \mathbb{S})$ is a two-level system of spheres, centered on $\|\mathbf{K}\|$.
(2) If $\|\alpha\| = \emptyset$, then $\mathbf{K} \odot \alpha = \mathbf{K}$;
(3) For α such that $\mathbf{K} \odot \alpha \nvdash \neg\alpha$ it holds for $S(\alpha) = \bigcup\{\|\mathbf{K} \odot \delta\| : \|\alpha\| \subseteq \|\delta\|\}$ that:
  (a) $S(\alpha) \in \mathbb{S}$;
  (b) $S(\alpha) = S_\alpha$ (i.e. $S(\alpha)$ is the minimal sphere that intersects with $\|\alpha\|$);

(c) The following equality holds (where $S_\alpha = S(\alpha)$):

$$\mathbf{K} \odot \alpha = \begin{cases} \bigcap(S_\alpha \cap \|\alpha\|) & \text{if } S_\alpha \in \mathbb{S}_{in} \\ \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) & \text{if } S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in} \\ \mathbf{K} & \text{if } X \cap \|\alpha\| = \emptyset, \text{ for all } X \in \mathbb{S} \end{cases}$$

□

We finish this section with a corollary stating that the classes of SS2LCL and E2LCL revision operators coincide. Furthermore, these classes also coincide with the class of 2LCL revision operators mentioned in statement (2) of Theorem 4.6. This corollary follows immediately from Theorem 4.5, 4.6, and 5.4.

Corollary 5.5. *Let* $\mathbf{K}$ *be a consistent and logically closed set and* $\odot$ *be an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

(1) $\odot$ *satisfies weak relative success, closure, inclusion, consistency preservation, vacuity, extensionality, strict improvement, N-persistence, N-recovery, disjunctive overlap, and disjunctive inclusion.*
(2) $\odot$ *is a SS2LCL revision operator.*
(3) $\odot$ *is an E2LCL revision operator.*
(4) $\odot$ *is a 2LCL revision operator induced by an AGM revision operator* $\star$ *for* $\mathbf{K}$ *and some sets* $C_H, C_L \subseteq \mathcal{L}$ *such that:* C_L *satisfy credibility of logical equivalents and element consistency,* $C_H \cap C_L = \emptyset$*,* C_H *satisfies element consistency, credibility lower bounding and single sentence closure and conditions* ($C_H \cup C_L$ - $\star$) *and* (C_H - $\star$) *hold.*

6 Related Works

In this section, we will briefly discuss other works closely related to topic of the present article.

— In [11], the 2LCL revision operators were defined in terms of basic AGM revision operators and sets C_H and C_L of credible sentences. Various properties have been proposed for these sets, and postulates have been introduced to describe 2LCL revision operators. Several results have been presented which illustrate the relationship between these postulates and the properties of C_H and C_L. Axiomatic characterizations have been provided for several classes of 2LCL revision operators, specifically for those induced by basic AGM revisions and by AGM revisions in which the associated sets of credible sentences satisfy certain properties.
— In [2], the 2LCL revision operators (there designated by filtered belief revision operators) were introduced in terms of basic AGM belief revision operators. The possibility that an item of information could still be "taken" seriously, even if it is not accepted as being fully credible (this type of information is there called allowable) was discussed. A syntactic analysis of filtered belief revision was provided.
— In [3], the works presented in [2] and [11] were extended by introducing the notion of partial belief revision structure, providing a characterization of filtered belief revision in terms of properties of these structures. There it is considered the notion of rationalizability of a choice structure in terms of a plausibility order and established a correspondence between rationalizability and AGM consistency in terms of the eight AGM postulates for revision. An interpretation of credibility, allowability and rejection of information in terms of the degree of implausibility of the information was provided.
— In [25], CL revision operators were introduced. When revising a belief set by a sentence by means of a CL revision operator, we need first to analyze whether that sentence is credible or not. When revising by a credible sentence, the operator works as a basic AGM revision operator,

otherwise it leaves the original belief set unchanged. In [25] several properties were proposed for C (the set of credible sentences) and this model was developed in terms of possible world models and entrenchment relations. Representation theorems were presented for different classes of CL revision operators. 2LCL revision operators can be seen as a generalization of CL revision operators. More precisely, if in the definition of 2LCL revision operators C_L is replaced by $\emptyset$, the resulting operator is a CL revision operator. Regarding the constructions based on systems of spheres, both models allow the existence of possible worlds outside all the spheres of the system, in contrast to the original system of spheres proposed by Grove. However, for SS2LCL, two systems of spheres are considered instead of one, as it is the case for CL revision operators. Regarding, the construction of 2LCL based on entrenchment, as for CL revisions, the entrenchment-based relation allows the existence of non-tautological sentences at the top level of the entrenchment, thereby relaxing condition (EE5). In contrast to the entrenchment-based relations for CL revision operators, those for 2LCL do not generally satisfy (EE4). This distinction allows us to treat sentences not contained in the belief set $\mathbf{K}$ in different ways, allowing the existence of more than one level of entrenchment concerning the sentences that do not belong to $\mathbf{K}$.

— In [6], shielded contraction operators were proposed. These operators, which can be seen as the contraction counterpart of the CL revision operators, are such that they behave as a standard contraction for sentences that are considered to be retractable and have no effect when the sentence to be contracted is considered irretractable. In that same article it has been shown that using the Harper identity and a variant of the Levi identity, named Consistency-preserving Levi identity, it is possible to define the shielded contraction operators from the CL revision operators and *vice versa*.
— CL revision operators on belief bases have been extensively studied in [9, 13, 15]. Several classes of shielded contraction operators for belief bases were proposed and axiomatically characterized in [14]. The interrelation among CL revisions and shielded contractions for belief bases has been presented in [16].

7 Conclusion

The CL revision model [25] can be regarded as an extension of the AGM framework [1] for belief revision. This extension addresses one of the main shortcomings pointed out to the AGM framework, namely the fact that it assumes that any new information has priority over previously held beliefs (as characterized by the *success* postulate for revisions). In the model of CL revisions, two classes of sentences are considered. Some sentences, referred to as *credible sentences*, are accepted as a result of the process of revision by them, while the remaining sentences are such that the process of revising by them does not alter the original belief set.

In the CL revision model, the agent is certain in the sense that the agent either fully accepts or rejects the input information. In contrast, the 2LCL revision model [2, 11] considers the concept of doubt by incorporating an additional class of sentences. A sentence belonging to this class is characterized by the fact that, while revising by it does not lead to its acceptance, it does result in the removal of its negation from the original belief set.

The present article introduces two new classes of operators, namely the SS2LCL and the E2LCL revision operators. The definition of these operators is based on two distinct structures. The definition of SS2LCL revision operators is based on a structure called two-level system of spheres, which extends the definition of the well-known systems of spheres proposed by Grove [22]. Whereas, the definition of the E2LCL revision operators is based on the notion of extended epistemic entrenchment which consists on a modified version of the entrenchment relations of [19, 20].

Two-level systems of spheres and extended epistemic entrenchment orderings provide a framework for determining how rational agents should change their beliefs upon receiving new information, in accordance with the 2LCL model. These two structures, much like the systems of spheres and entrenchments in the classical AGM setting, offer additional insights and intuitions, providing an elegant theoretical foundation for structured and rational belief revision. Additionally, they provide a more intuitive and rational method for revision, as they consist in specific types of orderings: one is a plausibility ordering of possible worlds, and the other is an ordering of sentences.

Extended epistemic entrenchments allow to assign different statuses to sentences not included in an agent's set of beliefs, thereby enabling a differential treatment of such beliefs. This feature is absent in classical epistemic entrenchments and may prove useful for defining other new belief revision operators. On the other hand, two-level systems of spheres allow the creation of different clusters within a spheres system, enabling differentiated treatment of worlds within these clusters. These clusters are formed by introducing an additional (innermost) system of spheres and by relaxing condition ($\mathbb{S}3$) from Definition 2.6. Based on the above, we believe that two-level systems of spheres and extended epistemic entrenchment orderings provide a richer and more refined structure for modelling belief change than the structures used in the classical AGM framework in which these new frameworks build upon.

The article presents an axiomatic characterization for SS2LCL and E2LCL revision operators and, in the spirit of AGM, shows that it coincides with the axiomatic characterization of a class of 2LCL revision operators in which the sets of credible sentences satisfy certain properties.

For future work, we intend to adapt the 2LCL revision operators to the context of belief bases, and to define non-prioritized revision operators that combine 2LCL and selective revision operators [5, 10]. The latter are non-prioritized revision operators in which the input information is either fully accepted or only partially incorporated. Additionally, we plan to investigate the relationships between the 2LCL revision operators and other non-prioritized belief change operators. Furthermore, we aim to develop a model for iterated 2LCL revision operators.

Acknowledgments

We would like to thank the anonymous reviewers for the time they devoted to reviewing this article and for their insightful and constructive comments, which have led to a substantial improvement.

References

[1] Carlos Alchourrón, Peter Gärdenfors, and David Makinson. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50, 2 (1985), 510–530.

[2] Giacomo Bonanno. 2019. Credible information, allowable information and belief revision—Extended abstract. In *Proceedings of the 17th Conference on Theoretical Aspects of Rationality and Knowledge (TARK '19)*. Lawrence S. Moss (Ed.), Vol. 297, EPTCS, 82–90. DOI: https://doi.org/10.4204/EPTCS.297.6

[3] Giacomo Bonanno. 2022. Filtered belief revision: Syntax and semantics. *Journal of Logic, Language and Information* 31, 4 (2022), 645–675.

[4] Eduardo Fermé. 1999. *Revising the AGM Postulates*. Ph.D. Dissertation. University of Buenos Aires.

[5] Eduardo Fermé and Sven Ove Hansson. 1999. Selective revision. *Studia Logica* 63, 3 (1999), 331–342.

[6] Eduardo Fermé and Sven Ove Hansson. 2001. Shielded contraction. In *Frontiers in Belief Revision*. H. Rott and M.-A. Williams (Eds.), Kluwer Academic Publishers, 85–107.

[7] Eduardo Fermé and Sven Ove Hansson. 2011. AGM 25 years: Twenty-Five years of research in belief change. *Journal of Philosophical Logic* 40, 2 (2011), 295–331.

[8] Eduardo Fermé and Sven Ove Hansson. 2018. *Belief Change:* Introduction and Overview. Springer.

[9] Eduardo Fermé, Juan Mikalef, and Jorge Taboada. 2003. Credibility-limited functions for belief bases. *Journal of Logic and Computation* 13, 1 (2003), 99–110.

[10] Marco Garapa. 2022. Selective base revisions. *Journal of Philosophical Logic* 51, 1 (2022), 1–26. DOI: https://doi.org/10.1007/s10992-021-09606-8

[11] Marco Garapa. 2022. Two level credibility-limited revisions. *The Review of Symbolic Logic* 15, 2 (2022), 388–408. DOI: https://doi.org/10.1017/S1755020320000283

[12] Marco Garapa, Eduardo Fermé, and MaurícioD. L. Reis. 2023. System of spheres-based two level credibility-limited revisions. In *Proceedings of the 19th Conference on Theoretical Aspects of Rationality and Knowledge (TARK '23)*. Rineke Verbrugge (Ed.), *Electronic Proceedings in Theoretical Computer Science*, Vol. 379, 287–298.

[13] Marco Garapa, Eduardo Fermé, and Maurício D. L. Reis. 2018. Studies in credibility-limited base revision. In *Proceedings of the 16th International Conference Principles of Knowledge Representation and Reasoning (KR '18)*. Michael Thielscher, Francesca Toni, and Frank Wolter (Eds.), AAAI Press, 240–247. Retrieved from https://aaai.org/ocs/index.php/KR/KR18/paper/view/17980

[14] Marco Garapa, Eduardo Fermé, and Maurício D. L. Reis. 2018. Shielded base contraction. *Artificial Intelligence* 259 (2018), 186–216. DOI: https://doi.org/10.1016/j.artint.2018.03.007

[15] Marco Garapa, Eduardo Fermé, and Maurício D. L. Reis. 2020. Credibility-limited base revision: New classes and their characterizations. *Journal of Artificial Intelligence Research* 69 (2020), 1023–1075. DOI: https://doi.org/10.1613/jair.1.12298

[16] Marco Garapa, Eduardo Fermé, and Maurício D. L. Reis. 2023. Levi and Harper identities for non-prioritized belief base change. *Artificial Intelligence* 319 (2023), 103907. DOI: https://doi.org/10.1016/j.artint.2023.103907

[17] Peter Gärdenfors. 1978. Conditionals and changes of belief. *Acta Philosophica Fennica* 30, (1978), 381–404.

[18] Peter Gärdenfors. 1982. Rules for rational changes of belief. In *Philosophical Essays Dedicated to Lennart Åqvist on His Fiftieth Birthday (Philosophical Studies*, 34). Tom Pauli (Ed.), Philosophical Society and Department of Philosophy, University of Uppsala, Uppsala, 88–101.

[19] Peter Gärdenfors. 1988. *Knowledge in Flux: Modeling the Dynamics of Epistemic States*. The MIT Press, Cambridge.

[20] Peter Gärdenfors and David Makinson. 1988. Revisions of Knowledge Systems using Epistemic Entrenchment. In *Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning About Knowledge*. Moshe Y. Vardi (Ed.), Morgan Kaufmann, Los Altos, 83–95.

[21] Peter Gärdenfors and Hans Rott. 1995. Belief revision. In *Handbook of Logic in Artificial Intelligence and Logic Programming*, Vol. 4. Dov M. Gabbay, C. J. Hogger, and J. A. Robinson (Eds.), Oxford University Press, Oxford, UK, 35–132. Retrieved from http://dl.acm.org/citation.cfm?id=216136.216138

[22] Adam Grove. 1988. Two modellings for theory change. *Journal of Philosophical Logic* 17, 2 (1988), 157–170.

[23] Sven Ove Hansson. 1999. A survey of non-prioritized belief revision. *Erkenntnis* 50 (1999), 413–427.

[24] Sven Ove Hansson. 1999. A Textbook of Belief Dynamics. Theory Change and Database Updating. Kluwer Academic Publishers, Dordrecht.

[25] Sven Ove Hansson, Eduardo Fermé, John Cantwell, and Marcelo Falappa. 2001. Credibility-limited revision. *Journal of Symbolic Logic* 66, 4 (2021), 1581–1596.

[26] William L. Harper. 1976. Rational conceptual change. In *Proceedings of the Biennial Meeting of the Philosophy of Science Association (PSA)*, 462–494.

[27] Hirofumi Katsuno and Alberto Mendelzon. 1991. Propositional knowledge base revision and minimal change. *Journal of Artificial Intelligence* 52, 3 (1991), 263–294.

[28] Isaac Levi. 1977. Subjunctives, dispositions, and chances. *Synthese* 34 (1977), 423–455.

[29] David Lewis. 1973. *Counterfactuals*. Blackwell, Oxford.

[30] Sten Lindström and Wlodek Rabinowicz. 1991. Epistemic entrenchment with incomparabilities and relational belief revision. In *The Logic of Theory Change*. Fuhrmann and Morreau (Eds.), Springer-Verlag, Berlin, 93–126.

[31] David Makinson. 1987. On the status of the postulate of recovery in the logic of theory change. *Journal of Philosophical Logic* 16, 4 (1987), 383–394.

[32] David Makinson. 1997. Screened revision. *Theoria* 63, 1–2 (1997), 14–23.

[33] Maurício D. L. Reis. 2013. On the interrelation between systems of spheres and epistemic entrenchment relations. *Logic Journal of the IGPL* 22, 1 (2013), 126–146. DOI: https://doi.org/10.1093/jigpal/jzt037

[34] Hans Rott. 1991. Two methods of constructing contractions and revisions of knowledge systems. *Journal of Philosophical Logic* 20, 2 (1991), 149–173.

Appendix

A Proofs

The following Lemmas will be used in the proofs of Theorems 4.5 and 5.4, presented below.

Lemma 8.1 (see, e.g., [24, Observation 2.75]). *Let $\leq$ be a relation that satisfies transitivity. Then $\preceq$ also satisfies:*

(a) *If* $\alpha \preceq \beta$ *and* $\beta \prec \delta$, *then* $\alpha \prec \delta$.
(b) *If* $\alpha \prec \beta$ *and* $\beta \preceq \delta$, *then* $\alpha \prec \delta$.
(c) *If* $\alpha \prec \beta$ *and* $\beta \prec \delta$, *then* $\alpha \prec \delta$.

Lemma 8.2 ([11]). *Let* $\mathbf{K}$ *be a belief set. Then:*

(a) $(\mathbf{K} + \alpha) \cap (\mathbf{K} + \beta) \subseteq \mathbf{K} + \alpha \vee \beta$.
(b) $\mathbf{K} + \alpha \vee \beta \subseteq \mathbf{K} + \alpha$.

Lemma 8.3 ([22]). *Let* $\mathbf{K}, \mathbf{H}$ *be belief sets and* U, V *be sets of possible worlds. Then:*

(a) $\bigcap(\|\mathbf{K}\|) = \mathbf{K}$ *(if the underlying logic is compact).*
(b) $\bigcap(V)$ *is consistent if and only if* V *is non-empty.*
(c) *For any* $\alpha \in \mathcal{L}$, $\bigcap(V \cap \|\alpha\|) = Cn(\bigcap(V) \cup \{\alpha\})$.
(d) *If* $U \subseteq V$, *then* $\bigcap V \subseteq \bigcap U$.
(e) *If* $\mathbf{K} \subseteq \mathbf{H}$, *then* $\|\mathbf{H}\| \subseteq \|\mathbf{K}\|$.
(f) $\|\mathbf{K} + \alpha\| = \|\mathbf{K}\| \cap \|\alpha\|$.

Proof of Theorem 4.5.
(1) to (2):
Let $\odot$ be an operator satisfying the postulates listed in (1) and let $\preceq$ be defined as follows:
$\alpha \preceq \beta$ if and only if either:

— $\alpha \notin \mathbf{K} \odot \alpha$ or
— $\beta \in \mathbf{K} \odot \beta$ and if $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$, then $\beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$.

It holds by Proposition 3.2 that $\odot$ satisfies N-relative success and confirmation.

We will now show that $\preceq$ is an extended epistemic entrenchment and that $\odot$ is an E2LCL revision operator induced by $\preceq$.

(EE1): Let $\alpha \preceq \beta$ and $\beta \preceq \delta$ we need to show that $\alpha \preceq \delta$. It holds trivially by definition of $\preceq$ if $\alpha \notin \mathbf{K} \odot \alpha$. Assume now that $\alpha \in \mathbf{K} \odot \alpha$. From $\alpha \preceq \beta$ it follows by definition of $\preceq$ that $\beta \in \mathbf{K} \odot \beta$. From the later and $\beta \preceq \delta$ it follows that $\delta \in \mathbf{K} \odot \delta$. If $\alpha \notin \mathbf{K} \odot \neg(\alpha \wedge \delta)$, then it follows, by definition of $\preceq$, that $\alpha \preceq \delta$. Assume now that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$, according to the definition of $\preceq$ we must show that $\delta \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$. We will consider two cases:

Case (1) $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. It holds that $\alpha \preceq \beta$, hence $\beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. By *closure* it follows that $\neg\neg(\alpha \wedge \beta) \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. By *N-persistence* it follows that $\neg\neg(\alpha \wedge \beta) \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. Hence by *closure* $\beta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. From $\beta \preceq \delta$ it follows by definition of $\preceq$ that $\delta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. Hence by *closure* $\neg\neg(\beta \wedge \delta) \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. It follows by *N-persistence* that $\neg\neg(\beta \wedge \delta) \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$. Thus, by *closure*, $\delta \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$.

Case (2) $\alpha \notin \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Assume by *reductio ad absurdum* that $\delta \notin \mathbf{K} \odot \neg(\alpha \wedge \delta)$. Hence by *closure* $\neg\neg\delta \notin \mathbf{K} \odot \neg(\alpha \wedge \delta)$. Thus, by *N-persistence* and *closure* it holds that $\delta \notin \mathbf{K} \odot \neg\delta$. Assume that $\beta \wedge \delta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. Then by *closure* $\neg\neg(\beta \wedge \delta) \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. It follows by *N-persistence* that $\neg\neg(\beta \wedge \delta) \in \mathbf{K} \odot \neg\delta$ and by *closure* that $\delta \in \mathbf{K} \odot \neg\delta$. Contradiction. Assume now that $\beta \wedge \delta \notin \mathbf{K} \odot \neg(\beta \wedge \delta)$. If $\beta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$, then, since $\beta \preceq \delta$, it follows that $\delta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. By *closure* it follows that $\beta \wedge \delta \in \mathbf{K} \odot \neg(\beta \wedge \delta)$. Contradiction. Consider now that $\beta \notin \mathbf{K} \odot \neg(\beta \wedge \delta)$. It follows from $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$, by *inclusion*, that $\alpha \in \mathbf{K} + \neg(\alpha \wedge \delta)$. Thus, by deduction, it follows that $\neg(\alpha \wedge \delta) \rightarrow \alpha \in \mathbf{K}$. Thus $\alpha \in \mathbf{K}$. By *N-recovery* it holds that $\alpha \in \mathbf{K} \odot (\alpha \wedge \neg\beta) + \neg(\alpha \wedge \neg\beta)$. Thus, by deduction it holds that $\neg(\alpha \wedge \neg\beta) \rightarrow \alpha \in \mathbf{K} \odot (\alpha \wedge \neg\beta)$. Thus, by *closure* it follows that $\alpha \in \mathbf{K} \odot (\alpha \wedge \neg\beta)$. By *disjunctive overlap* and *extensionality*, it holds that $\mathbf{K} \odot \neg(\alpha \wedge \delta) \cap \mathbf{K} \odot (\alpha \wedge \neg\beta) \subseteq \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$. Hence $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$. If $\alpha \wedge \beta \wedge \delta \in \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$, then by *N-persistence* and *closure* $\alpha \wedge \beta \wedge \delta \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$. Thus, it follows, by *closure*, that $\delta \in \mathbf{K} \odot \neg(\alpha \wedge \delta)$. Contradiction. Consider now that $\alpha \wedge \beta \wedge \delta \notin \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$. Since $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$, it holds that

$\beta \wedge \delta \notin \mathbf{K} \odot \neg(\alpha \wedge \beta \wedge \delta)$. Thus by *extensionality* and *closure* $\neg(\neg\beta \vee \neg\delta) \notin \mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta)$. Hence by *disjunctive inclusion* it follows that $\mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta) \subseteq \mathbf{K} \odot (\neg\beta \vee \neg\delta)$.

From $\beta \notin \mathbf{K} \odot (\neg\beta \vee \neg\delta)$ it follows that $\beta \notin \mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta)$. Thus, by *closure* it follows that $\alpha \wedge \beta \notin \mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta)$ and consequently, by *closure*, that $\neg(\neg\alpha \vee \neg\beta) \notin \mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta)$. Hence by *disjunctive inclusion* it follows that $\mathbf{K} \odot (\neg\alpha \vee \neg\beta \vee \neg\delta) \subseteq \mathbf{K} \odot (\neg\alpha \vee \neg\beta)$. Thus $\alpha \in \mathbf{K} \odot (\neg\alpha \vee \neg\beta)$. Therefore, it holds by *closure* and *extensionality* that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Contradiction.

(EE2). Assume that $\alpha \vdash \beta$. We will consider two cases:

Case (1) $\alpha \notin \mathbf{K} \odot \alpha$. Thus, by definition of $\preceq$, it follows that $\alpha \preceq \beta$.

Case (2) $\alpha \in \mathbf{K} \odot \alpha$. By strict improvement it follows that $\beta \in \mathbf{K} \odot \beta$. Assume that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Thus, by *closure* it follows that $\beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Thus, by definition of $\preceq$ it follows that $\alpha \preceq \beta$.

(wEE3). From $\perp \prec \alpha \wedge \beta$ it follows that $\alpha \wedge \beta \in \mathbf{K} \odot (\alpha \wedge \beta)$. We will prove by cases:

Case (1) $\alpha \notin \mathbf{K} \odot \alpha$. Thus $\alpha \preceq \alpha \wedge \beta$.

Case (2) $\beta \notin \mathbf{K} \odot \beta$. Thus $\beta \preceq \alpha \wedge \beta$.

Case (3) $\alpha \in \mathbf{K} \odot \alpha$ and $\beta \in \mathbf{K} \odot \beta$. We will consider two sub-cases:

Case (3.1) $\alpha \preceq \beta$. Assume that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge (\alpha \wedge \beta))$. Hence by *extensionality* it follows that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Hence, from $\alpha \preceq \beta$, it follows that $\beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$ and consequently, by *extensionality*, that $\beta \in \mathbf{K} \odot \neg(\alpha \wedge (\alpha \wedge \beta))$. Thus by *closure* it follows that $\alpha \wedge \beta \in \mathbf{K} \odot \neg(\alpha \wedge (\alpha \wedge \beta))$, from which it follows that $\alpha \preceq \alpha \wedge \beta$.

Case (3.2) $\alpha \npreceq \beta$. Hence, by definition of $\preceq$ it follows that $\beta \notin \mathbf{K} \odot \neg(\alpha \wedge \beta)$. By *extensionality* it follows that $\beta \notin \mathbf{K} \odot \neg(\beta \wedge (\alpha \wedge \beta))$. From which it follows by definition of $\preceq$ that $\beta \preceq \alpha \wedge \beta$.

(wEE4). Let $\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}$. By Proposition 3.2 it follows that $\beta \in \mathbf{K} \odot \beta$. From $\alpha \notin \mathbf{K}$ it follows that $\neg(\alpha \wedge \beta) \rightarrow \alpha \notin \mathbf{K}$ (since $\{\neg(\alpha \wedge \beta) \rightarrow \alpha\} \vdash \alpha$). Hence $\alpha \notin \mathbf{K} + \neg(\alpha \wedge \beta)$. Thus, by *inclusion* it follows that $\alpha \notin \mathbf{K} \odot \neg(\alpha \wedge \beta)$. On the other hand, from $\alpha \notin \mathbf{K}$ it follows that $\neg\neg(\alpha \wedge \beta) \notin \mathbf{K}$. Hence by *weak vacuity* it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Thus $\beta \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Hence $\alpha \prec \beta$.

(EEc). We will prove by cases:

Case (1) $\alpha \preceq \perp$. By *consistency preservation* it follows that $\perp \notin \mathbf{K} \odot \perp$. By definition of $\preceq$ it follows that $\alpha \notin \mathbf{K} \odot \alpha$, from which it follows, by definition of $\preceq$, that $\alpha \preceq \beta$.

Case (2) $\beta \preceq \perp$. It follows as in the previous case that $\beta \preceq \alpha$.

Case (3) $\alpha \npreceq \perp$ and $\beta \npreceq \perp$. Hence $\alpha \in \mathbf{K} \odot \alpha$ and $\beta \in \mathbf{K} \odot \beta$. Assume that $\alpha \npreceq \beta$. Hence, by definition of $\preceq$ it follows that $\alpha \in \mathbf{K} \odot \neg(\alpha \wedge \beta)$ and $\beta \notin \mathbf{K} \odot \neg(\alpha \wedge \beta)$. Hence, by definition of $\preceq$, it holds that $\beta \preceq \alpha$.

(EEv). Let $\neg\alpha \notin \mathbf{K}$. By $\odot$ vacuity and consistency preservation it follows, respectively, that $\alpha \in \mathbf{K} \odot \alpha$ and $\perp \notin \mathbf{K} \odot \perp$. Thus, by definition of $\preceq$, it follows that $\perp \prec \alpha$.

(EE6). Assume that $\alpha \preceq \perp$ and $\perp \prec \beta$. We intend to prove that and $\neg\beta \prec \neg\alpha$.

By $\odot$ consistency preservation it follows that $\perp \notin \mathbf{K} \odot \perp$. From which it follows by definition of $\preceq$ that $\alpha \notin \mathbf{K} \odot \alpha$. From which it follows by $\odot$ vacuity that $\neg\alpha \in \mathbf{K}$. If $\neg\beta \notin \mathbf{K}$, then it follows by (wEE4), proven above, that $\neg\beta \prec \neg\alpha$. Assume now that $\neg\beta \in \mathbf{K}$. Hence, by $\odot$ confirmation (Proposition 3.2) it follows that $\neg\beta \in \mathbf{K} \odot \neg\beta$. It also yields, by confirmation, that $\neg\alpha \in \mathbf{K} \odot \neg\alpha$. On the other hand, from $\perp \prec \beta$, it follows that $\beta \in \mathbf{K} \odot \beta$. From which it follows by $\odot$ strict improvement that $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$. Hence, by $\odot$ consistency preservation, it follows that $\mathbf{K} \odot (\alpha \vee \beta) \nvdash \neg(\alpha \vee \beta)$. From which it follows that either $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ or $\neg\beta \notin \mathbf{K} \odot (\alpha \vee \beta)$. Assume towards a contradiction that $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. Hence, by $\odot$ disjunctive inclusion it follows that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$. Hence $\alpha \vee \beta \in \mathbf{K} \odot \alpha$. On the other hand, from $\neg\beta \in \mathbf{K}$ it follows, by $\odot$ N-recovery that $\neg\beta \in \mathbf{K} \odot \alpha + \neg\alpha$. From which it follows, by deduction and $\odot$ closure that $\neg\alpha \rightarrow \neg\beta \in \mathbf{K} \odot \alpha$ and consequently $\beta \rightarrow \alpha \in \mathbf{K} \odot \alpha$. Therefore, $\{\beta \rightarrow \alpha, \alpha \vee \beta\} \subseteq \mathbf{K} \odot \alpha$. From which it follows, by $\odot$ closure, that $\alpha \in \mathbf{K} \odot \alpha$. Contradiction. Hence $\neg\alpha \in \mathbf{K} \odot (\alpha \vee \beta)$. Thus $\neg\beta \notin \mathbf{K} \odot (\alpha \vee \beta)$. Hence, it holds that $\neg\beta \in \mathbf{K} \odot \neg\beta$, $\neg\alpha \in \mathbf{K} \odot \neg\alpha$, $\neg\beta \notin \mathbf{K} \odot \neg(\neg\alpha \wedge \neg\beta)$, and $\neg\alpha \in \mathbf{K} \odot \neg(\neg\alpha \wedge \neg\beta)$ (the last two by $\odot$ extensionality). Thus, by definition of $\preceq$, it yields that $\neg\beta \prec \neg\alpha$.

We will now prove by cases that:

$$\mathbf{K} \odot \alpha = \begin{cases} \{\beta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \beta\} & \text{if } \perp \prec \alpha \\ \{\beta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \beta\} & \text{if } \alpha \preceq \perp \text{ and } \neg\alpha \prec \top \\ \mathbf{K} & \top \preceq \neg\alpha \end{cases}$$

Case (1) $\top \preceq \neg\alpha$. By $\odot$ closure it follows that $\top \in \mathbf{K}\odot\top$ and by extensionality that $\mathbf{K}\odot\neg(\neg\alpha\wedge\top) = \mathbf{K} \odot \alpha$. By $\odot$ closure it also follows that $\top \in \mathbf{K} \odot \neg(\neg\alpha \wedge \top)$. From the latter and $\top \preceq \neg\alpha$ it follows that $\neg\alpha \in \mathbf{K} \odot \neg(\neg\alpha \wedge \top) = \mathbf{K} \odot \alpha$. Thus, by $\odot$ N-relative success it follows that $\mathbf{K} \odot \alpha = \mathbf{K}$.

Case (2) $\alpha \preceq\perp$ and $\neg\alpha \prec \top$. Assume towards a contradiction that $\alpha \in \mathbf{K}\odot\alpha$. It holds by *consistency preservation* that $\perp\notin \mathbf{K}\odot \perp$. Thus $\alpha \not\preceq\perp$. Contradiction. Hence $\alpha \notin \mathbf{K} \odot \alpha$. Therefore, by $\odot$ weak relative success and vacuity it follows respectively that $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$ and $\neg\alpha \in \mathbf{K}$. Hence by $\odot$ confirmation it holds that $\mathbf{K} \odot \neg\alpha = \mathbf{K}$. Thus $\neg\alpha \in \mathbf{K} \odot \neg\alpha$.

Let $\beta \in \mathbf{K} \odot \alpha$. Hence $\beta \in \mathbf{K}$ and, consequently $\alpha \rightarrow \beta \in \mathbf{K}$. Suppose towards a contradiction that $\neg\alpha \not\prec \alpha \rightarrow \beta$. Hence by (EEc), proven above, it follows that $\alpha \rightarrow \beta \preceq \neg\alpha$. From $\alpha \rightarrow \beta \in \mathbf{K}$ it follows by $\odot$ confirmation that $\alpha \rightarrow \beta \in \mathbf{K} \odot (\alpha \rightarrow \beta)$. From which it follows, by definition of $\preceq$, from $\alpha \rightarrow \beta \preceq \neg\alpha$ that if $\alpha \rightarrow \beta \in \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha)$, then $\neg\alpha \in \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha)$. By $\odot$ extensionality it follows that $\mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha) = \mathbf{K} \odot \alpha$. On the other hand, by $\odot$ closure, it follows from $\beta \in \mathbf{K} \odot \alpha$ that $\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. Thus $\neg\alpha \in \mathbf{K} \odot \alpha$.

From $\neg\alpha \prec \top$ it follows, by definition, that either $\neg\alpha \notin \mathbf{K} \odot \neg\alpha$ or (since $\top \in \mathbf{K} \odot \top$) $\top \in \mathbf{K} \odot \neg(\neg\alpha \wedge \top)$ and $\neg\alpha \notin \mathbf{K} \odot \neg(\neg\alpha \wedge \top)$. Both cases lead to a contradiction (the latter since it follows, by $\odot$ extensionality, that $\neg\alpha \notin \mathbf{K} \odot \alpha$).

Hence $\mathbf{K} \odot \alpha \subseteq \{\beta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \beta\}$. To prove the reverse inclusion, let $\beta \in \mathbf{K}$ be such that $\neg\alpha \prec \alpha \rightarrow \beta$. By definition of $\preceq$ there are two cases to consider:

Case (a) $\neg\alpha \notin \mathbf{K} \odot \neg\alpha$. Hence by $\odot$ confirmation it holds that $\neg\alpha \notin \mathbf{K}$. From which it follows by $\odot$ vacuity that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Thus $\beta \in \mathbf{K} \odot \alpha$.

Case (b) $\neg\alpha \in \mathbf{K} \odot \neg\alpha$. Hence $\alpha \rightarrow \beta \in \mathbf{K} \odot (\alpha \rightarrow \beta)$, $\alpha \rightarrow \beta \in \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha)$ and $\neg\alpha \notin \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha)$. Thus by $\odot$ extensionality it follows that $\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. On the other hand, by $\odot$ N-recovery it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha + \neg\alpha$. Hence, by deduction and $\odot$ closure it holds that $\neg\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. Therefore $\{\neg\alpha \rightarrow \beta, \alpha \rightarrow \beta\} \subseteq \mathbf{K} \odot \alpha$. It holds that $\{\neg\alpha \rightarrow \beta, \alpha \rightarrow \beta\} \vdash \beta$, from which it follows by $\odot$ closure that $\beta \in \mathbf{K} \odot \alpha$.

Therefore $\mathbf{K} \odot \alpha = \{\beta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \beta\}$.

Case (3) $\perp\prec \alpha$. By definition of $\preceq$ it follows that $\alpha \in \mathbf{K} \odot \alpha$. Thus by $\odot$ consistency preservation it holds that $\neg\alpha \notin \mathbf{K} \odot \alpha$.

Let $\beta \in \mathbf{K} \odot \alpha$. By *inclusion* it follows, from $\beta \in \mathbf{K} \odot \alpha$, that $\beta \in \mathbf{K} + \alpha$. Thus $\alpha \rightarrow \beta \in \mathbf{K}$. By Proposition 3.2 it follows that $\alpha \rightarrow \beta \in \mathbf{K} \odot (\alpha \rightarrow \beta)$. On the other hand, by $\odot$ closure, it holds that $\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. We will consider two cases:

Case (a) $\neg\alpha \notin \mathbf{K}$. Hence by (wEE4), proven above, it yields that $\neg\alpha \prec \alpha \rightarrow \beta$.

Case (b) $\neg\alpha \in \mathbf{K}$. By (EE2) it follows that $\neg\alpha \preceq \alpha \rightarrow \beta$. Assume now towards a contradiction that $\alpha \rightarrow \beta \preceq \neg\alpha$. By $\odot$ extensionality it holds that $\mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha) = \mathbf{K} \odot \alpha$. Thus $\alpha \rightarrow \beta \in \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha)$. Therefore, from $\alpha \rightarrow \beta \preceq \neg\alpha$ it follows by definition of $\preceq$ that $\neg\alpha \in \mathbf{K} \odot \neg((\alpha \rightarrow \beta) \wedge \neg\alpha) = \mathbf{K} \odot \alpha$. Contradiction. Hence $\neg\alpha \prec \alpha \rightarrow \beta$. Therefore, $\mathbf{K} \odot \alpha \subseteq \{\beta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \beta\}$. To prove the reverse inclusion, let $\beta \in \mathbf{K} + \alpha$ be such that $\neg\alpha \prec \alpha \rightarrow \beta$. We intend to prove that $\beta \in \mathbf{K} \odot \alpha$. We will consider two cases:

Case (a) $\neg\alpha \notin \mathbf{K} \odot \neg\alpha$. Hence by $\odot$ confirmation it follows that $\neg\alpha \notin \mathbf{K}$. Thus by $\odot$ vacuity it follows that $\mathbf{K} + \alpha \subseteq \mathbf{K} \odot \alpha$. Thus $\beta \in \mathbf{K} \odot \alpha$.

Case (b) $\neg\alpha \in \mathbf{K}\odot\neg\alpha$. From $\neg\alpha \prec \alpha \rightarrow \beta$ it follows, by definition of $\preceq$, that $\neg\alpha \notin \mathbf{K}\odot\neg(\neg\alpha\wedge(\alpha \rightarrow \beta))$ and $\alpha \rightarrow \beta \in \mathbf{K} \odot \neg(\neg\alpha \wedge (\alpha \rightarrow \beta))$. Thus, by *extensionality* it follows that $\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. Hence, from $\alpha \in \mathbf{K} \odot \alpha$, it follows, by $\odot$ closure, that $\beta \in \mathbf{K} \odot \alpha$.

Therefore $\mathbf{K} \odot \alpha = \{\beta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \beta\}$.

(2) to (1).

Let $\odot$ be an E2LCL revision operator based on a relation $\preceq$. We start this part of the proof by noticing that according to (EEc) it holds that if $\alpha \npreceq \beta$, then $\beta \prec \alpha$ (and thus if $\alpha \nprec \beta$, then $\beta \preceq \alpha$). According to Proposition 4.2 and Proposition 4.3 it follows that $\preceq$ also satisfies (EE7) and (EE8).

Vacuity: Assume that $\neg\alpha \notin \mathbf{K}$. By (EEv) it follows that $\bot \prec \alpha$. Let $\beta \in \mathbf{K} + \alpha$. Hence $\alpha \rightarrow \beta \in \mathbf{K}$. Hence by (wEE4) it follows that $\neg\alpha \prec \alpha \rightarrow \beta$. Therefore $\beta \in \mathbf{K} \odot \alpha$. Hence $\mathbf{K} + \alpha \subseteq \mathbf{K} \odot \alpha$.

Inclusion: Follows trivially by definition.

Closure: Let $\beta \in Cn(\mathbf{K} \odot \alpha)$. Assume towards a contradiction that $\mathbf{K} \odot \alpha = \emptyset$. Hence $\vdash \beta$ and consequently, $\vdash \alpha \rightarrow \beta$ and $\beta \in \mathbf{K}$. Thus $\mathbf{K} \odot \alpha \neq \mathbf{K}$. Thus, by definition of $\odot$ it follows that $\top \npreceq \neg\alpha$. Thus $\neg\alpha \prec \top$. Thus, by Proposition 2.4 $\neg\alpha \prec \alpha \rightarrow \beta$. From which it follows by definition of $\odot$ that $\beta \in \mathbf{K} \odot \alpha$. Contradiction. Thus, $\mathbf{K} \odot \alpha \neq \emptyset$. From $\beta \in Cn(\mathbf{K} \odot \alpha)$ it follows by compactness there exists a finite subset $\{\beta_1, ..., \beta_n\}$ of $\mathbf{K} \odot \alpha$. Such that $\{\beta_1, \beta_n\} \vdash \beta$. We will first show that $\beta_1 \wedge . \wedge \beta_n \in \mathbf{K} \odot \alpha$. To do this, we are going to show that if $\beta_1 \in \mathbf{K} \odot \alpha$ and $\beta_2 \in \mathbf{K} \odot \alpha$, then $\beta_1 \wedge \beta_2 \in \mathbf{K} \odot \alpha$. The rest of the proof follows by iteration of the same process. We will consider three cases:

Case (1) $\top \preceq \neg\alpha$. Hence $\mathbf{K} \odot \alpha = \mathbf{K}$. Hence $\beta_1 \in \mathbf{K}$ and $\beta_2 \in \mathbf{K}$. Since $\mathbf{K}$ is a belief set it holds that $\beta_1 \wedge \beta_2 \in \mathbf{K} = \mathbf{K} \odot \alpha$.

Case (2) $\alpha \preceq \bot$ and $\neg\alpha \prec \top$. Then $\mathbf{K} \odot \alpha = \{\beta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \beta\}$. Hence it holds that $\beta_1 \in \mathbf{K}$, $\beta_2 \in \mathbf{K}$, $\neg\alpha \prec \alpha \rightarrow \beta_1$ and $\neg\alpha \prec \alpha \rightarrow \beta_2$. From $\beta_1 \in \mathbf{K}$, $\beta_2 \in \mathbf{K}$ it follows that $\beta_1 \wedge \beta_2 \in \mathbf{K}$ and $(\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2) \in \mathbf{K}$. It holds that $\bot \notin \mathbf{K}$. Thus by (wEE4) it follows that $\bot \prec (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$. By (wEE3) it follows that either $\alpha \rightarrow \beta_1 \preceq (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$ or $\alpha \rightarrow \beta_2 \preceq (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$. In both cases it holds by Lemma 8.1 that $\neg\alpha \prec (\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2)$. Therefore, by (EE2) and Lemma 8.1, it follows that $\neg\alpha \prec \alpha \rightarrow (\beta_1 \wedge \beta_2)$. Hence $\beta_1 \wedge \beta_2 \in \mathbf{K} \odot \alpha$.

Case (3) $\bot \prec \alpha$. Then $\mathbf{K} \odot \alpha = \{\beta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \beta\}$. Hence it holds that $\beta_1 \in \mathbf{K} + \alpha$, $\beta_2 \in \mathbf{K} + \alpha$, $\neg\alpha \prec \alpha \rightarrow \beta_1$, and $\neg\alpha \prec \alpha \rightarrow \beta_2$. From $\beta_1 \in \mathbf{K} + \alpha$, $\beta_2 \in \mathbf{K} + \alpha$ it follows that $\beta_1 \wedge \beta_2 \in \mathbf{K} + \alpha$ and $(\alpha \rightarrow \beta_1) \wedge (\alpha \rightarrow \beta_2) \in \mathbf{K}$. It follows as in the previous case that $\neg\alpha \prec \alpha \rightarrow (\beta_1 \wedge \beta_2)$. Hence $\beta_1 \wedge \beta_2 \in \mathbf{K} \odot \alpha$.

This finishes the first part of the proof. By repeated use of part 1 it holds that if $\beta_1, \beta_2, ..., \beta_n \in \mathbf{K} \odot \alpha$, then $\beta_1 \wedge \beta_2 \wedge . \wedge \beta_n \in \mathbf{K} \odot \alpha$.

It holds that $\{\beta_1 \wedge \beta_2 \wedge . \wedge \beta_n\} \vdash \beta$. We will consider three cases:

Case (1) $\top \preceq \neg\alpha$. Hence $\mathbf{K} \odot \alpha = \mathbf{K}$. Thus $\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n \in \mathbf{K}$. Therefore $\beta \in \mathbf{K} = \mathbf{K} \odot \alpha$.

Case (2) $\bot \prec \alpha$. Then from $\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n \in \mathbf{K} \odot \alpha$ it follows that $\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n \in \mathbf{K} + \alpha$ and $\neg\alpha \prec \alpha \rightarrow (\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n)$. From the former it follows that $\beta \in \mathbf{K} + \alpha$ and from the later, by (EE2) and $\{\alpha \rightarrow (\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n)\} \vdash \alpha \rightarrow \beta$, that $\alpha \rightarrow (\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n) \preceq \alpha \rightarrow \beta$. Thus, by Lemma 8.1 it follows that $\neg\alpha \prec \alpha \rightarrow \beta$. Hence $\beta \in \mathbf{K} \odot \alpha$.

Case (3) $\alpha \preceq \bot$ and $\neg\alpha \prec \top$. It follows by definition from $\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n \in \mathbf{K} \odot \alpha$ that $\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n \in \mathbf{K}$ and $\neg\alpha \prec \alpha \rightarrow (\beta_1 \wedge \beta_2 \wedge ... \wedge \beta_n)$. Thus, it holds that $\beta \in \mathbf{K}$ and (as proved in the previous case) $\neg\alpha \prec \alpha \rightarrow \beta$. Hence $\beta \in \mathbf{K} \odot \alpha$.

Weak relative success: It follows trivially by definition that $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$ if it holds that ($\alpha \preceq \bot$ and $\neg\alpha \prec \top$) or $\top \preceq \neg\alpha$. Assume now that $\bot \prec \alpha$, then it follows by (EE7) that $\neg\alpha \prec \top$. Thus $\neg\alpha \prec \alpha \rightarrow \alpha$. It also holds that $\alpha \in \mathbf{K} + \alpha$. Therefore $\alpha \in \mathbf{K} \odot \alpha$.

Consistency preservation: If $\top \preceq \neg\alpha$ or ($\alpha \preceq \bot$ and $\neg\alpha \prec \top$), then by definition of $\odot$ it follows that $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$. $\mathbf{K}$ is consistent, thus so is $\mathbf{K} \odot \alpha$. Consider now that $\bot \prec \alpha$. Hence by (EE7) it follows that $\neg\alpha \prec \top$. Assume by *reductio ad absurdum* that $\mathbf{K} \odot \alpha \vdash \bot$. Hence, by *closure* (proven above), it follows that $\bot \in \mathbf{K} \odot \alpha$. Thus $\neg\alpha \prec \alpha \rightarrow \bot$. By Proposition 2.4 it follows that $\neg\alpha \prec \neg\alpha$. This contradicts (EE2).

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. We need to prove that $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta$. This can be done by double inclusion, using the definition of $\odot$ and Proposition 2.4.

N-recovery: It follows trivially if $\top \preceq \neg\alpha$. Consider now that $\top \npreceq \neg\alpha$. Thus, by (EEc), it follows that $\neg\alpha \prec \top$. Let $\beta \in \mathbf{K}$. It holds that $\neg\alpha \rightarrow \beta \in \mathbf{K}$. It also holds that $\vdash (\alpha \rightarrow (\neg\alpha \rightarrow \beta)) \leftrightarrow \top$. Thus, by Proposition 2.4, from $\neg\alpha \prec \top$ it follows that $\neg\alpha \prec \alpha \rightarrow (\neg\alpha \rightarrow \beta)$. Hence $\neg\alpha \rightarrow \beta \in \mathbf{K} \odot \alpha$. Therefore $\beta \in \mathbf{K} \odot \alpha + \neg\alpha$.

N-persistence. Assume that $\neg\alpha \in \mathbf{K} \odot \alpha$ and $\beta \in \mathcal{L}$. By *N-relative success*[11] it follows that $\neg\alpha \in \mathbf{K}$. We intend to prove that $\neg\alpha \in \mathbf{K} \odot \beta$. Assume by *reductio ad absurdum* that $\top \npreceq \neg\alpha$. Hence by (EEc) it follows that $\neg\alpha \prec \top$. Thus by definition of $\odot$ it follows from $\neg\alpha \in \mathbf{K} \odot \alpha$ that $\neg\alpha \prec \alpha \rightarrow \neg\alpha$. It holds that $\{\alpha \rightarrow \neg\alpha\} \vdash \neg\alpha$. Thus by (EE2) $\alpha \rightarrow \neg\alpha \preceq \neg\alpha$. Hence by Lemma 8.1 it follows that $\neg\alpha \prec \neg\alpha$. Contradiction. Hence $\top \preceq \neg\alpha$. Thus $\neg\alpha \in \mathbf{K} \odot \alpha = \mathbf{K}$.

If $\top \preceq \neg\beta$, then it follows, by definition of $\odot$, that $\neg\alpha \in \mathbf{K} \odot \beta$. Consider now that $\top \npreceq \neg\beta$. Assume by *reductio ad absurdum* that $\neg\alpha \notin \mathbf{K} \odot \beta$. By definition of $\odot$, having in mind that $\neg\alpha \in \mathbf{K}$, it follows that $\neg\beta \not\prec \beta \rightarrow \neg\alpha$ thus, by (EEc), it follows that $\beta \rightarrow \neg\alpha \preceq \neg\beta$. On the other hand, it holds that $\{\neg\alpha\} \vdash \beta \rightarrow \neg\alpha$. Thus by (EE2) it follows that $\neg\alpha \preceq \beta \rightarrow \neg\alpha$. Hence by (EE1) (applied twice) it follows that $\top \preceq \neg\beta$. Contradiction.

Disjunctive inclusion. Assume that $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. As proven above $\odot$ satisfies inclusion and vacuity. We will prove by cases:

Case (1) $\top \preceq \neg(\alpha \vee \beta)$. Then by (EE1) and (EE2) it follows that $\top \preceq \neg\alpha$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} = \mathbf{K} \odot \alpha$.

Case (2) $\neg(\alpha \vee \beta) \prec \top$.

Case (2.1) $\neg\alpha \notin \mathbf{K}$. Then by $\odot$ vacuity it follows that $\mathbf{K} + \alpha \subseteq \mathbf{K} \odot \alpha$. By $\odot$ inclusion it follows that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} + (\alpha \vee \beta)$. Thus by Lemma 8.2 (b) it follows that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Case (2.2) $\neg\alpha \in \mathbf{K}$. By definition of $\odot$ it follows from $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ that $\neg(\alpha \vee \beta) \not\prec (\alpha \vee \beta) \rightarrow \neg\alpha$. Hence, by (EEc), it follows that $(\alpha \vee \beta) \rightarrow \neg\alpha \preceq \neg(\alpha \vee \beta)$. It holds that $\vdash ((\alpha \vee \beta) \rightarrow \neg\alpha) \leftrightarrow \neg\alpha$. Thus, by Proposition 2.4 it follows that $\neg\alpha \preceq \neg(\alpha \vee \beta)$. Let $\delta \in \mathbf{K} \odot (\alpha \vee \beta)$. By definition of $\odot$ it follows that $\neg(\alpha \vee \beta) \prec (\alpha \vee \beta) \rightarrow \delta$. On the other hand it holds that $\{(\alpha \vee \beta) \rightarrow \delta\} \vdash \alpha \rightarrow \delta$. From which it follows, by (EE2), that $(\alpha \vee \beta) \rightarrow \delta \preceq \alpha \rightarrow \delta$. Thus, by Lemma 8.1 (applied twice), it follows that $\neg\alpha \prec \alpha \rightarrow \delta$.

Case (2.2.1) $\alpha \vee \beta \preceq \bot$. From $\delta \in \mathbf{K} \odot (\alpha \vee \beta)$ it follows that $\delta \in \mathbf{K}$. Therefore, $\delta \in \mathbf{K} \odot \alpha$.

Case (2.2.2) $\bot \prec \alpha \vee \beta$. From $\delta \in \mathbf{K} \odot (\alpha \vee \beta)$ it follows that $\delta \in \mathbf{K} + \alpha \vee \beta$. Thus, by Lemma 8.2 (b) it follows that $\delta \in \mathbf{K} + \alpha$. Therefore, $\delta \in \mathbf{K} \odot \alpha$.

Thus $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Strict improvement. Let $\alpha \in \mathbf{K} \odot \alpha$ and $\alpha \vdash \beta$. Then $\neg\beta \vdash \neg\alpha$. Hence by (EE2) it follows that $\neg\beta \preceq \neg\alpha$. We will consider two cases:

Case (1) $\bot \prec \alpha$. Then, by (EE7), $\neg\alpha \prec \top$. By Lemma 8.1 it follows that $\neg\beta \prec \top$. Thus $\neg\beta \prec \beta \rightarrow \beta$. From $\alpha \vdash \beta$ it follows by (EE2) that $\alpha \preceq \beta$, from which it follows by Lemma 8.1 that $\bot \prec \beta$. It holds that $\beta \in \mathbf{K} + \beta$. Thus, by definition of $\odot$, it follows that $\beta \in \mathbf{K} \odot \beta$.

Case (2) $\bot \not\prec \alpha$. Hence, by (EEc), $\alpha \preceq \bot$. From $\alpha \in \mathbf{K} \odot \alpha$ it follows, by definition of $\odot$, that $\alpha \in \mathbf{K}$. Thus $\beta \in \mathbf{K}$. Hence by Proposition 3.2 it follows that $\mathbf{K} \odot \beta = \mathbf{K}$. Thus $\beta \in \mathbf{K} \odot \beta$.

Disjunctive overlap: We will prove by cases:

Case (1) $\top \preceq \neg\alpha$ and $\top \preceq \neg\beta$. Hence $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta = \mathbf{K}$. On the other hand, it follows by (wEE4) that $\neg\alpha \in \mathbf{K}$ and $\neg\beta \in \mathbf{K}$ (since $\top \in \mathbf{K}$). Thus $\neg\alpha \in \mathbf{K} \odot \alpha$ and $\neg\beta \in \mathbf{K} \odot \beta$. Therefore, by $\odot$ N-persistence and closure (proven above) $\neg(\alpha \vee \beta) \in \mathbf{K} \odot (\alpha \vee \beta)$. Thus, by N-relative success (Proposition 3.2) it follows that $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K}$. Hence $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta = \mathbf{K} \odot (\alpha \vee \beta)$.

[11] By Proposition 3.2, N-*Relative* success follows from *consistency preservation, weak relative success*, and *N-Recovery.* All these postulates were proven above.

Case (2) $\top \not\preceq \neg\alpha$ or $\top \not\preceq \neg\beta$. Hence by (EEc) it follows that $\neg\alpha \prec \top$ or $\neg\beta \prec \top$. If $\top \preceq \neg(\alpha \vee \beta)$, then by (EE2) and (EE1) it follows that $\top \preceq \neg\alpha$ and $\top \preceq \neg\beta$. Contradiction. Hence $\top \not\preceq \neg(\alpha \vee \beta)$. Hence by (EEc) it follows that $\neg(\alpha \vee \beta) \prec \top$.

Case (2.1) $\alpha \vee \beta \preceq \bot$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \{\delta \in \mathbf{K} : \neg(\alpha \vee \beta) \prec (\alpha \vee \beta) \rightarrow \delta\}$. From $\alpha \vee \beta \preceq \bot$ it follows by (EE2) and (EE1) that $\alpha \preceq \bot$ and $\beta \preceq \bot$.

We have three cases to consider:

Case (2.1.1) $\neg\alpha \prec \top$ and $\neg\beta \prec \top$. Hence $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \delta\}$ and $\mathbf{K} \odot \beta = \{\delta \in \mathbf{K} : \neg\beta \prec \beta \rightarrow \delta\}$. Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K} \odot \alpha$ and $\delta \in \mathbf{K} \odot \beta$, from which it follows that $\delta \in \mathbf{K}$, $\neg\alpha \prec \alpha \rightarrow \delta$ and $\neg\beta \prec \beta \rightarrow \delta$. By (EE2) it follows that $\neg(\alpha \vee \beta) \preceq \neg\alpha$ and $\neg(\alpha \vee \beta) \preceq \neg\beta$. Thus, by Lemma 8.1 it follows that $\neg(\alpha \vee \beta) \prec \alpha \rightarrow \delta$ and $\neg(\alpha \vee \beta) \prec \beta \rightarrow \delta$.

On the other hand, from $\delta \in \mathbf{K}$ it follows that $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \in \mathbf{K}$. It holds that $\bot \notin \mathbf{K}$. Thus by (wEE4) it follows that $\bot \prec (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$. By (wEE3) it follows that $(\alpha \rightarrow \delta) \preceq (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$ or $(\beta \rightarrow \delta) \preceq (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$. In both cases it holds by Lemma 8.1 that $\neg(\alpha \vee \beta) \prec (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$. By (EE2) it follows that $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \preceq (\alpha \vee \beta) \rightarrow \delta$. Thus, by Lemma 8.1 it follows that $\neg(\alpha \vee \beta) \prec (\alpha \vee \beta) \rightarrow \delta$. Therefore $\delta \in \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.1.2) $\neg\alpha \prec \top$ and $\top \preceq \neg\beta$. Hence $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \delta\}$ and $\mathbf{K} \odot \beta = \mathbf{K}$. It holds by Lemma 8.1 that $\neg\alpha \prec \neg\beta$. Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K}$ and $\neg\alpha \prec \alpha \rightarrow \delta$. On the other hand, it follows by (EE2) that $\neg\beta \preceq \beta \rightarrow \delta$. Thus, from $\neg\alpha \prec \neg\beta$, it follows by Lemma 8.1 that $\neg\alpha \prec \beta \rightarrow \delta$. By (EE2) it follows that $\neg(\alpha \vee \beta) \preceq \neg\alpha$. Thus, by Lemma 8.1, it follows that $\neg(\alpha \vee \beta) \prec \alpha \rightarrow \delta$ and $\neg(\alpha \vee \beta) \prec \beta \rightarrow \delta$. The rest of the proof for this case follows as in case 2.1.1.

Case (2.1.3) $\top \preceq \neg\alpha$ and $\neg\beta \prec \top$. This case is symmetrical to the previous one.

Case (2.2) $\bot \prec \alpha \vee \beta$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \{\delta \in \mathbf{K} + (\alpha \vee \beta) : \neg(\alpha \vee \beta) \prec (\alpha \vee \beta) \rightarrow \delta\}$. Note that by (EE7) it follows that if $\bot \prec \delta$, then $\neg\delta \prec \top$ (Proposition 4.2).

Case (2.2.1) $\bot \prec \alpha$ and $\bot \prec \beta$. Hence $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \delta\}$ and $\mathbf{K} \odot \beta = \{\delta \in \mathbf{K} + \beta : \neg\beta \prec \beta \rightarrow \delta\}$. Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K} + \alpha$, $\delta \in \mathbf{K} + \beta$, $\neg\alpha \prec \alpha \rightarrow \delta$ and $\neg\beta \prec \beta \rightarrow \delta$. It follows by Lemma 8.2. that $\delta \in \mathbf{K} + \alpha \vee \beta$. From $\delta \in \mathbf{K} + \alpha$ and $\delta \in \mathbf{K} + \beta$ it follows that $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \in \mathbf{K}$. The rest of the proof for this case follows as in case 2.1.1.

Case (2.2.2) $\bot \prec \alpha$ and $\beta \preceq \bot$. It holds that $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} + \alpha : \neg\alpha \prec \alpha \rightarrow \delta\}$. There are two sub-cases to consider:

Case (2.2.2.1) $\neg\beta \prec \top$. Hence $\mathbf{K} \odot \beta = \{\delta \in \mathbf{K} : \neg\beta \prec \beta \rightarrow \delta\}$.

Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K}$, $\neg\alpha \prec \alpha \rightarrow \delta$, and $\neg\beta \prec \beta \rightarrow \delta$. From $\delta \in \mathbf{K}$ it follows that $\delta \in \mathbf{K} + \alpha \vee \beta$ and $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \in \mathbf{K}$. The rest of the proof for this case follows as in case 2.1.1.

Case (2.2.2.2) $\top \preceq \neg\beta$. Hence $\mathbf{K} \odot \beta = \mathbf{K}$. Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K} \subseteq \mathbf{K} + \alpha \vee \beta$ and $\neg\alpha \prec \alpha \rightarrow \delta$. From $\bot \prec \alpha$ it follows by (EE7) that $\neg\alpha \prec \top$. The rest of the proof for this case follows as in case 2.1.2.

Case (2.2.3) $\alpha \preceq \bot$ and $\bot \prec \beta$. This case is symmetrical to 2.2.2.

Case (2.2.4) $\alpha \preceq \bot$ and $\beta \preceq \bot$. Since $\neg\alpha \prec \top$ or $\neg\beta \prec \top$ there are three cases to consider:

Case (2.2.4.1) $\neg\alpha \prec \top$ and $\neg\beta \prec \top$. Hence $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \delta\}$ and $\mathbf{K} \odot \beta = \{\delta \in \mathbf{K} : \neg\beta \prec \beta \rightarrow \delta\}$.

Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K}$, $\neg\alpha \prec \alpha \rightarrow \delta$ and $\neg\beta \prec \beta \rightarrow \delta$. From $\delta \in \mathbf{K}$ it follows that $\delta \in \mathbf{K} + \alpha \vee \beta$ and $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \in \mathbf{K}$. The rest of the proof for this case follows as in case 2.1.1.

Case (2.2.4.2) $\neg\alpha \prec \top$ and $\top \preceq \neg\beta$. Hence $\mathbf{K} \odot \alpha = \{\delta \in \mathbf{K} : \neg\alpha \prec \alpha \rightarrow \delta\}$ and $\mathbf{K} \odot \beta = \mathbf{K}$. Let $\delta \in \mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. Hence $\delta \in \mathbf{K} \subseteq \mathbf{K} + \alpha \vee \beta$ and $\neg\alpha \prec \alpha \rightarrow \delta$. The rest of the proof for this case follows as in case 2.1.2.

Case (2.2.4.3) $\top \preceq \neg\alpha$ and $\neg\beta \prec \top$. This case is symmetrical to 2.2.4.2. □

Proof of Theorem 5.4.

(2) to (1).

Note that if it holds for some $X \in \mathbb{S}$ that $X \cap \|\alpha\| \neq \emptyset$, then by ($\mathbb{S}$4) there is a minimal sphere S_α in $\mathbb{S}$ that intersects with $\|\alpha\|$. It follows from $\mathbb{S}_{in} \subseteq \mathbb{S}$ that either $S_\alpha \in \mathbb{S}_{in}$ or $S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$.

Closure: Follows by the definition of $\odot$ and the fact that the intersection of sets of possible worlds is a belief set.

Vacuity: Assume that $\neg\alpha \notin \mathbf{K}$. Hence $\|\mathbf{K}\| \cap \|\alpha\| \neq \emptyset$. Hence $S_\alpha = \|\mathbf{K}\| \in \mathbb{S}_{in}$. Hence $\mathbf{K} \odot \alpha = \bigcap(S_\alpha \cap \|\alpha\|) = \bigcap(\|\mathbf{K}\| \cap \|\alpha\|) = \bigcap(\|\mathbf{K}\|) + \alpha = \mathbf{K} + \alpha$.

We will now prove that $\odot$ satisfies *inclusion, weak relative success, consistency preservation, strict improvement,* and *N-recovery*. By definition of $\odot$ we have to consider three cases:

Case (A) It holds for all $X \in \mathbb{S}$ that $X \cap \|\alpha\| = \emptyset$. Hence $\mathbf{K} \odot \alpha = \mathbf{K}$.

Case (B) $S_\alpha \in \mathbb{S}_{in}$. It holds that $\|\mathbf{K}\| \subseteq S_\alpha$. Thus $\|\mathbf{K}\| \cap \|\alpha\| \subseteq S_\alpha \cap \|\alpha\| \subseteq \|\alpha\|$. From which it follows that $\bigcap \|\alpha\| \subseteq \bigcap(S_\alpha \cap \|\alpha\|) \subseteq \bigcap(\|\mathbf{K}\| \cap \|\alpha\|)$. Thus, by definition of $\odot$ and Lemma 8.3 (a) and (c) it follows that $Cn(\alpha) \subseteq \mathbf{K} \odot \alpha \subseteq \mathbf{K} + \alpha$.

Case (C) $S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$. It holds that $\|\mathbf{K}\| \subseteq \|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)$. Thus $\mathbf{K} \odot \alpha = \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \subseteq \bigcap(\|\mathbf{K}\|) = \mathbf{K}$.

From the above it follows that $\odot$ satisfies **inclusion** and **weak relative success**.

Consistency Preservation: Assume that $\mathbf{K} \nvdash \bot$. In case (A) and case (C), above, it follows trivially that $\mathbf{K} \odot \alpha \nvdash \bot$. In case (B), it holds that $S_\alpha \cap \|\alpha\| \neq \emptyset$. From which it follows, by definition of $\odot$ and Lemma 8.3 (b) that $\mathbf{K} \odot \alpha \nvdash \bot$.

Strict Improvement. Let $\vdash \alpha \to \beta$ and assume that $\alpha \in \mathbf{K} \odot \alpha$. In case (A) and case (C), above, it follows that $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$, hence $\beta \in \mathbf{K}$. Thus by $\odot$ confirmation (that follows from (weak) vacuity and inclusion, proven above) it follows that $\beta \in \mathbf{K} \odot \beta$. In case (B), from $\vdash \alpha \to \beta$ it follows that $\|\alpha\| \subseteq \|\beta\|$. Thus $S_\alpha \cap \|\beta\| \neq \emptyset$. By the minimality of S_β it follows that $S_\beta \subseteq S_\alpha$. Hence $S_\beta \in \mathbb{S}_{in}$. Thus, as seen above (in case B)), it holds that $\beta \in \mathbf{K} \odot \beta$.

N-recovery. We will consider the cases (A), (B), and (C) above:

Case (A) It holds that $\mathbf{K} \odot \alpha = \mathbf{K}$. Hence $\mathbf{K} \subseteq \mathbf{K} \odot \alpha + \neg\alpha$.

Case (B) It holds that $\alpha \in \mathbf{K} \odot \alpha$. Thus $\mathbf{K} \subseteq \mathbf{K} \odot \alpha + \neg\alpha = \mathcal{L}$.

Case (C) Let $\beta \in \mathbf{K}$. Hence $\|\mathbf{K}\| \subseteq \|\beta\|$. Thus $\|\mathbf{K}\| \cup \|\alpha\| \subseteq \|\alpha\| \cup \|\beta\|$. Therefore $(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \subseteq \|\mathbf{K}\| \cup \|\alpha\| \subseteq \|\alpha\| \cup \|\beta\| = \|\neg\alpha \to \beta\|$. Thus $\neg\alpha \to \beta \in \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) = \mathbf{K} \odot \alpha$. Therefore, $\beta \in \mathbf{K} \odot \alpha + \neg\alpha$.

Now we will show that $\odot$ satisfies the remaining postulates.

Extensionality: If $\vdash \alpha \leftrightarrow \beta$, then $\|\alpha\| = \|\beta\|$. If it holds that $X \cap \|\alpha\| = \emptyset$, for all $X \in \mathbb{S}$, then $\mathbf{K} \odot \alpha = \mathbf{K}$ and $\mathbf{K} \odot \beta = \mathbf{K}$. If $Y \cap \|\alpha\| \neq \emptyset$ for some $Y \in \mathbb{S}$, then there exists S_α and S_β i.e., there exist minimal spheres that intersect with $\|\alpha\|$ and with $\|\beta\|$). From $\vdash \alpha \leftrightarrow \beta$ it follows that $S_\alpha = S_\beta$. Therefore, by definition of $\odot$, $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta$.

N-persistence: Assume that $\neg\beta \in \mathbf{K} \odot \beta$. We will consider two cases:

Case (1) $\vdash \neg\beta$. Hence by $\odot$ closure (proven above) $\neg\beta \in \mathbf{K} \odot \alpha$.

Case (2) $\nvdash \neg\beta$. If $S_\beta \in \mathbb{S}_{in}$, then by definition of $\odot$ it follows that $\beta \in \mathbf{K} \odot \beta$. Thus $\mathbf{K} \odot \beta \vdash \bot$. This contradicts $\odot$ consistency preservation (proven above).

Assume now that $S_\beta \in \mathbb{S} \setminus \mathbb{S}_{in}$. Thus, by definition of $\odot$ it follows, from $\neg\beta \in \mathbf{K} \odot \beta$, that $S_\beta \cap \|\beta\| \subseteq \|\neg\beta\|$. Contradiction. Hence, it holds that $\|\beta\| \cap X = \emptyset$, for all $X \in \mathbb{S}$. Therefore $\mathbf{K} \odot \beta = \mathbf{K}$. Thus $\neg\beta \in \mathbf{K}$.

Case (2.1) It holds that $\|\alpha\| \cap X = \emptyset$, for all $X \in \mathbb{S}$. Therefore $\mathbf{K} \odot \alpha = \mathbf{K}$. Thus $\neg\beta \in \mathbf{K} \odot \alpha$.

Case (2.2) $S_\alpha \in \mathbb{S}_{in}$. It holds that $(S_\alpha \cap \|\alpha\|) \cap \|\beta\| = \emptyset$ (otherwise it would follow that a sphere in $\mathbb{S}$ would intersect with $\|\beta\|$). Hence $S_\alpha \cap \|\alpha\| \subseteq \|\neg\beta\|$. Thus $\neg\beta \in \mathbf{K} \odot \alpha$.

Case (2.3) $S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$. Suppose towards a contradiction that $\neg\beta \notin \mathbf{K} \odot \alpha$. Thus $(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \cap \|\beta\| \neq \emptyset$. Hence $\|\mathbf{K}\| \cap \|\beta\| \neq \emptyset$ or $(S_\alpha \cap \|\alpha\|) \cap \|\beta\| \neq \emptyset$. It follows, in both cases, that a sphere in $\mathbb{S}$ intersects with $\|\beta\|$. Contradiction.

Disjunctive overlap: We will prove by cases:

Case (1) It holds for all $X \in \mathbb{S}$ that $X \cap \|\alpha \vee \beta\| = \emptyset$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K}$. From $\|\alpha\| \subseteq \|\alpha \vee \beta\|$ and $\|\beta\| \subseteq \|\alpha \vee \beta\|$ it follows that no sphere in $\mathbb{S}$ intersects $\|\alpha\|$ or intersects $\|\beta\|$. By definition of $\odot$ it follows that $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta = \mathbf{K}$. Therefore, $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2) It holds for some $X \in \mathbb{S}$ that $X \cap \|\alpha \vee \beta\| \neq \emptyset$. Hence by ($\mathbb{S}4$) there is a minimal sphere $S_{\alpha\vee\beta}$ in $\mathbb{S}$ that intersects $\|\alpha \vee \beta\|$. From $\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$ it follows that $S_{\alpha\vee\beta} \cap \|\alpha\| \neq \emptyset$ or $S_{\alpha\vee\beta} \cap \|\beta\| \neq \emptyset$. In the first case there exists a sphere in $\mathbb{S}$ that intersects $\|\alpha\|$, thus by ($\mathbb{S}4$), there is a minimal sphere S_α in $\mathbb{S}$ that intersect $\|\alpha\|$. By the minimality of S_α it follows that $S_\alpha \subseteq S_{\alpha\vee\beta}$. In this case it also holds that $S_\alpha \cap \|\alpha \vee \beta\| \neq \emptyset$. Thus by the minimality of $S_{\alpha\vee\beta}$ it follows that $S_{\alpha\vee\beta} \subseteq S_\alpha$. Thus $S_{\alpha\vee\beta} = S_\alpha$. In the second case it follows that there exists a minimal sphere S_β in $\mathbb{S}$ that intersect $\|\beta\|$. As in the first case we can conclude that $S_{\alpha\vee\beta} = S_\beta$.

Thus $S_{\alpha\vee\beta} = S_\alpha$ or $S_{\alpha\vee\beta} = S_\beta$. Assume without loss of generality that $S_{\alpha\vee\beta} = S_\alpha$.

Case (2.1) It holds for all $X \in \mathbb{S}$ that $X \cap \|\beta\| = \emptyset$.

Hence $S_{\alpha\vee\beta} \cap \|\beta\| = \emptyset$. Therefore $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_{\alpha\vee\beta} \cap \|\alpha\|$. Thus $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap \|\alpha\|$.

Case (2.1.1) $S_{\alpha\vee\beta} = S_\alpha \in \mathbb{S}_{in}$. Thus $\mathbf{K} \odot (\alpha \vee \beta) = \bigcap(S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|) = \bigcap(S_\alpha \cap \|\alpha\|) = \mathbf{K} \odot \alpha$. Hence $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.1.2) $S_{\alpha\vee\beta} = S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$. From $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap \|\alpha\|$ it follows that $\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|) = \|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)$. Thus $\mathbf{K} \odot \alpha = \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) = \bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)) = \mathbf{K} \odot (\alpha \vee \beta)$. Therefore $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.2) It holds for some $X \in \mathbb{S}$ that $X \cap \|\beta\| \neq \emptyset$. By ($\mathbb{S}4$) it follows there is a minimal sphere S_β in $\mathbb{S}$ that intersect $\|\beta\|$. Thus $S_\beta \cap \|\alpha \vee \beta\| \neq \emptyset$. By the minimality of $S_{\alpha\vee\beta}$ it follows that $S_{\alpha\vee\beta} \subseteq S_\beta$. Hence it holds that $S_{\alpha\vee\beta} \cap \|\beta\| \subseteq S_\beta \cap \|\beta\|$. Hence $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\beta\| \subseteq S_\beta \cap \|\beta\|$. It also holds that $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\alpha\| \subseteq S_\alpha \cap \|\alpha\|$.

Case (2.2.1) $S_{\alpha\vee\beta} = S_\alpha \in \mathbb{S}_{in}$. By definition of $\odot$, it follows that $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.2.1.1) $S_\beta \in \mathbb{S}_{in}$. Thus, from $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\beta\| \subseteq S_\beta \cap \|\beta\|$ it follows that $\bigcap(S_\beta \cap \|\beta\|) \subseteq \bigcap(S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\beta\|)$. Thus, by definition of $\odot$ and Lemma 8.3 (c) it follows that $\mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta) + \beta$. By symmetry of the case, it follows from $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\alpha\| \subseteq S_\alpha \cap \|\alpha\|$ that $\mathbf{K} \odot \alpha \subseteq \mathbf{K} \odot (\alpha \vee \beta) + \alpha$. Thus $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq (\mathbf{K} \odot (\alpha \vee \beta) + \alpha) \cap (\mathbf{K} \odot (\alpha \vee \beta) + \beta)$. Therefore, by Lemma 8.2, $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta) + \alpha \vee \beta = \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.2.1.2) $S_\beta \in \mathbb{S} \setminus \mathbb{S}_{in}$. Hence $S_{\alpha\vee\beta} \cap \|\beta\| = \emptyset$. Thus $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_{\alpha\vee\beta} \cap \|\alpha\|$. Hence $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap \|\alpha\|$. From which it follows that $\bigcap(S_\alpha \cap \|\alpha\|) = \bigcap(S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)$. Therefore, $\mathbf{K} \odot \alpha = \mathbf{K} \odot (\alpha \vee \beta)$ and, consequently, $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.2.2) $S_{\alpha\vee\beta} = S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$. From $S_{\alpha\vee\beta} \subseteq S_\beta$ it follows by condition (c) of Definition 8 that $S_\beta \in \mathbb{S} \setminus \mathbb{S}_{in}$.

Case (2.2.2.1) $S_{\alpha\vee\beta} \cap \|\beta\| \neq \emptyset$. By the minimality of S_β it follows that $S_\beta \subseteq S_{\alpha\vee\beta}$. Thus $S_\alpha = S_{\alpha\vee\beta} = S_\beta$. It holds that $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap \|\alpha \vee \beta\|$. Thus $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap (\|\alpha\| \cup \|\beta\|) = (S_\alpha \cap \|\alpha\|) \cup (S_\alpha \cap \|\beta\|) \subseteq (S_\alpha \cap \|\alpha\|) \cup (S_\beta \cap \|\beta\|)$. Thus $\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|) \subseteq (\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \cup (\|\mathbf{K}\| \cup (S_\beta \cap \|\beta\|))$. Thus $\bigcap((\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \cup (\|\mathbf{K}\| \cup (S_\beta \cap \|\beta\|))) \subseteq \bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|))$, from which it follows that $\bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) \cap \bigcap(\|\mathbf{K}\| \cup (S_\beta \cap \|\beta\|)) \subseteq \bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|))$. Hence $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Case (2.2.2.2) $S_{\alpha\vee\beta} \cap \|\beta\| = \emptyset$. Hence $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_{\alpha\vee\beta} \cap \|\alpha\|$. Thus $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| = S_\alpha \cap \|\alpha\|$. Hence $\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|) = \|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)$. Therefore, $\bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)) = \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|))$. Hence $\mathbf{K} \odot \alpha = \mathbf{K} \odot (\alpha \vee \beta)$ and, consequently, $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta \subseteq \mathbf{K} \odot (\alpha \vee \beta)$.

Disjunctive inclusion: Assume that $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. We will prove by cases:

Case (1) It holds for all $X \in \mathbb{S}$ that $X \cap \|\alpha \vee \beta\| = \emptyset$. Hence $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K}$. Thus $\neg\alpha \notin \mathbf{K}$. From which it follows by $\odot$ *vacuity* (proven above) that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Thus $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$.

Case (2) It holds for some $X \in \mathbb{S}$ that $X \cap \|\alpha \vee \beta\| \neq \emptyset$. Hence by ($\mathbb{S}4$) there is a minimal sphere $S_{\alpha\vee\beta}$ in $\mathbb{S}$ that intersects $\|\alpha \vee \beta\|$.

Case (2.1) $S_{\alpha\vee\beta} \in \mathbb{S}_{in}$. Thus $\mathbf{K} \odot (\alpha \vee \beta) = \bigcap(S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)$. From $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ it follows that $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\alpha\| \neq \emptyset$. Thus $S_{\alpha\vee\beta} \cap \|\alpha\| \neq \emptyset$. By the minimality of S_α it follows that $S_\alpha \subseteq S_{\alpha\vee\beta}$. Hence by Proposition 5.2 it follows that $S_\alpha \in \mathbb{S}_{in}$. Thus, $S_\alpha \cap \|\alpha\| \subseteq S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|$. Therefore, $\mathbf{K} \odot (\alpha \vee \beta) = \bigcap(S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|) \subseteq \bigcap(S_\alpha \cap \|\alpha\|) = \mathbf{K} \odot \alpha$.

Case (2.2) $S_{\alpha\vee\beta} \in \mathbb{S} \setminus \mathbb{S}_{in}$. Thus $\mathbf{K} \odot (\alpha \vee \beta) = \bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|))$. Hence $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K}$. On the other hand, from $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$ it follows that $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\alpha\| \neq \emptyset$ or $\|\mathbf{K}\| \cap \|\alpha\| \neq \emptyset$. In the latter case it follows by $\neg\alpha \notin \mathbf{K}$. Thus by $\odot$ *vacuity* it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Thus, $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$. Assume now that $S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\| \cap \|\alpha\| \neq \emptyset$. Hence $S_{\alpha\vee\beta} \cap \|\alpha\| \neq \emptyset$. Thus, by the minimality of S_α it follows that $S_\alpha \subseteq S_{\alpha\vee\beta}$. On the other hand, $S_\alpha \cap \|\alpha \vee \beta\| \neq \emptyset$. Thus, $S_{\alpha\vee\beta} \subseteq S_\alpha$. Hence, $S_{\alpha\vee\beta} = S_\alpha$. Therefore, $\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|) \subseteq \|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha \vee \beta\|) = \|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)$. Hence, $\mathbf{K} \odot (\alpha \vee \beta) = \bigcap(\|\mathbf{K}\| \cup (S_{\alpha\vee\beta} \cap \|\alpha \vee \beta\|)) \subseteq \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) = \mathbf{K} \odot \alpha$.

(1) to (2).
Assume that $\odot$ satisfies all the postulates listed in statement (1). By Propositions 3.2 and 3.3 it follows that $\odot$ satisfies N-relative success, confirmation, and containment. Consider the following constructions for $\mathbb{S}$ and $\mathbb{S}_{in}$:

$S \in \mathbb{S}_{in}$ iff (i) $S = \|\mathbf{K}\|$ or (ii) $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|\}$ and $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$.

$S \in \mathbb{S}$ iff (i) $S = \|\mathbf{K}\|$ or (ii) $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset\}$, $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$ and if $S \cap \|\alpha\| = \emptyset$ and $S \notin \mathbb{S}_{in}$, then $\|\mathbf{K} \odot \alpha\| \cap S = \|\mathbf{K}\|$.

We will show that:

(1) $(\mathbb{S}_{in}, \mathbb{S})$ is a two-level system of spheres, centered on $\|\mathbf{K}\|$. To do so, it is necessary to prove that:
   (a) $\mathbb{S}$ and $\mathbb{S}_{in}$ satisfy conditions ($\mathbb{S}1$) to ($\mathbb{S}4$), of Definition 3, with the exception of ($\mathbb{S}3$); (b) $\mathbb{S}_{in} \subseteq \mathbb{S}$; (c) If $X \in \mathbb{S}_{in}$, then $X \subseteq Y$ for all $Y \in \mathbb{S} \setminus \mathbb{S}_{in}$.
(2) If $\|\alpha\| = \emptyset$, then $\mathbf{K} \odot \alpha = \mathbf{K}$;
(3) For α such that $\mathbf{K} \odot \alpha \nvdash \neg\alpha$ it holds for $S(\alpha) = \bigcup\{\|\mathbf{K} \odot \delta\| : \|\alpha\| \subseteq \|\delta\|\}$ that:
   (a) $S(\alpha) \in \mathbb{S}$;
   (b) $S(\alpha) = S_\alpha$ (i.e., $S(\alpha)$ is the minimal sphere that intersects with $\|\alpha\|$);
   (c) The following equality holds (where $S_\alpha = S(\alpha)$):

$$\mathbf{K} \odot \alpha = \begin{cases} \bigcap(S_\alpha \cap \|\alpha\|) & \text{if } S_\alpha \in \mathbb{S}_{in} \\ \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) & \text{if } S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in} \\ \mathbf{K} & \text{if } X \cap \|\alpha\| = \emptyset, \text{ for all } X \in \mathbb{S} \end{cases}$$

(1) We start by showing that $\mathbb{S}_{in}$ satisfies conditions ($\mathbb{S}1$), ($\mathbb{S}2$) and ($\mathbb{S}4$).

($\mathbb{S}2$) That $\|\mathbf{K}\| \in \mathbb{S}_{in}$ follows by definition of $\mathbb{S}_{in}$. Let $S \in \mathbb{S}_{in}$, we intend to show that $\|\mathbf{K}\| \subseteq S$. It follows trivially if $S = \|\mathbf{K}\|$. Assume now that $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that $\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|\}$ and $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$. It holds that $S \cap \|\top\| \neq \emptyset$, from which it follows that $\|\mathbf{K} \odot \top\| \subseteq S$. By $\odot$ confirmation (that according to Proposition 3.2 follows from weak vacuity and inclusion) it follows that $\mathbf{K} \odot \top = \mathbf{K}$. Thus $\|\mathbf{K}\| \subseteq S$.

($\mathbb{S}4$) Let $S \in \mathbb{S}_{in}$ be such that $\|\alpha\| \cap S \neq \emptyset$. Let S' be the intersection of all spheres in $S'' \in \mathbb{S}_{in}$ such that $\|\alpha\| \cap S'' \neq \emptyset$. It holds that $S' \subseteq S$. We must prove that $S' \in \mathbb{S}_{in}$. If $\|\alpha\| \cap \|\mathbf{K}\| \neq \emptyset$, then

$S' = \|\mathbf{K}\|$. Thus $S' \in \mathbb{S}_{in}$. Assume now that $\|\alpha\| \cap \|\mathbf{K}\| = \emptyset$. We will prove that $S' \in \mathbb{S}_{in}$ by showing that: (i) $S' \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|\}$; (ii) $S' \neq \emptyset$; (iii) $\|\mathbf{K} \odot \beta\| \subseteq S'$ for all β such that $S' \cap \|\beta\| \neq \emptyset$.

(i) Follows trivially, since $S'' \subseteq S \in \mathbb{S}_{in}$ and $S \neq \|\mathbf{K}\|$.

(ii) It holds that all spheres in $\mathbb{S}_{in}$ contain $\|\mathbf{K}\|$. On the other hand, it holds that $\|\mathbf{K}\| \neq \emptyset$ (since $\mathbf{K}$ is consistent). Thus $\emptyset \neq \|\mathbf{K}\| \subseteq S''$.

(iii) Let β be such that $S' \cap \|\beta\| \neq \emptyset$. It holds for all $S \in \mathbb{S}_{in}$ such that $S' \subseteq S$ that $S \cap \|\beta\| \neq \emptyset$. Therefore $\|\mathbf{K} \odot \beta\| \subseteq S$. Hence the intersection of all such spheres contains $\|\mathbf{K} \odot \beta\|$. Thus $\|\mathbf{K} \odot \beta\| \subseteq S'$.

($\mathbb{S}1$) Let S_1 and S_2 be elements of $\mathbb{S}_{in}$. We have to show that either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. As seen in the proof of ($\mathbb{S}2$) this holds if $S_1 = \|\mathbf{K}\|$ or $S_2 = \|\mathbf{K}\|$. Suppose the contrary and assume towards a contradiction that $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. Hence there exists $w_1 \in S_1 \setminus S_2$. Then $w_1 \in \|\mathbf{K} \odot \alpha\|$ for some α such that $\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|$. Thus $w_1 \in S_1 \cap \|\alpha\|$. Therefore $\|\mathbf{K} \odot \alpha\| \subseteq S_1$. On the other hand, it follows that $S_2 \cap \|\alpha\| = \emptyset$, otherwise it would follow that $w_1 \in \|\mathbf{K} \odot \alpha\| \subseteq S_2$. Thus $S_2 \subseteq \|\neg\alpha\|$. There exists also $w_2 \in S_2 \setminus S_1$, from which it follows in the same way that there exists some β such that $w_2 \in \|\mathbf{K} \odot \beta\| \subseteq S_2$ and $S_1 \subseteq \|\neg\beta\|$.

It holds that $\|\alpha\| \subseteq \|\alpha \vee \beta\|$ and $\|\beta\| \subseteq \|\alpha \vee \beta\|$. Thus $S_1 \cap \|\alpha \vee \beta\| \neq \emptyset$ and $S_2 \cap \|\alpha \vee \beta\| \neq \emptyset$. From which it follows that $\|\mathbf{K} \odot (\alpha \vee \beta)\| \subseteq S_1 \cap S_2$. Hence $\|\mathbf{K} \odot (\alpha \vee \beta)\| \subseteq \|\neg\alpha\| \cap \|\neg\beta\| = \|\neg\alpha \wedge \neg\beta\|$. Thus $\mathbf{K} \odot (\alpha \vee \beta) \vdash \neg\alpha \wedge \neg\beta$. By $\odot$ closure it follows that $\neg(\alpha \vee \beta) \in \mathbf{K} \odot (\alpha \vee \beta)$. From which it follows, by $\odot$ N-Persistence, that $\neg(\alpha \vee \beta) \in \mathbf{K} \odot \alpha$. Thus $\mathbf{K} \odot \alpha \vdash \neg\alpha$. On the other hand, it holds that $\mathbf{K} \odot \alpha \vdash \alpha$ (since $\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|$). From which it follows by $\odot$ consistency preservation that $\mathbf{K} \vdash \perp$. Contradiction.

We will show that $\mathbb{S}$ satisfies conditions ($\mathbb{S}1$), ($\mathbb{S}2$), ($\mathbb{S}4$) and that conditions (b) and (c) of Definition 8 hold.

We start by showing that $\mathbb{S}_{in} \subseteq \mathbb{S}$ (condition (b) of Definition 8). Let $S \in \mathbb{S}_{in}$. If $S = \|\mathbf{K}\|$, then $S \in \mathbb{S}$. Assume now that $S \neq \|\mathbf{K}\|$. Hence $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that $\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|\}$ and $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$. Let $w \in S$. Hence $w \in \|\mathbf{K} \odot \beta\|$, for some β such that$\|\mathbf{K} \odot \beta\| \subseteq \|\beta\|$. Thus $\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$. Hence $w \in \|\mathbf{K} \odot \beta\|$, for some β such that$\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$. Thus $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that $\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset\}$ and $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$. Thus $S \in \mathbb{S}$. Therefore $\mathbb{S}_{in} \subseteq \mathbb{S}$.

($\mathbb{S}2$) The proof that $\mathbb{S}$ satisfies ($\mathbb{S}2$) is similar to the proof that $\mathbb{S}_{in}$ satisfies ($\mathbb{S}2$) presented above.

We will now show that if $S \in \mathbb{S}_{in}$, then $S \subseteq S'$ for all $S' \in \mathbb{S} \setminus \mathbb{S}_{in}$ (condition (c) of Definition 5.1).

Case (1) $S = \|\mathbf{K}\|$. Let $S' \in \mathbb{S}$. Hence $S \subseteq S'$, for all $S' \in \mathbb{S} \setminus \mathbb{S}_{in}$ (since as proven above $\mathbb{S}$ satisfies ($\mathbb{S}2$)).

Case (2) $\emptyset \neq S \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \subset \|\alpha\|\}$, $\|\mathbf{K} \odot \alpha\| \subseteq S$ for all α such that $S \cap \|\alpha\| \neq \emptyset$.

Let $S' \in \mathbb{S} \setminus \mathbb{S}_{in}$ and suppose towards a contradiction that $S \nsubseteq S'$. Hence there exists $w \in S \setminus S'$. Then there exists α such that $w \in \|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|$. From $w \notin S'$ it follows that $S' \cap \|\alpha\| = \emptyset$ (otherwise it would follow that $w \in \|\mathbf{K} \odot \alpha\| \subseteq S'$). Hence $S' \cap \|\mathbf{K} \odot \alpha\| = \|\mathbf{K}\|$. Thus $\|\mathbf{K}\| = \emptyset$. Contradiction, since $\mathbf{K}$ is consistent.

Now we show that $\mathbb{S}$ satisfies conditions ($\mathbb{S}1$) and ($\mathbb{S}4$).

($\mathbb{S}1$) Let S_1 and S_2 be elements of $\mathbb{S}$. We have to show that either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. As proven above, it holds that $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$ if $\{S_1, S_2\} \subseteq \mathbb{S}_{in}$. It remains to consider three cases:

Case (1) $S_1 \in \mathbb{S}_{in}$ and $S_2 \in \mathbb{S} \setminus \mathbb{S}_{in}$. Condition (c) of Definition 8 holds. Thus $S_1 \subseteq S_2$.

Case (2) $S_1 \in \mathbb{S} \setminus \mathbb{S}_{in}$ and $S_2 \in \mathbb{S}_{in}$. This case is symmetric with case 1, thus $S_2 \subseteq S_1$.

Case (3) $S_1 \notin \mathbb{S}_{in}$ and $S_2 \notin \mathbb{S}_{in}$. Suppose the contrary and assume towards a contradiction that $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. Hence there exists $w_1 \in S_1 \setminus S_2$. Then $w_1 \in \|\mathbf{K} \odot \alpha\|$ for some α such that $\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset$. If $S_1 \cap \|\alpha\| = \emptyset$, then $S_1 \cap \|\mathbf{K} \odot \alpha\| = \|\mathbf{K}\|$. Thus $w_1 \in \|\mathbf{K}\|$. It yields that $\|\mathbf{K}\| \subseteq S_2$ from which it follows that $w_1 \in S_2$. Contradiction. Assume now that $S_1 \cap \|\alpha\| \neq \emptyset$. Therefore $\|\mathbf{K} \odot \alpha\| \subseteq S_1$. On the other hand, it follows that $S_2 \cap \|\alpha\| = \emptyset$, otherwise it would follow that $w_1 \in \|\mathbf{K} \odot \alpha\| \subseteq S_2$. Thus $S_2 \subseteq \|\neg\alpha\|$. There exists also $w_2 \in S_2 \setminus S_1$. Then $w_2 \in \|\mathbf{K} \odot \beta\|$ for some β such that $\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$. Assuming that $S_2 \cap \|\beta\| = \emptyset$ leads to a contradiction. Assume now that $S_2 \cap \|\beta\| \neq \emptyset$. Hence $w_2 \in \|\mathbf{K} \odot \beta\| \subseteq S_2$ and $S_1 \subseteq \|\neg\beta\|$.

Reasoning in the same way as in the proof that $\mathbb{S}_{in}$ satisfies ($\mathbb{S}1$), we conclude that $\mathbf{K} \odot \alpha \vdash \neg\alpha$. This leads to a contradiction, since it holds that $\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset$.

($\mathbb{S}4$) Let $S \in \mathbb{S}$ be such that $\|\alpha\| \cap S \neq \emptyset$. Let S' be the intersection of all spheres in $S'' \in \mathbb{S}$ such that $\|\alpha\| \cap S'' \neq \emptyset$. It holds that $S' \subseteq S$. We must prove that $S' \in \mathbb{S}$.

If $\|\alpha\| \cap \|\mathbf{K}\| \neq \emptyset$, then $S' = \|\mathbf{K}\|$ and we are done. Assume now that $\|\alpha\| \cap \|\mathbf{K}\| = \emptyset$. To prove that $S' \in \mathbb{S}$ we need to show:

(i) $S' \subseteq \{w : w \in \|\mathbf{K} \odot \alpha\|$, for some α such that$\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset\}$; (ii) $S' \neq \emptyset$; (iii) $\|\mathbf{K} \odot \beta\| \subseteq S'$ for all β such that $S' \cap \|\beta\| \neq \emptyset$; (iv) if $S' \cap \|\beta\| = \emptyset$ and $S' \notin \mathbb{S}_{in}$, then $\|\mathbf{K} \odot \beta\| \cap S' = \|\mathbf{K}\|$.

(i) Follows trivially, since $S' \subseteq S$ and $S \neq \|\mathbf{K}\|$.

(ii) It holds that all spheres in $\mathbb{S}$ contain $\|\mathbf{K}\|$. On the other hand, it holds that $\|\mathbf{K}\| \neq \emptyset$ (since $\mathbf{K}$ is consistent). Thus $\emptyset \neq \|\mathbf{K}\| \subseteq S'$.

(iii) Let β be such that $S' \cap \|\beta\| \neq \emptyset$. It holds for all $S \in \mathbb{S}$ such that $S' \subseteq S$ that $S \cap \|\beta\| \neq \emptyset$. Therefore $\|\mathbf{K} \odot \beta\| \subseteq S$. Hence the intersection of all such spheres contains $\|\mathbf{K} \odot \beta\|$. Thus $\|\mathbf{K} \odot \beta\| \subseteq S'$.

(iv) Assume that $S' \cap \|\beta\| = \emptyset$ and $S' \notin \mathbb{S}_{in}$. If there exists $Y \in \mathbb{S}_{in}$ such that $\|\alpha\| \cap Y \neq \emptyset$, then $S' \subseteq Y$. From which it follows, by Proposition 5.2, that $S' \in \mathbb{S}_{in}$. Contradiction. Therefore, it holds that if Y is such that $\|\alpha\| \cap Y \neq \emptyset$, then $Y \in \mathbb{S} \setminus \mathbb{S}_{in}$. We will consider two cases:

Case (1) For some $Y \in \mathbb{S} \setminus \mathbb{S}_{in}$ such that $\|\alpha\| \cap Y \neq \emptyset$ it holds that $\|\beta\| \cap Y = \emptyset$, then $Y \cap \|\mathbf{K} \odot \beta\| = \|\mathbf{K}\|$. Hence $\|\mathbf{K}\| \subseteq \|\mathbf{K} \odot \beta\|$. Therefore, by ($\mathbb{S}2$), $\|\mathbf{K}\| \subseteq \|\mathbf{K} \odot \beta\| \cap S'$. On the other hand, it holds that $Y \subseteq \|\neg\beta\|$. By N-recovery it holds that $\mathbf{K} \subseteq \mathbf{K} \odot \beta + \neg\beta$. Thus $\|\mathbf{K} \odot \beta\| \cap S' \subseteq \|\mathbf{K} \odot \beta\| \cap Y \subseteq \|\mathbf{K} \odot \beta\| \cap \|\neg\beta\| = \|\mathbf{K} \odot \beta + \neg\beta\| \subseteq \|\mathbf{K}\|$. Thus $\|\mathbf{K} \odot \beta\| \cap S' = \|\mathbf{K}\|$.

Case (2) For all $Y \in \mathbb{S} \setminus \mathbb{S}_{in}$ such that $\|\alpha\| \cap Y \neq \emptyset$ it holds that $\|\beta\| \cap Y \neq \emptyset$. Then $\|\mathbf{K} \odot \beta\| \subseteq Y$ for all such Y. Thus $\|\mathbf{K} \odot \beta\| \subseteq S'$. Hence $\|\mathbf{K} \odot \beta\| \cap S' = \|\mathbf{K} \odot \beta\|$. If $\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$, then $S' \cap \|\beta\| \neq \emptyset$. Contradiction.

Assume now that $\|\mathbf{K} \odot \beta\| \cap \|\beta\| = \emptyset$. Then $\mathbf{K} \odot \beta \vdash \neg\beta$. Thus, by $\odot$ closure and N-relative success, it follows that $\mathbf{K} \odot \beta = \mathbf{K}$. Therefore $\|\mathbf{K} \odot \beta\| = \|\mathbf{K}\|$. Hence, $\|\mathbf{K} \odot \beta\| \cap S' = \|\mathbf{K}\|$.

(2) Assume that $\|\alpha\| = \emptyset$. Thus $S \cap \|\alpha\| = \emptyset$ for all $S \in \mathbb{S}$. On the other hand $\vdash \neg\alpha$. Thus by $\odot$ closure it holds that $\neg\alpha \in \mathbf{K} \odot \alpha$. Hence by $\odot$ N-relative success $\mathbf{K} \odot \alpha = \mathbf{K}$.

(3) (a) Assume now that $\mathbf{K} \odot \alpha \nvdash \neg\alpha$ and let $S(\alpha) = \bigcup\{\|\mathbf{K} \odot \delta\| : \|\alpha\| \subseteq \|\delta\|\}$. We will now prove that $S(\alpha) \in \mathbb{S}$.

It holds that $\|\alpha\| \subseteq \|\top\|$. Thus $\|\mathbf{K} \odot \top\| \subseteq S(\alpha)$. Hence $\|\mathbf{K}\| \subseteq S(\alpha)$. Thus $S(\alpha) \neq \emptyset$ (since $\mathbf{K}$ is consistent).

Let $w \in S(\alpha)$. Hence there exists β such that $w \in \|\mathbf{K} \odot \beta\|$ and $\|\alpha\| \subseteq \|\beta\|$. Suppose towards a contradiction that $\|\mathbf{K} \odot \beta\| \cap \|\beta\| = \emptyset$. Hence $\mathbf{K} \odot \beta \vdash \neg\beta$. By $\odot$ N-persistence it follows that $\mathbf{K} \odot \alpha \vdash \neg\beta$. Thus $\mathbf{K} \odot \alpha \vdash \neg\alpha$ (since $\|\alpha\| \subseteq \|\beta\|$). Contradiction. Thus $\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$. Therefore $S(\alpha) \subseteq \{w : w \in \|\mathbf{K} \odot \beta\|$, for some β such that$\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset\}$.

Suppose now that $S(\alpha) \cap \|\beta\| \neq \emptyset$. It follows by definition of $S(\alpha)$ that there exists some δ such that $\|\alpha\| \subseteq \|\delta\|$ and $\|\mathbf{K} \odot \delta\| \cap \|\beta\| \neq \emptyset$. From $\|\alpha\| \subseteq \|\delta\|$ it follows that $\|\alpha\| \subseteq \|\delta \vee \beta\|$. Thus $\|\mathbf{K} \odot (\delta \vee \beta)\| \subseteq S(\alpha)$. Assume towards a contradiction that $\neg(\delta \vee \beta) \in \mathbf{K} \odot (\delta \vee \beta)$, then by $\odot$ N-persistence it follows that $\neg(\delta \vee \beta) \in \mathbf{K} \odot \delta$, from which it follows by $\odot$ closure that $\neg\delta \in \mathbf{K} \odot \delta$. As seen above it follows from the latter that $\neg\alpha \in \mathbf{K} \odot \alpha$. Contradiction. Hence, by $\odot$ closure, either $\neg\delta \notin \mathbf{K} \odot (\delta \vee \beta)$ or $\neg\beta \notin \mathbf{K} \odot (\delta \vee \beta)$. Assume towards a contradiction that $\neg\beta \in \mathbf{K} \odot (\delta \vee \beta)$. Hence $\neg\delta \notin \mathbf{K} \odot (\delta \vee \beta)$, from which it follows by $\odot$ disjunctive inclusion that $\mathbf{K} \odot (\delta \vee \beta) \subseteq \mathbf{K} \odot \delta$.

Hence $\neg\beta \in \mathbf{K} \odot \delta$. Contradiction since $\|\mathbf{K} \odot \delta\| \cap \|\beta\| \neq \emptyset$. Therefore $\neg\beta \notin \mathbf{K} \odot (\delta \vee \beta)$. Thus, by $\odot$ disjunctive inclusion it follows that $\mathbf{K} \odot (\delta \vee \beta) \subseteq \mathbf{K} \odot \beta$. Hence $\|\mathbf{K} \odot \beta\| \subseteq \|\mathbf{K} \odot (\delta \vee \beta)\|$. Thus $\|\mathbf{K} \odot \beta\| \subseteq S(\alpha)$.

Suppose that $S(\alpha) \cap \|\beta\| = \emptyset$ and $S(\alpha) \notin \mathbb{S}_{in}$. We intend to prove that $\|\mathbf{K} \odot \beta\| \cap S(\alpha) = \|\mathbf{K}\|$. It follows from $S(\alpha) \cap \|\beta\| = \emptyset$ that $\|\mathbf{K} \odot \delta\| \cap \|\beta\| = \emptyset$ for all δ such that $\|\alpha\| \subseteq \|\delta\|$. Thus $\mathbf{K} \odot \delta \vdash \neg\beta$, for all δ such that $\|\alpha\| \subseteq \|\delta\|$.

On the other hand, it follows from $S(\alpha) \notin \mathbb{S}_{in}$ that there exists some ξ such that $\|\alpha\| \subseteq \|\xi\|$ and $\|\mathbf{K} \odot \xi\| \nsubseteq \|\xi\|$. Therefore $\alpha \notin \mathbf{K} \odot \alpha$, otherwise it would follow by $\odot$ strict improvement that $\|\mathbf{K} \odot \xi\| \subseteq \|\xi\|$.

Let $w \in S(\alpha) \cap \|\mathbf{K} \odot \beta\|$. Hence $w \in \|\neg\beta\| \cap \|\mathbf{K} \odot \beta\|$. Thus $w \in \|\mathbf{K} \odot \beta + \neg\beta\|$. By N-recovery it holds that $\mathbf{K} \subseteq \mathbf{K} \odot \beta + \neg\beta$. Thus $\|\mathbf{K} \odot \beta + \neg\beta\| \subseteq \|\mathbf{K}\|$. Hence $S(\alpha) \cap \|\mathbf{K} \odot \beta\| \subseteq \|\mathbf{K}\|$. We will now prove the other inclusion.

It holds that $\|\alpha\| \subseteq \|\top\|$. Thus $\|\mathbf{K} \odot \top\| \subseteq S(\alpha)$. Therefore, $\|\mathbf{K}\| \subseteq S(\alpha)$.

Assume towards a contradiction that $\beta \in \mathbf{K} \odot \beta$. Thus $\|\mathbf{K} \odot \beta\| \subseteq \|\beta\|$. By $\odot$ strict improvement it follows that $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$. It also holds that $\|\alpha\| \subseteq \|\alpha \vee \beta\|$. Thus $\|\mathbf{K} \odot (\alpha \vee \beta)\| \subseteq S(\alpha)$. We will consider two cases:

Case (1) $\neg\beta \notin \mathbf{K} \odot (\alpha \vee \beta)$. Then $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \beta$. Hence $\|\mathbf{K} \odot \beta\| \subseteq \|\mathbf{K} \odot (\alpha \vee \beta)\|$. It holds that $\|\mathbf{K} \odot \beta\| \neq \emptyset$ (since, by $\odot$ consistency preservation $\mathbf{K} \odot \beta \nvdash \perp$). Thus $S(\alpha) \cap \|\mathbf{K} \odot \beta\| \neq \emptyset$. Therefore, $S(\alpha) \cap \|\beta\| \neq \emptyset$. Contradiction.

Case (2) $\neg\beta \in \mathbf{K} \odot (\alpha \vee \beta)$. By $\odot$ consistency preservation it follows that $\neg\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. Hence by $\odot$ disjunctive inclusion it follows that $\mathbf{K} \odot (\alpha \vee \beta) \subseteq \mathbf{K} \odot \alpha$. Therefore, $\{\neg\beta, \alpha \vee \beta\} \subseteq \mathbf{K} \odot \alpha$. Hence $\mathbf{K} \odot \alpha \vdash \alpha$. Contradiction.

Both cases lead to a contradiction. Therefore $\beta \notin \mathbf{K} \odot \beta$. Hence, by $\odot$ weak relative success, it follows that $\mathbf{K} \odot \beta \subseteq \mathbf{K}$. Thus $\|\mathbf{K}\| \subseteq \|\mathbf{K} \odot \beta\|$. Hence $\|\mathbf{K}\| \subseteq S(\alpha) \cap \|\mathbf{K} \odot \beta\|$. Therefore $S(\alpha) \in \mathbb{S}$.

(3) (b) We will now show that $S(\alpha) \cap \|\alpha\| \neq \emptyset$ and $S(\alpha) \subseteq S$ for all $S \in \mathbb{S}$ such that $S \cap \|\alpha\| \neq \emptyset$.

It holds that $\|\mathbf{K} \odot \alpha\| \subseteq S(\alpha)$ (by letting $\delta = \alpha$). From $\mathbf{K} \odot \alpha \nvdash \neg\alpha$, it follows that $\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \neq \emptyset$. From the former it follows that $\|\mathbf{K} \odot \alpha\| \cap \|\alpha\| \subseteq S(\alpha) \cap \|\alpha\|$. Therefore, $S(\alpha) \cap \|\alpha\| \neq \emptyset$.

Let $S \in \mathbb{S}$ be such that $S \cap \|\alpha\| \neq \emptyset$. We intend to prove that $S(\alpha) \subseteq S$. Let $w \in S(\alpha)$. Hence there exists β such that $w \in \|\mathbf{K} \odot \beta\|$ and $\|\alpha\| \subseteq \|\beta\|$.

From $S \cap \|\alpha\| \neq \emptyset$ it follows that $S \cap \|\beta\| \neq \emptyset$. Thus, by definition of $\mathbb{S}$ it follows that $\|\mathbf{K} \odot \beta\| \subseteq S$. Therefore, $w \in S$. Hence $S(\alpha) \subseteq S$. Therefore, $S(\alpha)$ is the minimal sphere in $\mathbb{S}$ that intersects with $\|\alpha\|$, i.e. $S(\alpha) = S_\alpha$.

(3) (c) We will now show that the following equality holds:

$$\mathbf{K} \odot \alpha = \begin{cases} \bigcap(S_\alpha \cap \|\alpha\|) & \text{if } S_\alpha \in \mathbb{S}_{in} \\ \bigcap(\|\mathbf{K}\| \cup (S_\alpha \cap \|\alpha\|)) & \text{if } S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in} \\ \mathbf{K} & \text{if } X \cap \|\alpha\| = \emptyset, \text{ for all } X \in \mathbb{S} \end{cases}$$

where $S_\alpha = S(\alpha)$.

If for all $X \in \mathbb{S}$ it holds that $X \cap \|\alpha\| = \emptyset$, then it holds that $\mathbf{K} \odot \alpha \vdash \neg\alpha$, otherwise it would follow that $S(\alpha) \cap \|\alpha\| \neq \emptyset$ and, as proven above, it holds that $S(\alpha) \in \mathbb{S}$. From $\mathbf{K} \odot \alpha \vdash \neg\alpha$ it follows, by $\odot$ N-relative success, that $\mathbf{K} \odot \alpha = \mathbf{K}$.

Assume now that there exists $X \in \mathbb{S}$ such that $X \cap \|\alpha\| \neq \emptyset$. Hence there exists $w \in X \cap \|\alpha\|$. Thus, by definition of $\mathbb{S}$ it follows that $w \in \|\mathbf{K} \odot \beta\|$, for some β such that $\|\mathbf{K} \odot \beta\| \cap \|\beta\| \neq \emptyset$. Hence $\|\mathbf{K} \odot \beta\| \cap \|\alpha\| \neq \emptyset$. From which it follows that $\mathbf{K} \odot \beta \nvdash \neg\alpha$. Thus, by $\odot$ N-persistence $\mathbf{K} \odot \alpha \nvdash \neg\alpha$. There are two cases to consider:

Case (1) $S_\alpha \in \mathbb{S}_{in}$.

Case (1.1) $S_\alpha = \|\mathbf{K}\|$. Hence $\neg\alpha \notin \mathbf{K}$. Thus by $\odot$ vacuity and inclusion it follows that $\mathbf{K} \odot \alpha = \mathbf{K} + \alpha$. On the other hand, $\bigcap(S_\alpha \cap \|\alpha\|) = \bigcap(\|\mathbf{K}\| \cap \|\alpha\|) = \bigcap(\|\mathbf{K} + \alpha\|) = \mathbf{K} + \alpha$. Thus $\bigcap(S_\alpha \cap \|\alpha\|) = \mathbf{K} \odot \alpha$.

Case (1.2) $S_\alpha \neq \|\mathbf{K}\|$. It holds that $S_\alpha \cap \|\alpha\| \neq \emptyset$. Thus there exists $w \in S_\alpha \cap \|\alpha\|$. Thus, by definition of $\mathbb{S}_{in}$ it follows that there exists β such that $w \in \|\mathbf{K} \odot \beta\| \subseteq \|\beta\|$. Hence $w \in \|\mathbf{K} \odot \beta\| \cap \|\alpha\|$. Thus $\beta \in \mathbf{K} \odot \beta$ and $\mathbf{K} \odot \beta \nvdash \neg\alpha$. Thus, by Proposition 3.4 it follows that $\alpha \in \mathbf{K} \odot \alpha$. Hence $\|\mathbf{K} \odot \alpha\| \subseteq \|\alpha\|$.

It holds that $\|\mathbf{K} \odot \alpha\| \subseteq S(\alpha)$ (letting $\delta = \alpha$). Hence $\|\mathbf{K} \odot \alpha\| \subseteq S(\alpha) \cap \|\alpha\|$. For the other inclusion, let $w \in S(\alpha) \cap \|\alpha\|$. Hence there exists β such that $\|\alpha\| \subseteq \|\beta\|$ and $w \in \|\mathbf{K} \odot \beta\|$. Thus $\vdash \alpha \rightarrow \beta$ and $w \in \|\mathbf{K} \odot \beta\| \cap \|\alpha\|$.

We will now show that $\mathbf{K} \odot \alpha \subseteq \mathbf{K} \odot \beta + \alpha$. If $\neg\alpha \notin \mathbf{K}$, then from $\vdash \alpha \rightarrow \beta$ it follows that $\neg\beta \notin \mathbf{K}$. By $\odot$ vacuity it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \beta$. Thus $\mathbf{K} + \alpha \subseteq \mathbf{K} \odot \beta + \alpha$. From which it follows by $\odot$ inclusion that $\mathbf{K} \odot \alpha \subseteq \mathbf{K} \odot \beta + \alpha$. Assume now that $\neg\alpha \in \mathbf{K}$. By $\odot$ containment it follows that $\mathbf{K} \cap ((\mathbf{K} \odot (\neg\alpha \wedge \beta)) + (\neg\alpha \wedge \beta)) \subseteq \mathbf{K} \odot (\neg\alpha \wedge \beta)$. Thus, by $\odot$ closure, $\neg\alpha \in \mathbf{K} \odot (\neg\alpha \wedge \beta)$. Let $\xi \in \mathbf{K} \odot \alpha$ we intend to prove that $\xi \in \mathbf{K} \odot \beta + \alpha$. It holds that $\alpha \rightarrow \xi \in \mathbf{K} \odot \alpha$. By $\odot$ extensionality it follows from $\vdash \alpha \rightarrow \beta$ that $\mathbf{K} \odot \alpha = \mathbf{K} \odot (\alpha \wedge \beta)$. Hence $\alpha \rightarrow \xi \in \mathbf{K} \odot (\alpha \wedge \beta)$. On the other hand, it follows from $\neg\alpha \in \mathbf{K} \odot (\neg\alpha \wedge \beta)$, by $\odot$ closure, that $\alpha \rightarrow \xi \in \mathbf{K} \odot (\neg\alpha \wedge \beta)$. Thus, by $\odot$ disjunctive overlap, it follows that $\alpha \rightarrow \xi \in \mathbf{K} \odot ((\neg\alpha \wedge \beta) \vee (\alpha \wedge \beta))$. From which it follows by $\odot$ extensionality that $\alpha \rightarrow \xi \in \mathbf{K} \odot \beta$. Hence $\xi \in \mathbf{K} \odot \beta + \alpha$. Therefore, it holds (in both of the cases considered) that $\mathbf{K} \odot \alpha \subseteq \mathbf{K} \odot \beta + \alpha$. Thus (by Lemma 8.3) $\|\mathbf{K} \odot \beta\| \cap \|\alpha\| \subseteq \|\mathbf{K} \odot \alpha\|$. Hence $w \in \|\mathbf{K} \odot \alpha\|$. Thus $S(\alpha) \cap \|\alpha\| \subseteq \|\mathbf{K} \odot \alpha\|$.

Therefore $\|\mathbf{K} \odot \alpha\| = S(\alpha) \cap \|\alpha\|$. Thus $\mathbf{K} \odot \alpha = \bigcap(S(\alpha) \cap \|\alpha\|)$.

Case (2) $S_\alpha \in \mathbb{S} \setminus \mathbb{S}_{in}$. If $\alpha \in \mathbf{K} \odot \alpha$, then it holds that $S_\alpha \in \mathbb{S}_{in}$ (since it follows by $\odot$ strict improvement that $\|\mathbf{K} \odot \beta\| \subseteq \|\beta\|$, for all β such that $\|\alpha\| \subseteq \|\beta\|$, from which it follows that $S_\alpha \subseteq \{w : w \in \|\mathbf{K} \odot \delta\|$ for some δ such that $\|\mathbf{K} \odot \delta\| \subseteq \|\delta\|\}$). Thus $\alpha \notin \mathbf{K} \odot \alpha$. From which it follows, by $\odot$ weak relative success, that $\mathbf{K} \odot \alpha \subseteq \mathbf{K}$. Thus $\|\mathbf{K}\| \subseteq \|\mathbf{K} \odot \alpha\|$.

Let $w \in \|\mathbf{K}\| \cup (S(\alpha) \cap \|\alpha\|)$. Hence $w \in \|\mathbf{K}\|$ or $w \in S(\alpha) \cap \|\alpha\|$. In the first case it follows that $w \in \|\mathbf{K} \odot \alpha\|$. In the second case it follows that $w \in S(\alpha) \cap \|\alpha\|$. Thus, there exists β such that $\|\alpha\| \subseteq \|\beta\|$ and $w \in \|\mathbf{K} \odot \beta\| \cap \|\alpha\|$. Reasoning as above it follows that $\mathbf{K} \odot \alpha \subseteq \mathbf{K} \odot \beta + \alpha$. Hence $\|\mathbf{K} \odot \beta\| \cap \|\alpha\| \subseteq \|\mathbf{K} \odot \alpha\|$. Thus $w \in \|\mathbf{K} \odot \alpha\|$. Therefore $\|\mathbf{K}\| \cup (S(\alpha) \cap \|\alpha\|) \subseteq \|\mathbf{K} \odot \alpha\|$.

We will now prove the other inclusion. Let $w \in \|\mathbf{K} \odot \alpha\|$. Hence $w \in \|\mathbf{K} \odot \alpha\| \cap (\|\alpha\| \cup \|\neg\alpha\|) = (\|\mathbf{K} \odot \alpha\| \cap \|\alpha\|) \cup (\|\mathbf{K} \odot \alpha\| \cap \|\neg\alpha\|) \subseteq (S(\alpha) \cap \|\alpha\|) \cup (\|\mathbf{K} \odot \alpha\| \cap \|\neg\alpha\|)$. By $\odot$ N-recovery it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \alpha + \neg\alpha$. Thus $\|\mathbf{K} \odot \alpha + \neg\alpha\| \subseteq \|\mathbf{K}\|$. Hence $\|\mathbf{K} \odot \alpha\| \cap \|\neg\alpha\| \subseteq \|\mathbf{K}\|$. From which it follows that $w \in (S(\alpha) \cap \|\alpha\|) \cup (\|\mathbf{K} \odot \alpha\| \cap \|\neg\alpha\|) \subseteq (S(\alpha) \cap \|\alpha\|) \cup \|\mathbf{K}\|$. Hence $\|\mathbf{K} \odot \alpha\| \subseteq \|\mathbf{K}\| \cup (S(\alpha) \cap \|\alpha\|)$. Thus $\|\mathbf{K} \odot \alpha\| = \|\mathbf{K}\| \cup (S(\alpha) \cap \|\alpha\|)$. And so, $\mathbf{K} \odot \alpha = \bigcap(\|\mathbf{K}\| \cup (S(\alpha) \cap \|\alpha\|))$.

□

Received 16 April 2024; revised 7 January 2025; accepted 14 April 2025