Selective Contraction

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Abstract. *Selective revision* operators are well-known non-prioritized revision operators that allow for the acceptance of only part of the new information while rejecting the rest. In this paper, we introduce their contraction counterparts, which we refer to as *selective contraction* operators. These operators enable the removal of only part of the input information. As such, selective contractions can be viewed as a method for removing some of the beliefs that support the input belief without eliminating the belief itself. In this sense, selective contraction operators allow for a weakening of the input belief's strength without discarding it. We provide representation theorems for various classes of selective contraction operators.

Keywords: Belief contraction, Non-prioritized belief change, Selective contractions, Transformation functions.

1. Introduction

The area of research that investigates the dynamics of belief is known as *belief change*, and is also referred to as *belief revision* or *belief dynamics*.

An important milestone in belief change is the so-called AGM model. In this model, each belief is represented by a sentence, and an agent's belief state is represented by a logically closed set of sentences, called a belief set. The AGM model considers three types of belief change operators: *expansion, contraction* and *revision*. The expansion (typically denoted by $+$) of a belief set by a sentence is a two-step process: first, the sentence is added to the belief set, and then the resulting set is closed under logical consequence. As a result, the outcome of an expansion is a belief set that contains all the sentences from the original belief set, along with the sentence representing the new information. Consequently, the resulting belief set may be inconsistent. A contraction involves removing beliefs from the agent's belief set $\mathbf{K}$ in order to obtain a subset of $\mathbf{K}$ which does not entail the input information. A revision occurs when the input information is incorporated

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in a consistent manner, provided that the information itself is consistent. Contrary to expansions, the operators of contraction and revision are not defined in a unique way, but are constrained by a set of postulates (that will be recalled in Subsections 2.2 and 2.3). As stated by Meyer [19, p. 18]: "The idea is that these are the rational choices to be made". A feature of the AGM revision is the unconditional acceptance of the new information. In AGM revisions, the incoming information outweighs the previous one. Therefore, in case of a conflict between the new and old information, the new one will prevail. This feature may be a reasonable assumption in many situations. For instance, if a new observation contradicts a scientific theory, the theory should be revised to account for it. Similarly, if a law changes, then the new decisions should be guided by the newest version of the law. However, there are cases in which this feature, characterized by the *success* postulate, is unrealistic. This may happen for various reasons. For example, the new information may lack on credibility or it may contradict previous beliefs that the agent is not willing to remove. Belief change operators that do not satisfy the *success postulate* are called *non-prioritized belief change* operators. For an overview of these operators see [1].

In particular, there are in the literature several classes of *non-prioritized belief revision* operators. One of those classes consists of the so-called *selective revision* operators [38]. Contrary to AGM revisions, selective revisions allow the acceptance of only part of the new information and the non-acceptance of the rest of it. Formally, a selective revision operator, $\odot$, is constructed using a basic AGM revision $\star$ and a function $f : \mathcal{L} \rightarrow \mathcal{L}$, called *transformation function*, as follows:

$$\mathbf{K} \odot \alpha = \mathbf{K} \star f(\alpha).$$

Intuitively, the transformation function f selects the credible part of every sentence. When adding properties to the transformation function f, several additional properties are obtained for the selective revision operator which is based on f. In [38] several representation theorems for operators of selective revision were presented depending on the properties that the associated transformation function f satisfies. In [20, 34] the selective revision operators were adapted to the belief base context.

To the best of our knowledge, there aren't many works on non-prioritized contractions (that, contrary to AGM contractions, are operations in which the input information is not always removed). In [11], Fermé and Hansson proposed the shielded contraction operators. These operators are defined by means of an AGM contraction and a set of sentences, designated by *set of retractable sentences* which is denoted by R. If a sentence α belongs to R, then the outcome of the shielded contraction by it coincides with the outcome of the associated AGM contraction, otherwise the original belief set is left unchanged. Hence in this model the new information is either fully removed or fully kept. Other works on non-prioritized contraction operators include the decrement operators proposed in [21] and the non-prioritized kernel contraction operators proposed in [4]. In Section 5, we recall these operators and briefly relate them to selection contraction operators.

In the present paper we propose another kind of non-prioritized contraction operator that we designate by *selective contraction*, which can be considered as the dual of the selective revision. These are operators that, contrary to AGM and shielded contractions, allow the removal of only part of the incoming information. The following example motivates this kind of operator.

EXAMPLE 1. Consider a medical diagnosis scenario. The initial belief set of a doctor includes:

p: The patient has Disease A.
q: The patient has Disease B.

Both diagnoses are independently supported by early symptoms. After prescribing a medical exam to confirm his diagnosis, the doctor receives new information from a highly reliable specialist report stating: "It is not the case that the patient has either Disease A or Disease B." However, the prescribed medical exam is known to be very reliable for detecting Disease A, but often fails to detect Disease B in its early stages. A rational response is therefore to drop p (Disease A) while keeping q (Disease B) as a plausible hypothesis. Dropping both would amount to concluding that the patient has neither disease, which is not warranted given the limitations of the test. Standard AGM contraction by $p \vee q$ would remove both p and q, whereas a selective contraction operator may remove only p, thus preserving useful diagnostic information.

In standard AGM contraction, the input sentence is identical to the sentence to be removed from the belief set.[1] In contrast, in the selective contraction framework the input information and the contraction target are distinct. The received input α is not itself necessarily removed from the original belief set. Instead, it triggers the removal of a sentence $g(\alpha)$, which

[1] In standard AGM revision, the input sentence is the one that is added to the original belief set in a consistent manner (whenever possible), while in the context of AGM contraction the input sentence is the one to be removed from the belief set.

is actually removed from the belief set (unless it is a tautology). Accordingly, contraction is performed with respect to $g(\alpha)$ rather than α. Intuitively, g is a function responsible for selecting the part of each sentence that the agent is willing to lose (typically chosen so that $\vdash g(\alpha) \rightarrow \alpha$).

The rest of the paper is organized as follows: In Section 2 we introduce the notations and recall the main background concepts that will be needed throughout this article. In Section 3 we recall the definition of selective revision operators and present representation theorems for classes of such operators. In Section 4 we present the definition of the selective contraction operators and axiomatically characterize several classes of such operators. In Section 5, we present a brief survey of related works. In Section 6 we summarize the main contributions of the paper, briefly discuss their relevance and present some potential future research topics. In the Appendix we provide proofs for all the original results presented.

2. Background

2.1. Formal Preliminaries

We consider a propositional language $\mathcal{L}$ that contains the usual truth functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence). The set of all logical consequences of a subset A of $\mathcal{L}$ is denoted by $Cn(A)$. We will assume that Cn satisfies the standard Tarskian properties, namely (i) $A \subseteq Cn(A)$ (*inclusion*); (ii) If $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$ (*monotony*); (iii) $Cn(A) = Cn(Cn(A))$ (*iteration*). Furthermore, we will assume that Cn satisfies the following three properties: (iv) If α can be derived from A by classical truth-functional logic, then $\alpha \in Cn(A)$ (*supraclassicality*); (v) $\beta \in Cn(A \cup \{\alpha\})$ if and only if $\alpha \rightarrow \beta \in Cn(A)$ (*deduction*); (vi) If $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite subset $A' \subseteq A$ (*compactness*). We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. The letters $\alpha, \beta, \ldots$ will be used to denote sentences of $\mathcal{L}$. Lowercase Latin letters such as $p, q, \ldots$ will be used to denote atomic sentences of $\mathcal{L}$. Uppercase Latin letters such as $A, B, \ldots$ shall denote sets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (*i.e.* $\mathbf{K} = Cn(\mathbf{K})$) — such a set is called a *belief set* or *theory*. Given a set A we will denote $Cn(A \cup \{\alpha\})$ by $A + \alpha$. We will use $\perp$ and $\top$ to denote, respectively, a contradiction and a tautology of $\mathcal{L}$.

2.2. AGM Revisions

The operation of revision of a belief set consists of the incorporation of new beliefs in that set. In a revision process, some previous beliefs may be retracted in order to obtain, as output, a consistent belief set. The following six postulates, which were originally presented in [37], are known as the *basic AGM postulates for revision*[2]:

($\star$1) $\mathbf{K}\star\alpha = Cn(\mathbf{K}\star\alpha)$ (*i.e.* $\mathbf{K}\star\alpha$ is a belief set). (Closure)
($\star$2) $\alpha \in \mathbf{K}\star\alpha$. (Success)
($\star$3) $\mathbf{K}\star\alpha \subseteq \mathbf{K} + \alpha$. (Inclusion)
($\star$4) If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} + \alpha \subseteq \mathbf{K}\star\alpha$. (Vacuity)
($\star$5) If α is consistent, then $\mathbf{K}\star\alpha$ is consistent. (Consistency)
($\star$6) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}\star\alpha = \mathbf{K}\star\beta$. (Extensionality)

The following two postulates proposed in [8, 24] are usually referred to as supplementary AGM revision postulates.

($\star$7) $\mathbf{K}\star\alpha \cap \mathbf{K}\star\beta \subseteq \mathbf{K}\star(\alpha \vee \beta)$. (Disjunctive overlap)
($\star$8) If $\neg\alpha \notin \mathbf{K}\star(\alpha \vee \beta)$, then $\mathbf{K}\star(\alpha \vee \beta) \subseteq \mathbf{K}\star\alpha$. (Disjunctive inclusion)

Definition 1. An operator $\star$ for a belief set $\mathbf{K}$ is a basic AGM revision if and only if it satisfies postulates ($\star$1) to ($\star$6). It is an AGM revision if and only if it satisfies postulates ($\star$1) to ($\star$8).

2.3. AGM Contractions

A contraction of a belief set occurs when some beliefs are removed from it (and no new beliefs are added to that set). The following postulates, which were presented in [32] (following [8, 24]), are known as the *basic AGM postulates for contraction*:

($\div$1) $\mathbf{K} \div \alpha = Cn(\mathbf{K} \div \alpha)$ (*i.e.* $\mathbf{K} \div \alpha$ is a belief set). (Closure)
($\div$2) $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. (Inclusion)
($\div$3) If $\alpha \notin \mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. (Vacuity)
($\div$4) If $\nvdash \alpha$, then $\alpha \notin \mathbf{K} \div \alpha$. (Success)
($\div$5) $\mathbf{K} \subseteq (\mathbf{K} \div \alpha) + \alpha$. (Recovery)
($\div$6) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. (Extensionality)

[2] The postulates were already presented in [32] but with slightly different formulations.

The following two postulates proposed in [32] are usually referred to as supplementary AGM contraction postulates.
($\div$7) $(\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta) \subseteq \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive overlap)
($\div$8) $\mathbf{K} \div (\alpha \wedge \beta) \subseteq \mathbf{K} \div \alpha$ whenever $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive inclusion)

Operators that satisfy postulates ($\div$1) to ($\div$4) and ($\div$6) are designated by *withdrawals*. Withdrawal operators satisfying *recovery* are basic AGM contractions.

Definition 2. ([18]) An operator $\div$ for a belief set $\mathbf{K}$ is a withdrawal if and only if it satisfies postulates ($\div$1) to ($\div$4) and ($\div$6). It is a basic AGM contraction if and only if it satisfies postulates ($\div$1) to ($\div$6) and an AGM contraction if and only if it satisfies postulates ($\div$1) to ($\div$8).

The following observation relates some of the AGM postulates for contraction with another well known contraction postulate.

Observation 2. ([2]) *Let* $\mathbf{K}$ *be a belief set and* $\div$ *an operator on* $\mathbf{K}$. *If* $\div$ *satisfies closure, inclusion and recovery, then it satisfies:*
(Failure) [2] If $\vdash \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K}$.

3. Selective Revisions

In [38] Fermé and Hansson proposed an operator that allows the acceptance of only part of the new information and the non-acceptance of the rest of it. They called this operator *selective revision*. An operator of selective revision is constructed from a basic AGM revision and a function f from $\mathcal{L}$ to $\mathcal{L}$ as follows.

Definition 3. ([38]) Let $\mathbf{K}$ be a belief set, $\star$ be a basic AGM revision operator for $\mathbf{K}$ and f be a function from $\mathcal{L}$ to $\mathcal{L}$. The selective revision based on $\star$ and f, is the operator $\odot$ such that for all sentences α:

$$\mathbf{K} \odot \alpha = \mathbf{K} \star f(\alpha).$$

f is called transformation function (on which $\odot$ is based).

Intuitively, the transformation function f selects the credible part of every sentence. A natural restriction is that $f(\alpha)$ should not contain more information than the one that is contained in α (*i.e.*, $\vdash \alpha \rightarrow f(\alpha)$). The following properties were proposed in [38] for transformation functions:

$\vdash \alpha \to f(\alpha)$. (Implication)
$\vdash f(f(\alpha)) \leftrightarrow f(\alpha)$ (Idempotence)
If $\vdash \alpha \leftrightarrow \beta$, then $\vdash f(\alpha) \leftrightarrow f(\beta)$. (Extensionality)
If $\alpha \nvdash \perp$, then $f(\alpha) \nvdash \perp$. (Consistency Preservation)
$f(\alpha) \nvdash \perp$. (Consistency)
If $\mathbf{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$. (Weak Maximality)

3.1. Postulates for Selective Revision

We start this subsection by presenting some postulates for selective revisions.

Proxy success: There is a sentence β, such that $\mathbf{K} \odot \alpha \vdash \beta$, $\vdash \alpha \to \beta$ and $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta$.
Weak proxy success: There is a sentence β, such that $\mathbf{K} \odot \alpha \vdash \beta$ and $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta$.
Proxy success and *weak proxy success*, both proposed in [38], are weaker versions of success. These postulates capture the intuition that the revision should take the form of accepting some part of the incoming information.

The following three postulates are related to consistency.

Consistent expansion: If $\mathbf{K} \nsubseteq \mathbf{K} \odot \alpha$, then $\mathbf{K} \cup (\mathbf{K} \odot \alpha) \vdash \perp$.
Strong consistency: $\mathbf{K} \odot \alpha \nvdash \perp$.
Consistency preservation: If $\mathbf{K} \nvdash \perp$, then $\mathbf{K} \odot \alpha \nvdash \perp$.

Consistent expansion proposed in [38], expresses the idea that previous beliefs are given up in order to avoid inconsistency. *Strong consistency* was proposed in [9] and assures that the outcome of a selective revision is consistent. *Consistency preservation* was proposed in [3] and states that the outcome of a selective revision on a belief set is consistent whenever that belief set is consistent.

We now recall from [38] representation theorems for two classes of selective revision operators.

THEOREM 3. ([38]) *Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

1. $\odot$ *satisfies closure, inclusion, vacuity, consistency, extensionality, and weak proxy success;*
2. $\odot$ *is a selective revision operator based on a basic AGM revision operator* $\star$ *for* $\mathbf{K}$ *and a transformation function* f *that satisfies extensionality, consistency preservation, weak maximality, and idempotence.*

THEOREM 4. ([38]) *Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator on* $\mathbf{K}$*. The following conditions are equivalent:*

1. $\odot$ *satisfies closure, inclusion, vacuity, consistency, extensionality, and proxy success;*
2. $\odot$ *is a selective revision operator based on a basic AGM revision operator* $\star$ *for* $\mathbf{K}$ *and a transformation function* f *that satisfies extensionality, consistency preservation, weak maximality, idempotence and implication.*

For further developments regarding selective revision operators, see [1, 38].

4. Selective Contractions

In this section, we present the definition of selective contraction operators as well as axiomatic characterizations for some classes of such operators. This type of operator is defined through the use of a withdrawal operator and a function from $\mathcal{L}$ to $\mathcal{L}$ (that we call transformation function).

DEFINITION 4. Let $\mathbf{K}$ be a belief set, $\div$ be a withdrawal on $\mathbf{K}$ and g a function from $\mathcal{L}$ to $\mathcal{L}$. The selective contraction based on $\div$ and g, is the operator $\ominus$ such that for all sentences α[3]:

$$\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha).$$

g is called transformation function (on which $\ominus$ is based).

We now present a list of properties of transformation functions in the context of selective contraction.

$\vdash g(\alpha) \to \alpha$.	(Reverse implication)
$\vdash g(g(\alpha)) \leftrightarrow g(\alpha)$.	(Idempotence)
If $\vdash \alpha \leftrightarrow \beta$, then $\vdash g(\alpha) \leftrightarrow g(\beta)$.	(Extensionality)
If $\nvdash \alpha$, then $\nvdash g(\alpha)$.	(Non-validity preservation)
$\nvdash g(\alpha)$.	(Non-tautological choice)
If $\mathbf{K} \vdash \perp$, then $\nvdash g(\alpha)$.	(Consistency recoverability)
If $\mathbf{K} \nvdash \alpha$, then $\vdash g(\alpha) \leftrightarrow \alpha$.	(Weak minimality)
If $\alpha \vdash \perp$, then $\vdash g(\alpha) \leftrightarrow \alpha$.	(Full selectivity)
If $\vdash \alpha$, then $\vdash g(\alpha) \leftrightarrow \alpha$.	(Tautological propagation)

Intuitively, the transformation function g is responsible for selecting the part of each information to be removed. Therefore, it is reasonable to assume

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[3] We define selective contraction operators based on withdrawal operators, which are more general than basic AGM contraction operators. Later, we examine classes of selective contraction operators based on basic AGM contractions.

that $g(\alpha)$ is more general than α, *i.e.*, that $\vdash g(\alpha) \rightarrow \alpha$. This assumption allows us to remove $g(\alpha)$ while keeping α. Although removing $g(\alpha)$ may also remove some beliefs that support α, it does not necessarily remove α itself. *Idempotence* and *extensionality* are properties that were already proposed for transformation functions in the selective revision context. *Non-validity preservation* is a weaker version of the *reverse implication* property. *Non-tautological choice* states that the part of α to be removed is not a tautology. *Consistency recoverability* states the same whenever $\mathbf{K}$ is inconsistent. A transformation function that satisfies either of the latter two properties ensures the consistency of the outcome of the selective contraction it induces. *Weak minimality* states that if α is not an agent's belief, then the part of α to be removed should be α itself (in this case, the agent's belief set is left unchanged).[4] *Full selectivity* informally states that contradictions should be fully removed. Full selectivity follows from weak minimality whenever the belief set is consistent. It also follows from reverse implication. *Tautological propagation* states that if α is a tautology, then $g(\alpha)$ should also be a tautology. This property reflects the principle of *doing nothing when demanded the impossible.*

The following observation highlights several relationships among the properties of transformation functions presented above.

Observation 5. *Let* $\mathbf{K}$ *be a belief set and* g *be a transformation function. It holds that:*

1. *If* g *satisfies reverse implication, then it satisfies non-validity preservation.*
2. *If* g *satisfies non-tautological choice, then it satisfies non-validity preservation and consistency recoverability.*
3. *If* $\mathbf{K}$ *is consistent and* g *satisfies weak minimality, then* g *satisfies full selectivity.*

When the selective contraction is based on a transformation function that satisfies tautological propagation and a withdrawal operator that satisfies *failure*,[5] the process of contracting by a tautology using that selective

[4] We note that, more rigorously, the expression "with respect to $\mathbf{K}$" should be added to the designation of the properties consistency recoverability and weak minimality, as they depend on the belief set $\mathbf{K}$. However, we will omit this reference when the context makes it clear which belief set is being considered. The same applies to the weak minimality property discussed in the previous section.

[5] A withdrawal operator that satisfies failure is a Levi contraction [10,31].

contraction leaves the belief set unchanged, as the following observation illustrates.

Observation 6. *Let* $\mathbf{K}$ *be belief set,* α *be a tautology,* $\div$ *be a withdrawal on* $\mathbf{K}$ *that satisfies failure and* g *be a function from* $\mathcal{L}$ *to* $\mathcal{L}$ *that satisfies tautological propagation. Let* $\ominus$ *be the selective contraction based on* $\div$ *and* g*. It holds that* $\mathbf{K} \ominus \alpha = \mathbf{K}$*.*

Non-tautological choice and tautological propagation reflect two different perspectives regarding tautologies. Non-tautological choice requires that the part of the information to be removed is never a tautology. This choice ensures the consistency of the outcome in every selective contraction, even if the initial belief set is inconsistent and the sentence by which that belief set is contracted is a tautology. The same holds true with regard to the property of consistency recoverability. Tautological propagation, on the other hand demands that $g(\alpha)$ is a tautology, whenever α is a tautology. In this case, the consistency of the outcome is not obtained when performing a selective contraction of an inconsistent belief set by a tautology, whenever the selective contraction operator satisfies *failure*. This raises the question of which of these conflicting properties is more suitable. We believe that the answer to this question depends on the context. Therefore, we will consider cases in which the transformation function g satisfies non-tautological choice, cases in which it satisfies consistency recoverability, and cases in which it satisfies tautological propagation, exploring each of these options in turn.

4.1. Postulates for Selective Contraction

In this subsection we propose postulates for selective contractions. We start by presenting postulates related to consistency.

Strong consistency: $\mathbf{K} \ominus \alpha \nvdash \perp$.
Non-tautological consistency: If $\nvdash \alpha$, then $\mathbf{K} \ominus \alpha \nvdash \perp$.

Strong consistency assures that the outcome of a selective contraction is consistent. *Non-tautological consistency* states the same, but has the precondition that the information by which we contract is not a tautology. Note that if $\mathbf{K}$ is inconsistent, $\ominus$ satisfies *failure* and α is a tautology, then the outcome of $\mathbf{K} \ominus \alpha$ is inconsistent. We note also that *non-tautological consistency* follows from *success* and *closure*. Moreover, if the belief set $\mathbf{K}$ is consistent, then both consistency-related postulates mentioned above can be derived from the inclusion postulate.

We now present versions of *proxy success* for selective contractions. These incorporate the principle that selective contractions should remove some part of the input information. That part serves as a proxy for the input.

Weak proxy success: If $\nvdash \alpha$, then there is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$ and $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$.

Proxy success: If $\nvdash \alpha$, then there is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$, $\vdash \beta \rightarrow \alpha$ and $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$.

Strong proxy success: There is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$, $\vdash \beta \rightarrow \alpha$ and $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$.

The first two of the postulates presented above are weaker versions of the *success* postulate for contractions. *Weak proxy success* states that if $\nvdash \alpha$, then there exists a sentence β such that the outcome of contracting by it is successful and coincides with the outcome of the contraction by α. *Proxy success* requires additionally that the sentence β also implies α. *Strong proxy success*, as the name suggests, is a stronger version of proxy success, since their formulation are similar but *strong proxy success* has no preconditions. This postulate taken together with *closure* implies *strong consistency*.

The following observation states that if $\mathbf{K}$ is a consistent belief set, then *strong proxy success* follows from *proxy success* taken together with the contraction postulates *failure, vacuity* and *inclusion*. Thus, in the presence of the latter, *strong proxy success* and *proxy success* are equivalent, whenever $\mathbf{K}$ is consistent.

Observation 7. *Let* $\mathbf{K}$ *be a consistent belief set and* $\ominus$ *be an operator on* $\mathbf{K}$. *If* $\ominus$ *satisfies proxy success, failure, vacuity and inclusion, then it satisfies strong proxy success.*

From the above observation, we can also conclude that, provided the belief set under consideration is consistent, *strong proxy success* follows from the well-known contraction postulates: *success, failure, vacuity* and *inclusion*.

Alongside *success*, the *recovery* postulate has also been the subject to criticism [12,28,36]. According to the AGM trio, the postulate of *recovery* is an expression of the *principle of minimal change* [16,17,37]. While the principle of *recovery* is intuitively appealing as it allows for the reinstatement of beliefs that were previously removed, there are examples of contractions where *recovery* seems implausible.

Example 8. ([35]) I previously entertained the two beliefs "George is a criminal" (α) and "George is a mass murderer" (β). When I received information that induced me to give up the first of these beliefs (α), the second

(β) had to go as well (since α would otherwise follow from β). I then received new information that made me accept the belief "George is a shoplifter" (δ). The resulting new belief set is the expansion of $\mathbf{K} \div \alpha$ by δ, $(\mathbf{K} \div \alpha) + \delta$. Since α follows from δ, $(\mathbf{K} \div \alpha) + \alpha$ is a subset of $(\mathbf{K} \div \alpha) + \delta$. By recovery, $(\mathbf{K} \div \alpha) + \alpha$ includes β, from which follows that $(\mathbf{K} \div \alpha) + \delta$ includes β. Thus, since I previously believed George to be a mass murderer, I cannot any longer believe him to be a shoplifter without believing him to be a mass murderer.

This example illustrates that retaining the sentence $\alpha \rightarrow \beta$ (with $\beta \in \mathbf{K}$) after contracting $\mathbf{K}$ by α leads to unintuitive results. There seem to be cases when this sentence has to be removed, but due to the *recovery* postulate, AGM contraction cannot eliminate it. It appears that in order to properly recover the initial belief set, the outcome of the contraction sometimes needs to be expanded by more than the sentence that motivated the contraction. Therefore, a more general form of recovery seems reasonable: $\mathbf{K} \subseteq Cn((\mathbf{K} \div \alpha) \cup h(\alpha))$ for some function h, subject to some possible conditions (*e.g.* $\vdash h(\alpha) \rightarrow \alpha$). This idea is expressed by the following postulate proposed originally in [6]:

Proxy recovery: If $\mathbf{K} \neq \mathbf{K} \div \alpha$ then there exists some $\beta \in \mathbf{K}$, such that $\mathbf{K} \div \alpha \nvdash \beta$ and $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \beta$.

In what follows, we introduce postulates that can be regarded as *recovery* variants appropriate in the context of selective contractions.

Weak proxy recovering success: If $\nvdash \alpha$, then there is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$.

Proxy recovering success: If $\nvdash \alpha$ then there is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$, $\vdash \beta \rightarrow \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$.

Strong proxy recovering success: There is a sentence β, such that $\mathbf{K} \ominus \alpha \nvdash \beta$, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$, $\vdash \beta \rightarrow \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$.

The three postulates above capture the same intuitions underlying *proxy recovery*. The first two of these postulates are implied by *success* and *recovery*. *Strong proxy recovering success* is a stronger version of *proxy recovering success*. Each of the three postulates above constitutes a stronger version of, respectively, *weak proxy success, proxy success*, and *strong proxy success*.

Proxy recovery follows from *weak proxy recovering success* when taken together with other well-known contraction postulates, as stated in the following observation:

Observation 9. *Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *be an operator on* $\mathbf{K}$*. If* $\ominus$ *satisfies closure, inclusion, failure, vacuity and weak proxy recovering success, then it also satisfies proxy recovery.*

The next observation presents a similar result to that of Observation 7. It relates *proxy recovering success* with *strong proxy recovering success.*

Observation 10. *Let* $\mathbf{K}$ *be a consistent belief set and* $\ominus$ *be an operator on* $\mathbf{K}$*. If* $\ominus$ *satisfies proxy recovering success, failure, vacuity and inclusion, then it satisfies strong proxy recovering success.*

From the above observation, we can also conclude that whenever the belief set under consideration is consistent, *proxy recovering success* and *strong proxy recovering success* are equivalent, provided that the well-known contraction postulates of *success, failure, vacuity*, and *inclusion* are satisfied. Furthermore, in light of Observation 2, if the belief set is consistent, we can also conclude that *strong proxy recovering success* follows from *success, inclusion, vacuity* and *recovery.*

The following observation illustrates some of the properties that an operator of selective contraction, based on a withdrawal operator $\div$ and a transformation function g, satisfies whenever $\div$ and g satisfy certain properties.

Observation 11. *Let* $\mathbf{K}$ *be a belief set,* $\div$ *be a withdrawal operator on* $\mathbf{K}$ *and* g *be a transformation function. Let* $\ominus$ *be the selective contraction operator on* $\mathbf{K}$ *based on* $\div$ *and* g*. Then:*

1. $\ominus$ *satisfies closure and inclusion.*
2. *If* g *satisfies extensionality, then* $\ominus$ *satisfies extensionality.*
3. *If* g *satisfies weak minimality, then* $\ominus$ *satisfies vacuity.*
4. *If* g *satisfies reverse implication, then* $\ominus$ *satisfies vacuity and non-tautological consistency.*
5. *If* g *satisfies non-validity preservation and idempotence, then* $\ominus$ *satisfies weak proxy success.*
6. *If* g *satisfies idempotence and reverse implication, then* $\ominus$ *satisfies proxy success.*
7. *If* $\div$ *satisfies failure and* g *satisfies tautological propagation, then* $\ominus$ *satisfies failure.*
8. *If* g *satisfies non-tautological choice, then* $\ominus$ *satisfies strong consistency.*
9. *If* g *satisfies consistency recoverability, then* $\ominus$ *satisfies strong consistency.*

10. *If g satisfies idempotence, reverse implication and non-tautological choice, then $\ominus$ satisfies strong proxy success.*
11. *If $\div$ satisfies failure and g satisfies idempotence, reverse implication and consistency recoverability, then $\ominus$ satisfies strong proxy success.*
12. *If $\div$ satisfies recovery and g satisfies non-validity preservation and idempotence, then $\ominus$ satisfies weak proxy recovering success.*
13. *If $\div$ satisfies recovery, g satisfies reverse implication and idempotence, then $\ominus$ satisfies proxy recovering success.*
14. *If $\div$ satisfies recovery, g satisfies reverse implication, idempotence and non-tautological choice, then $\ominus$ satisfies strong proxy recovering success.*
15. *If $\div$ satisfies recovery, g satisfies reverse implication, idempotence and consistency recoverability, then $\ominus$ satisfies strong proxy recovering success.*
16. *If g satisfies non-validity preservation, then $\ominus$ satisfies non-tautological consistency.*

The following theorems provide axiomatic characterizations of several classes of selective contraction operators. The first three representation theorems specifically characterize classes of selective contractions that are based on withdrawal operators.

THEOREM 12. *Let $\mathbf{K}$ be a belief set and $\ominus$ an operator on $\mathbf{K}$. The following conditions are equivalent:*

1. *$\ominus$ satisfies closure, inclusion, vacuity, extensionality and weak proxy success;*
2. *$\ominus$ is a selective contraction operator based on a withdrawal operator $\div$ for $\mathbf{K}$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, and non-validity preservation.*

THEOREM 13. *Let $\mathbf{K}$ be a belief set and $\ominus$ an operator on $\mathbf{K}$. The following conditions are equivalent:*

1. *$\ominus$ satisfies closure, inclusion, vacuity, extensionality and proxy success;*
2. *$\ominus$ is a selective contraction operator based on a withdrawal operator $\div$ for $\mathbf{K}$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, and reverse implication.*

The selective contractions characterized in the following theorem yield a consistent belief set as output, even when the original belief set is inconsistent and the sentence with respect to which the selective contraction is performed is a tautology.

THEOREM 14. *Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *an operator on* $\mathbf{K}$. *The following conditions are equivalent:*

1. $\ominus$ *satisfies closure, inclusion, vacuity, extensionality and strong proxy success;*
2. $\ominus$ *is a selective contraction operator based on a withdrawal operator* $\div$ *for* $\mathbf{K}$ *and a transformation function* g *that satisfies weak minimality, extensionality, idempotence, reverse implication, and non-tautological choice.*
3. $\ominus$ *is a selective contraction operator based on a withdrawal operator* $\div$ *for* $\mathbf{K}$ *that satisfies failure and a transformation function* g *that satisfies weak minimality, extensionality, idempotence, reverse implication, and consistency recoverability.*

The two following theorems provide axiomatic characterizations of distinct classes of selective contraction operators, each based on basic AGM contraction operators. The first theorem characterizes selective contraction operators based on a transformation function that satisfies a given set of properties along with non-validity preservation. The second theorem characterizes selective contraction operators based on a transformation function that satisfies the same set of properties, but replaces non-validity preservation with the stronger property of reverse implication. This strengthening corresponds to replacing *weak proxy recovering success* in the first theorem with the stronger postulate *proxy recovering success* in the second.

THEOREM 15. *Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *an operator on* $\mathbf{K}$. *The following conditions are equivalent:*

1. $\ominus$ *satisfies closure, inclusion, vacuity, extensionality, failure and weak proxy recovering success;*
2. $\ominus$ *is a selective contraction operator based on a basic AGM operator* $\div$ *for* $\mathbf{K}$ *and a transformation function* g *that satisfies weak minimality, extensionality, idempotence, non-validity preservation and tautological propagation.*

THEOREM 16. *Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *an operator on* $\mathbf{K}$. *The following conditions are equivalent:*

1. *$\ominus$ satisfies closure, inclusion, vacuity, extensionality, failure and proxy recovering success;*
2. *$\ominus$ is a selective contraction operator based on a basic AGM operator $\div$ for $\mathbf{K}$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, reverse implication and tautological propagation.*

The selective contractions characterized in the following theorem invariably produce a consistent belief set as output.

THEOREM 17. *Let $\mathbf{K}$ be a belief set and $\ominus$ an operator on $\mathbf{K}$. The following conditions are equivalent:*

1. *$\ominus$ satisfies closure, inclusion, vacuity, extensionality, and strong proxy recovering success;*
2. *$\ominus$ is a selective contraction operator based on a basic AGM operator $\div$ for $\mathbf{K}$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, reverse implication and non-tautological choice.*
3. *$\ominus$ is a selective contraction operator based on a basic AGM operator $\div$ for $\mathbf{K}$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, reverse implication and consistency recoverability.*

Note that additional representation theorems can be obtained by modifying the properties of g and the postulates of $\div$, while modifying the postulates satisfied by $\ominus$ in the the first statement of those theorems. For example, a different representation theorem can be obtained from Theorems 12 and 13 by requiring that $\ominus$ and $\div$ also satisfy *failure*, and that g satisfies tautological propagation.[6]

In the following definition, we assign designations to the six different kinds of selective contraction operators that were axiomatically characterized in the above theorems.

DEFINITION 5. Let $\ominus$ be an operator on a belief set $\mathbf{K}$. Table 1 defines the classes of the operator $\ominus$ according to the postulates that axiomatically characterize them.

[6]The proof of this claim is analogous to the proof of Theorems 12 and 13. To prove the 1. to 2. part, use the same constructions as in the proofs of those theorems. The 2. to 1. part follows directly from Observation 11.

Table 1. Designations of selective contraction operators

$\ominus$ **is a**	$\ominus$ **is a selective contraction operator based on a withdrawal operator $\div$ for K and a transformation function g that satisfies weak minimality, extensionality, idempotence, and**
Weak-Proxy Selective Contraction Operator (**WPSelCO**)	non-validity preservation.
Proxy Selective Contraction Operator (**PSelCO**)	reverse implication.
Strong-Proxy Selective Contraction Operator (**SPSelCO**)	either (i) reverse implication and non-tautological choice; or (ii) reverse implication, consistency recoverability, and additionally $\div$ satisfies failure.
Weak-Proxy Recovering Selective Contraction Operators (**WPRSelCO**)	non-validity preservation and tautological propagation, and additionally $\div$ satisfies recovery.
Proxy Recovering Selective Contraction Operators (**PRSelCO**)	reverse implication and tautological propagation, and additionally $\div$ satisfies recovery.
Strong-Proxy Recovering Selective Contraction Operators (**SPRSelCO**)	reverse implication and either non-tautological choice or consistency recoverability, and $\div$ additionally satisfies *recovery*.

In the following example, we[7] illustrate the use of the operators introduced in the above definition.

EXAMPLE 18. Consider a propositional language $\mathcal{L}$ generated by the finite set of propositional symbols $\{p, q\}$ and the Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Let $\mathbf{K} = Cn(p \wedge q)$, and let $\ominus$ be a selective contraction operator induced by a withdrawal operator $\div$ and a transformation function g that satisfies weak minimality, extensionality, idempotence, and non-validity preservation.

Suppose that we want to perform the selective contraction of $\mathbf{K}$ by $p \vee q$, using $\ominus$.

- If $g(p \vee q) = p \leftrightarrow q$, then g fails to satisfy reverse implication. In this case, $\ominus$ may be a WPSelCO, but it is not a PSelCO, SPSelCO, PRSelCO, or

[7]Hence $\div$ is a basic AGM contraction operator.

SPRSelCO. Moreover, $\ominus$ may be a WPRSelCO if, in addition, g satisfies tautological propagation and $\div$ satisfies *recovery* (in which case $\div$ is a basic AGM contraction operator).

- If g additionally satisfies reverse implication[8] and $g(p \vee q) = p$, then $\ominus$ is a PSelCO, and
$$\mathbf{K} \ominus (p \vee q) = \mathbf{K} \div p.$$
In this case, both $Cn(q)$ and $Cn(p \vee q)$ are admissible outcomes of $\mathbf{K} \ominus (p \vee q)$. If, furthermore, g satisfies tautological propagation and $\div$ satisfies *recovery* (so that $\ominus$ is a PRSelCO), then by $\div$ *recovery* and the deduction property of Cn,
$$\neg p \vee q \in \mathbf{K} \ominus (p \vee q),$$
since $\neg p \vee q$ is logically equivalent to $p \rightarrow (p \wedge q)$. Hence, in this case, $\mathbf{K} \ominus (p \vee q)$ may be $Cn(q)$, but it cannot be $Cn(p \vee q)$.

If $\ominus$ is an SPSelCO or an SPRSelCO, then contracting any belief set by any sentence using $\ominus$ always yields a consistent outcome, even in the limiting case where the belief set is inconsistent and the contracted sentence is a tautology. This property does not generally hold for the remaining four types of selective contraction operators.

The difference between an SPSelCO and an SPRSelCO is that the latter is induced by an operator that satisfies *recovery*, whereas this property does not necessarily hold for the former. In the present language, if we perform a selective contraction—induced by a transformation function that maps every tautology to p—of an inconsistent belief set by a tautology, then $\neg p$ must be an element of the outcome of such a selective contraction whenever it is an SPRSelCO. By contrast, this inclusion is not guaranteed if the selective contraction is an SPSelCO.

In Figure 1 we present a diagram that summarizes the representation theorems presented in this section. In that diagram a solid arrow between two boxes symbolizes that the class of selective contractions at the origin of the arrow is a subclass of the class of selective contractions at the end of that arrow.

In all the theorems presented in this subsection, the transformation functions that induce selective contraction satisfy weak minimality, extensionality, idempotence, and non-validity preservation (note that the latter is implied by reverse implication). As Observation 11 illustrates, these properties of transformation functions are directly related to the selective contraction

[8] Note that reverse implication implies non-validity preservation.

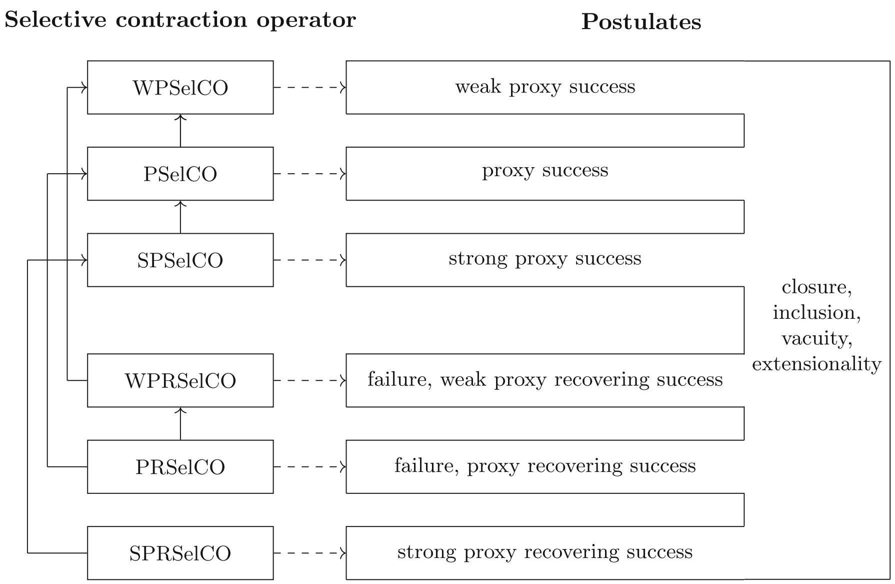

Figure 1. Schematic representation of the axiomatic characterizations of the various classes of selective contraction operators presented in this section, together with the relationships among those classes

postulates *vacuity, extensionality*, and *proxy success* versions, respectively (the latter when taken together with other properties of transformation functions). Therefore, these properties can be regarded as the core properties of transformation functions. Nevertheless, a transformation function satisfying these properties does not prevent the sentence that is effectively removed from being independent of the sentence that triggers the removal. A less permissive property is reverse implication. It requires that the sentence to be removed is logically weaker than the sentence that triggers the selective contraction. As a result, the latter can be retained while a sentence that supports it is eliminated, as seen in Example 18. This property is satisfied by the transformation functions that induce four of the six types of operators specified in Definition 5. The non-tautological choice and consistency recoverability properties are directly related to the postulate of strong consistency (Observation 11, statements 8 and 9), which ensures the consistency of the outcome of selective contraction.

Unlike the first three selective contraction operators introduced in Definition 5, which are induced by withdrawal operators, the last three are induced by basic AGM contraction operators. This distinction is reflected in the proxy recovering success versions of WPRSelCO, PRSelCO, and SPRSelCO.

These proxy recovering success versions can be seen as expressions of the principle of minimal change, according to which the new epistemic state should retain as much of the previous state as possible.

The operators introduced in Definition 5 reflect different epistemic attitudes toward the sentence that triggers the selective contraction, and can be seen as progressively refining the notion of selective contraction by imposing increasingly restrictive constraints on the transformation function and on the operator in which the selective contraction is based.

5. Related Works

As mentioned in the introduction of this paper, selective contractions are non-prioritized contraction operators. To the best of our knowledge, there are few operators of this kind so far presented in the literature. A relevant contribution to this topic is found in [11], where the operators of shielded contraction are introduced. When contracting a belief set by a sentence using a shielded contraction operator, we need first to analyse whether that sentence is retractable or not (a retractable sentence is a belief that the agent is willing to remove). When contracting by a retractable sentence, the operator works as a basic AGM contraction operator, otherwise it leaves the original belief set unchanged. The set of retractable sentences is modelled by a set R. Thus a shielded contraction $\sim$ can be defined, from a basic AGM contraction $\div$ and R as follows:

$$\mathbf{K} \sim \alpha = \begin{cases} \mathbf{K} \div \alpha & \text{if } \alpha \in R \\ \mathbf{K} & \text{otherwise} \end{cases}$$

In [11] several properties were proposed for R (the set of retractable sentences) and this model was developed in terms of possible worlds and entrenchment relations. Representation theorems for different classes of shielded contraction operators were presented. In [13,39] shielded contractions were adapted to the belief base context (where the agent's set of beliefs is modelled by a non-necessarily logically closed set of sentences).

The selective contractions proposed in the present paper can be seen as a generalization of shielded contractions. In fact, if $\ominus$ is an operator of selective contraction on a belief set $\mathbf{K}$ induced by a transformation function g and a basic AGM operator $\div$ and $\sim$ is a shielded contraction operator on $\mathbf{K}$ induced by $\div$ and a set of retractable sentence R, both operators coincide if either of the following conditions holds:

i. g is a transformation function such that $\vdash g(\alpha)$, whenever $\alpha \notin R$ and $g(\alpha) = \alpha$, otherwise (in this case the equality between $\ominus$ and $\sim$ holds by $\div$ *failure*);

ii. $\mathbf{K}$ is consistent and g is a transformation function such that $g(\alpha) \notin \mathbf{K}$, whenever $\alpha \notin R$ and $g(\alpha) = \alpha$, otherwise (in this case the equality between $\ominus$ and $\sim$ holds by $\div$ *vacuity* and *inclusion*).

Another work on non-prioritized contraction operators is presented in [21]. This paper introduces a non-prioritized generalization of iterated contraction. In particular, it introduces *hesitant contraction operators*, whose sole application does not necessarily remove the input sentence, but applying it for a certain number of steps leads to its removal (provided that the sentence is not a tautology). Less general classes of operators are also presented, namely the (weak) decrement operators. These can be seen as the contraction counterparts of improvement operators [26], which form a class of belief revision operators that gradually increase the plausibility of new information rather than accepting it immediately. Over successive steps, the input sentence becomes sufficiently plausible to be accepted.

Other non-prioritized contraction operators are the so-called *non-prioritized kernel contraction* operators proposed by Aravanis [4]. These operators are defined for finite belief bases, and their constructions are based on the kernel contraction operators introduced by Hansson [25]. When performing a kernel contraction of a set A by a sentence α, one first selects at least one formula from each α-kernel–that is, from each minimal (under set inclusion) subset of A that implies α–and then removes all the selected sentences from A. In contrast, non-prioritized kernel contraction allows the selection of sentences to be removed from A to come from only a single α-kernel. In this sense, non-prioritized kernel contraction can be seen as a way of weakening the strength of the sentence α with respect to which the operation is performed, as it always reduces the number of α-kernels without necessarily producing a set that fails to imply α.

When considering finite belief bases, one may view the non-prioritized kernel contraction of a set A by α as a form of selective contraction, in which the transformation function g maps α to the conjunction of sentences in the α-kernels from which the sentences to be removed are selected. The corresponding (kernel) contraction operator, that induces that selective contraction operator, then removes from A exactly the sentences selected by the non-prioritized kernel contraction, since contracting by a conjunction requires removing at least one of its conjuncts.

Credibility-limited revision operators, introduced in [15] can be seen as the revision counterpart to shielded contractions. These operators first assess the credibility of the input sentence: if it is deemed credible, revision proceeds via a basic AGM operator; otherwise, the belief set remains unchanged. In [11], the Levi identity ([27]) was adapted and used to establish the inter-definability of shielded contractions and credibility-limited revisions. Credibility-limited operators were subsequently extended to the belief base context in [30,33,39], and their relationship with shielded contractions in this setting was further investigated in [23]. In [5,22], *two level credibility-limited revision* operators were presented. In these papers, operators similar to credibility-limited revision operators were defined, but different degrees of credibility were considered. These operators have the following behaviour: a belief at the highest level of credibility is always incorporated when revising by it; if it is considered to be at the second level of credibility, then that sentence is not incorporated in the revision process, but its negation is removed from the original belief set; when revising by a non-credible sentence, the operator leaves the original belief set unchanged. Formally, this operator is defined by means of an AGM revision $*$ and two sets of sentences C_H and C_L as follows:

$$\mathbf{K} \circledast \alpha = \begin{cases} \mathbf{K} * \alpha & \text{if } \alpha \in C_H \\ (\mathbf{K} * \alpha) \cap \mathbf{K} & \text{if } \alpha \in C_L \\ \mathbf{K} & \text{if } \alpha \notin (C_L \cup C_H) \end{cases}$$

Note that, according to the Harper identity [29] (and the extensionality of $*$), the operator $\div$ defined by $\mathbf{K} \div \neg\alpha = (\mathbf{K} * \alpha) \cap \mathbf{K}$ is an AGM contraction operator. The intuition regarding a sentence in C_L is that it is not credible enough to be incorporated when revising by it, but its reception introduces sufficient doubt to compel the agent to remove from the belief set $\mathbf{K}$ those beliefs that are inconsistent with it. In [7,14] these operators have been characterized in terms of systems of spheres and epistemic entrenchment relations.

6. Conclusion

Standard contraction operators are always successful. However, as previously discussed, this is not a realistic feature of belief contraction. Agents may hold a set of beliefs (not necessarily tautologies) that they are unwilling to relinquish, regardless of the contraction being performed. Selective contraction are non-prioritized contraction operators constructed using a withdrawal operator $\div$ and a transformation function g, which is responsible for selecting the portion of the incoming information that shall be removed from the original belief set. One fundamental property of transformation functions is reverse implication. Given a sentence α, a transformation function satisfying reverse implication returns a sentence that is more general than α. Thus, applying a withdrawal operator to $g(\alpha)$ may preserve α while removing $g(\alpha)$. This can be understood as a weakening of α, as it involves eliminating a sentence that supports α.

Comparing the postulates that characterize withdrawal operators with those presented in Theorems 12 and 13, which characterize selective contraction operators based on withdrawal operators, we observe that the latter satisfy exactly the same postulates as the former, except that success is replaced by weaker variants. This also holds true regarding the selective contraction operators characterized in Theorem 14, if the belief set is consistent (Observation 7).

Regarding basic AGM contractions and the selective contraction operators based on them that were characterized in Theorems 15 and 16, we see that both share the same characterizing postulates, except for *recovery* and *success*, both of which are replaced by weaker versions. The same holds true regarding Theorem 17, provided that the belief set under consideration is consistent (Observation 10). The absence of *recovery* can be seen as a strength of selective contractions, as *recovery* is one of the most controversial AGM postulates.

The study presented in this paper opens several directions for future research. One such direction is the establishment of identities interrelating selective contractions and selective revisions, similar to the well-established Levi and Harper identities ([27,29]) that link AGM contractions and AGM revisions. Another possibility of research is the definition of selective contractions for belief bases (sets of sentences not necessarily closed under logical consequence). A similar study, concerning selective revisions, was conducted in [20,34]. Additionally, a promising topic is the development of a model analogous to two-level credibility-limited revision, incorporating selective revisions and, implicitly, selective contractions.

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7. Appendix

LEMMA 19. *Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator on* $\mathbf{K}$*. If* $\odot$ *satisfies weak proxy success and vacuity, then it also satisfies consistent expansion.*

PROOF. Let $\mathbf{K}$ be a belief set and $\odot$ an operator on $\mathbf{K}$. Assume that $\mathbf{K} \not\subseteq \mathbf{K} \odot \alpha$ By *weak proxy success* it follows that there exists a sentence β such that $\mathbf{K} \odot \alpha \vdash \beta$ and $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta$. If $\neg\beta \notin \mathbf{K}$, then by $\odot$ *vacuity* it follows that $\mathbf{K} \subseteq \mathbf{K} \odot \beta = \mathbf{K} \odot \alpha$. Thus $\mathbf{K} \subseteq \mathbf{K} \odot \alpha$. Contradiction. Hence $\neg\beta \in \mathbf{K}$. On the other hand it holds that $\mathbf{K} \odot \alpha \vdash \beta$. Therefore $\mathbf{K} \cup (\mathbf{K} \odot \alpha) \vdash \perp$. ■

PROOF OF OBSERVATION 5. The statements are immediate consequences of the formulation of the transformation properties under consideration. ■

PROOF OF OBSERVATION 6. Let $\vdash \alpha$. It holds that g satisfies tautological propagation. Hence $\vdash g(\alpha)$. Thus, by $\div$ *failure*, it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha) = \mathbf{K}$. ■

PROOF OF OBSERVATION 7. Let $\mathbf{K}$ be a consistent belief set and $\ominus$ be an operator on $\mathbf{K}$ that satisfies *proxy success, vacuity, failure* and *inclusion*. We intent to prove that $\ominus$ satisfies *strong proxy success*. We will consider two cases:

Case 1) $\vdash \alpha$. By hypothesis $\mathbf{K}$ is a consistent belief set, thus $\mathbf{K} \neq \mathcal{L}$. Hence there exists a sentence β such that $\beta \notin \mathbf{K}$. By $\ominus$ *inclusion* and *vacuity* it follows that $\mathbf{K} \ominus \beta = \mathbf{K}$. On the other hand, by $\ominus$ *failure* it follows that $\mathbf{K} \ominus \alpha = \mathbf{K}$. Furthermore, it holds that $\vdash \beta \to \alpha$.

Case 2) $\nvdash \alpha$. Then by *proxy success* it follows that there is a sentence β, such that $\beta \notin \mathbf{K} \ominus \alpha$, $\vdash \beta \to \alpha$ and $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$. ∎

Proof of Observation 9. Assume that $\mathbf{K} \neq \mathbf{K} \ominus \alpha$. If $\vdash \alpha$, then by $\ominus$ *failure* it follows that $\mathbf{K} = \mathbf{K} \ominus \alpha$. Contradiction. Hence $\nvdash \alpha$. By $\ominus$ *weak proxy recovering success* it follows that there is a sentence β, such that $\beta \notin \mathbf{K} \ominus \alpha$, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$. By *closure* it follows that $\mathbf{K} \ominus \alpha \nvdash \beta$. If $\beta \notin \mathbf{K}$, then by $\ominus$ *vacuity* and *inclusion* it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta = \mathbf{K}$. Contradiction. Hence $\beta \in \mathbf{K}$. ∎

Proof of Observation 10. Let $\mathbf{K}$ be a consistent belief set and $\ominus$ be an operator on $\mathbf{K}$ that satisfies *proxy recovering success, vacuity, failure* and *inclusion*. We intent to prove that $\ominus$ satisfies strong proxy recovering success. We will consider two cases:

Case 1) $\vdash \alpha$. The proof for this case is similar to that presented in Case 1 of the proof of Observation 7.

Case 2) $\nvdash \alpha$. Then by *proxy recovering success* it follows that there is a sentence β, such that $\beta \notin \mathbf{K} \ominus \alpha$, $\vdash \beta \to \alpha$, $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$. ∎

Proof of Observation 11.

1. Follows trivially by definition of $\ominus$ and $\div$ *inclusion* and *closure*.
2. Let $\vdash \alpha \leftrightarrow \beta$. It holds that g satisfies extensionality, thus $\vdash g(\alpha) \leftrightarrow g(\beta)$, from which it follows, by $\div$ *extensionality* that $\mathbf{K} \div g(\alpha) = \mathbf{K} \div g(\beta)$. Thus $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$.
3. Let $\alpha \notin \mathbf{K}$. It holds that g satisfies weak minimality. Hence $g(\alpha) \notin \mathbf{K}$. From which it follows, by $\div$ vacuity, that $\mathbf{K} \subseteq \mathbf{K} \div g(\alpha)$. Thus $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha$.
4. Assume that g satisfies reverse implication. Hence $\vdash g(\alpha) \to \alpha$.

   Vacuity: Let $\alpha \notin \mathbf{K}$. Hence $g(\alpha) \notin \mathbf{K}$. From which it follows, by $\div$ vacuity, that $\mathbf{K} \subseteq \mathbf{K} \div g(\alpha)$. Thus $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha$.

   Non-tautological consistency: Let $\nvdash \alpha$. Thus $\nvdash g(\alpha)$. By $\div$ *success* and *closure* it follows that $\mathbf{K} \div g(\alpha) \nvdash g(\alpha)$. Thus $\mathbf{K} \ominus \alpha \nvdash g(\alpha)$, from which it follows that $\mathbf{K} \ominus \alpha \nvdash \bot$.
5. Let $\alpha \notin Cn(\emptyset)$. By definition of $\ominus$ it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$. On the other hand g satisfies non-validity preservation thus $g(\alpha) \notin Cn(\emptyset)$. By $\div$ *success* it follows that $g(\alpha) \notin \mathbf{K} \div g(\alpha)$. Thus $g(\alpha) \notin \mathbf{K} \ominus \alpha$. It

holds that g satisfies idempotence, thus by $\div$ *extensionality* it follows that $\mathbf{K} \div g(g(\alpha)) = \mathbf{K} \div g(\alpha)$. By definition of $\ominus$ it follows that $\mathbf{K} \ominus g(\alpha) = \mathbf{K} \div g(g(\alpha))$. Hence $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus g(\alpha)$. Thus, there exists $\beta (= g(\alpha))$ such that $\beta \notin \mathbf{K} \ominus \beta = \mathbf{K} \ominus \alpha$.

6. Let $\alpha \notin Cn(\emptyset)$. By definition of $\ominus$ it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$. On the other hand g satisfies reverse inclusion thus $\vdash g(\alpha) \rightarrow \alpha$. Hence $g(\alpha) \notin Cn(\emptyset)$. The rest of the proof follows as in 5.
7. Let $\vdash \alpha$. By definition of $\ominus$ it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$. On the other hand g satisfies tautological propagation thus $\vdash g(\alpha)$. Hence by $\div$ *failure* it follows that $\mathbf{K} \ominus \alpha = \mathbf{K}$.
8. It holds that g satisfies non-tautological choice. Thus $\nvdash g(\alpha)$. By $\div$ *success* it follows that $g(\alpha) \notin \mathbf{K} \div g(\alpha)$. From which it follows by $\div$ *closure* that $\mathbf{K} \div g(\alpha) \nvdash g(\alpha)$. Thus $\mathbf{K} \div g(\alpha) \nvdash \bot$. Therefore $\mathbf{K} \ominus \alpha \nvdash \bot$.
9. We will prove by cases:

   Case 1) $\mathbf{K} \vdash \bot$. By consistency recoverability $\nvdash g(\alpha)$. The proof follows that of Statement 8.

   Case 2) $\mathbf{K} \nvdash \bot$. By $\ominus$ *inclusion* (Statement 1) it follows that $\mathbf{K} \ominus \alpha \nvdash \bot$.
10. By non-tautological choice it follows that $\nvdash g(\alpha)$. Thus by $\div$ *success*, it follows that $\mathbf{K} \div g(\alpha) \nvdash g(\alpha)$. By idempotence it follows that $\vdash g(g(\alpha)) \leftrightarrow g(\alpha)$. Thus, by $\div$ *extensionality*, $\mathbf{K} \div g(\alpha) = \mathbf{K} \div g(g(\alpha))$. Hence $g(\alpha) \notin \mathbf{K} \ominus g(\alpha) = \mathbf{K} \ominus \alpha$. On the other hand, it holds that $\vdash g(\alpha) \rightarrow \alpha$.
11. We will prove by cases:

    Case 1) $\mathbf{K} \vdash \bot$. By g consistency recoverability, it follows that $\nvdash g(\alpha)$. The remainder of the proof for this case proceeds as in the proof of Statement 10.

    Case 2) $\mathbf{K} \nvdash \bot$. There exists β such that $\beta \notin \mathbf{K}$. By g reverse implication it follows that $g(\beta) \notin \mathbf{K}$. We will consider two sub-cases.

    Case 2.1) $\nvdash g(\alpha)$. The proof follows that of Statement 10.

    Case 2.2) $\vdash g(\alpha)$. By g reverse implication it follows that $\vdash \alpha$. Hence $\vdash \beta \rightarrow \alpha$. On the other hand, by $\div$ *failure* and the definition of $\ominus$ it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha) = \mathbf{K}$. On the other hand, by the definition of $\ominus$ and $\div$ *vacuity* and *inclusion* it follows that $\mathbf{K} \ominus \beta = \mathbf{K} \div g(\beta) = \mathbf{K}$. Thus $\beta \notin \mathbf{K} \ominus \beta = \mathbf{K} \ominus \alpha$.

12. Assume that $\nvdash \alpha$. By definition of $\ominus$ it holds that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$. On the other hand, by g non-validity preservation it follows that $\nvdash g(\alpha)$. Thus, by $\div$ *success* it follows that $g(\alpha) \notin \mathbf{K} \ominus \alpha$. By $\div$ *recovery* it follows that $\mathbf{K} \subseteq \mathbf{K} \div g(\alpha) + g(\alpha) = \mathbf{K} \ominus \alpha + g(\alpha)$. On the other hand, it follows by definition of $\ominus$ that $\mathbf{K} \ominus g(\alpha) = \mathbf{K} \div g(g(\alpha))$. From which it follows by definition of $\ominus$, g idempotence and $\div$ *extensionality* that $\mathbf{K} \ominus g(\alpha) = \mathbf{K} \div g(g(\alpha)) = \mathbf{K} \div g(\alpha) = \mathbf{K} \ominus \alpha$. Thus $\ominus$ satisfies *weak proxy recovering success* (consider $\beta = g(\alpha)$).
13. Assume that $\nvdash \alpha$. By reverse implication it follows that $\vdash g(\alpha) \rightarrow \alpha$. Therefore, $\nvdash g(\alpha)$. The rest of the proof follows as presented in proof of Statement 12.
14. By non-tautological choice it holds that $\nvdash g(\alpha)$. The rest of the proof follows as presented in proof of Statement 13.
15. We will prove by cases:

    Case 1) $\mathbf{K} \vdash \perp$. By g consistency recoverability, it follows that $\nvdash g(\alpha)$. The remainder of the proof for this case proceeds as in the proof of Statement 14.

    Case 2) $\mathbf{K} \nvdash \perp$. There exists β such that $\beta \notin \mathbf{K}$. By g reverse implication it follows that $g(\beta) \notin \mathbf{K}$. We will consider two sub-cases.

    Case 2.1) $\nvdash g(\alpha)$. The proof follows that of Statement 14.

    Case 2.2) $\vdash g(\alpha)$. By g reverse implication it follows that $\vdash \alpha$. Hence $\vdash \beta \rightarrow \alpha$. On the other hand, by $\div$ *failure* (Observation 2) and the definition of $\ominus$ it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha) = \mathbf{K}$. On the other hand, by the definition of $\ominus$ and $\div$ *vacuity* and *inclusion* it follows that $\mathbf{K} \ominus \beta = \mathbf{K} \div g(\beta) = \mathbf{K}$. Thus $\beta \notin \mathbf{K} \ominus \beta = \mathbf{K} \ominus \alpha$, and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \beta$.
16. Assume that $\nvdash \alpha$. Hence by non-validity preservation it follows that $\nvdash g(\alpha)$. Thus, by $\div$ *success* and *closure* it follows that $\mathbf{K} \div g(\alpha) \nvdash g(\alpha)$. Hence $\mathbf{K} \ominus \alpha \nvdash \perp$. ■

Proof of Theorem 12. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \text{ or } \vdash \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} \text{ such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha) & \\ \text{ and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *weak proxy success* and *extensionality.*

Let

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} \ominus \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \text{ or } \vdash \alpha \\ \mathbf{K} \div' \alpha & \text{otherwise} \\ \text{where } \div' \text{ is any basic} & \\ \text{AGM contraction operator.} & \end{cases}$$

We need to show that:

(1) g satisfies the properties listed in Statement 2.;

(2) $\div$ is a withdrawal;

(3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

(1) We will start by showing that g satisfies non-validity preservation. Assume that $\nvdash \alpha$. If $\alpha \notin \mathbf{K} \ominus \alpha$, then it follows by definition of g that $g(\alpha) = \alpha$, from which it follows that $\nvdash g(\alpha)$. Consider now that $\alpha \in \mathbf{K} \ominus \alpha$. It holds by definition of g that $g(\alpha) \notin \mathbf{K} \ominus \alpha$. From which it follows by $\ominus$ *closure* that $\nvdash g(\alpha)$.

To show that g satisfies idempotence:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ or $\vdash \alpha$. Thus $g(\alpha) = \alpha$. Hence $g(g(\alpha)) = g(\alpha)$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ and $\nvdash \alpha$. Thus $g(\alpha) = r(\alpha)$ and $r(\alpha) \notin \mathbf{K} \ominus r(\alpha)$. Hence $g(r(\alpha)) = r(\alpha)$. From the latter and $g(\alpha) = r(\alpha)$ it follows that $g(g(\alpha)) = g(\alpha)$.

To show that g satisfies weak minimality:

Let $\mathbf{K} \nvdash \alpha$. By $\ominus$ *inclusion* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$, from which it follows by definition of g that $\vdash g(\alpha) \leftrightarrow \alpha$.

To show that g satisfies extensionality:

Let $\vdash \alpha \leftrightarrow \beta$. It holds by $\ominus$ *extensionality* that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$. Thus by $\ominus$ *closure* it holds that $\alpha \in \mathbf{K} \ominus \alpha$ if and only if $\beta \in \mathbf{K} \ominus \beta$. We will consider three cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$. Hence $\beta \notin \mathbf{K} \ominus \beta$. Thus $g(\alpha) = \alpha$ and $g(\beta) = \beta$, from wich it follows that $\vdash g(\alpha) \leftrightarrow g(\beta)$.

case 2) $\vdash \alpha$. Hence $\vdash \beta$. Thus $g(\alpha) = \alpha$ and $g(\beta) = \beta$, from wich it follows that $\vdash g(\alpha) \leftrightarrow g(\beta)$.

case 3) $\alpha \in \mathbf{K} \ominus \alpha$ and $\nvdash \alpha$. Hence $\beta \in \mathbf{K} \ominus \beta$ and $\nvdash \beta$. It follows by definition of g that $\vdash g(\alpha) \leftrightarrow g(\beta)$.

(2) That $\div$ satisfies **closure** and **inclusion** follows trivially by definition of $\div$ and by $\ominus$ *closure* and *inclusion*, respectively.

Success: Let $\nvdash \alpha$. If $\alpha \notin \mathbf{K} \ominus \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$, and by $\ominus$ *closure* $\mathbf{K} \ominus \alpha \nvdash \alpha$. Hence $\mathbf{K} \div \alpha \nvdash \alpha$. If $\alpha \in \mathbf{K} \ominus \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$. It holds that $\div'$ satisfies *success*. Thus $\mathbf{K} \div \alpha \nvdash \alpha$.

Vacuity: Let $\alpha \notin \mathbf{K}$. By $\ominus$ *inclusion* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$. Hence $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$, from which it follows by $\ominus$ *vacuity* that $\mathbf{K} \subseteq \mathbf{K} \div \alpha$.

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. By $\ominus$ *extensionality* it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus \beta$. Thus by $\ominus$ *closure* it holds that $\alpha \in \mathbf{K} \ominus \alpha$ if and only if $\beta \in \mathbf{K} \ominus \beta$.

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$. Thus $\beta \notin \mathbf{K} \ominus \beta$. Hence $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$ and $\mathbf{K} \div \beta = \mathbf{K} \ominus \beta$. Therefore, $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.

case 2) $\vdash \alpha$. Thus $\vdash \beta$. This case follows as in the previous one.

case 3) $\alpha \in \mathbf{K} \ominus \alpha$ and $\nvdash \alpha$. Thus $\beta \in \mathbf{K} \ominus \beta$ and $\nvdash \beta$. Hence $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$ and $\mathbf{K} \div \beta = \mathbf{K} \div' \beta$. It holds that $\div'$ satisfies *extensionality*, from which it follows that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.

(3) We will now prove that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$.

case 1) $\alpha \in \mathbf{K} \ominus \alpha$ and $\nvdash \alpha$. By definition of g it holds that $g(\alpha) \notin \mathbf{K} \ominus \alpha = \mathbf{K} \ominus g(\alpha)$. Hence by definition of $\div$ it follows that $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus g(\alpha)$, from which it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$.

case 2) $\alpha \notin \mathbf{K} \ominus \alpha$ or $\vdash \alpha$. Hence $g(\alpha) = \alpha$ and $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Thus $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus \alpha$.

(2. to 1.)

This direction of the proof follows from Observation 11. ∎

Proof of Theorem 13. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \text{ or } \vdash \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} \text{ such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha, \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha), \vdash r(\alpha) \rightarrow \alpha & \\ \text{ and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *proxy success* and *extensionality*.

Let $\div$ be defined as in in the (1. to 2.) part of proof of Theorem 13.

We need to show that: (1) g satisfies the properties listed in Statement 2.; (2) $\div$ is a withdrawal; and (3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

That g satisfies reverse implication follows trivially by definition of g.

The rest of the proof for this part is similar to the one present for the Theorem 13.

(2. to 1.)

This direction of the proof follows from Observation 11. ■

Proof of Theorem 14. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} \text{ such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha, \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha), \vdash r(\alpha) \to \alpha & \\ \text{ and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *strong proxy success* and *extensionality*.

Let

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} \ominus \alpha & \alpha \notin \mathbf{K} \ominus \alpha \\ \mathbf{K} \div' \alpha & \text{otherwise} \\ \text{where } \div' \text{ is any basic} & \\ \text{AGM contraction operator.} & \end{cases}$$

We need to show that:

(1) g satisfies the properties listed in Statement 2.;

(2) $\div$ is a withdrawal;

(3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

(1) That g satisfies reverse implication follows trivially by definition of g. For weak minimality: Let $\mathbf{K} \nvdash \alpha$. By $\ominus$ *inclusion* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$. From which it follows, by definition of g, that $\vdash g(\alpha) \leftrightarrow \alpha$.

The proof that g satisfies extensionality and idempotence is similar to the one presented in the corresponding part of Theorem 12. To prove that g satisfies non-tautological choice. If $\vdash \alpha$, then $\alpha \in \mathbf{K} \ominus \alpha$ by $\ominus$ *closure*. By definition of g it follows that $g(\alpha) \notin \mathbf{K} \ominus \alpha$. From which it follows by $\ominus$ *closure* that $\nvdash g(\alpha)$. If $\nvdash \alpha$, then by reverse implication, $\nvdash g(\alpha)$.

(2) That $\div$ satisfies **closure** and **inclusion** follows trivially by definition of $\div$ and by $\ominus$ *closure* and *inclusion*. That $\div$ satisfies **vacuity**, follows by definition of $\div$ and by $\ominus$ *inclusion* and *vacuity*, respectively.

The proof of **success** is similar to the proof of this postulate presented in part (2) of the (1. to 2.) part of the proof of Theorem 12.

The proof that $\div$ satisfies **extensionality** is similar to the one presented in the corresponding part of Theorem 12.

(3) This part of the proof is similar to the one presented in the corresponding part of Theorem 12 (considering Case 1) and Case 2) as corresponding to $\alpha \in \mathbf{K} \ominus \alpha$ and $\alpha \notin \mathbf{K} \ominus \alpha$, respectively).

(2. to 3.)

Consider the same constructions as in the (1. to 2.) part. Regarding the proof presented in the (1. to 2.) part it only remains to prove that: 1) g satisfies consistency recoverability and 2) $\div$ satisfies *failure*.

1) Follows from the fact that g satisfies non-tautological choice, which in turn implies that g satisfies consistency recoverability.

2) Assume that $\vdash \alpha$. By $\ominus$ *closure* it follows that $\alpha \in \mathbf{K} \ominus \alpha$. From which it follows, by definition of $\div$, that $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$. It holds that $\div'$ is a basic AGM contraction operator. Hence, $\div'$ satisfies *failure* (Observation 2). Thus, $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha = \mathbf{K}$.

(3. to 1.)

This direction of the proof follows from Observation 11. ■

Proof of Theorem 15. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } (\alpha \notin \mathbf{K} \ominus \alpha \text{ and } \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha) \\ & \text{or } \vdash \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} & \\ \text{such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha), & \\ \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + r(\alpha) & \\ \text{and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *weak proxy recovering success* and *extensionality*.

Let

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} \ominus \alpha & \text{if } (\alpha \notin \mathbf{K} \ominus \alpha \text{ and } \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha) \\ & \text{or } \vdash \alpha \\ \mathbf{K} \div' \alpha & \text{otherwise} \\ \text{where } \div' \text{ is any} & \\ \text{basic AGM contraction operator.} & \end{cases}$$

We need to show that:

(1) g satisfies the properties listed in Statement 2.;

(2) $\div$ is a basic AGM contraction;

(3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

(1) That g satisfies tautological propagation follows by definition of g. For weak minimality consider $\mathbf{K} \nvdash \alpha$. By *inclusion* and *vacuity* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha$. Hence, by definition of g, $\vdash g(\alpha) \leftrightarrow \alpha$. We now show that g satisfies non-validity preservation. Assume that $\nvdash \alpha$. We will prove by cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence $g(\alpha) = \alpha$. Thus, $g(\alpha) \notin \mathbf{K} \ominus \alpha$. Therefore, by $\odot$ *closure*, $\nvdash g(\alpha)$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \nsubseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, $g(\alpha) \notin \mathbf{K} \ominus \alpha$. Therefore, by $\odot$ *closure*, $\nvdash g(\alpha)$.

To show that g satisfies idempotence:

case 1) $(\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha)$ or $\vdash \alpha$. Thus $g(\alpha) = \alpha$. Hence $g(g(\alpha)) = g(\alpha)$.

case 2) $(\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \nsubseteq \mathbf{K} \ominus \alpha + \alpha)$ and $\nvdash \alpha$. Thus $g(\alpha) = r(\alpha)$ where $r(\alpha) \notin \mathbf{K} \ominus r(\alpha) = \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + r(\alpha) = \mathbf{K} \ominus r(\alpha) + r(\alpha)$. Hence, by definition of g, it follows that $g(r(\alpha)) = r(\alpha)$. From the latter and $g(\alpha) = r(\alpha)$ it follows that $g(g(\alpha)) = g(\alpha)$.

To show that g satisfies extensionality:

Let $\vdash \alpha \leftrightarrow \beta$. It follows by $\ominus$ *extensionality* and *closure* that

$(\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha)$ or $\vdash \alpha$ if and only if

$(\beta \notin \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \beta + \beta)$ or $\vdash \beta$.

Hence, by definition of g, it follows that $\vdash g(\alpha) \leftrightarrow g(\beta)$.

(2) We need to show that $\div$ satisfies *closure*, *inclusion*, *success*, *vacuity*, *extensionality* and *recovery*.

That $\div$ satisfies **closure**, **inclusion** follows by definition of $\div$ and $\ominus$ and $\div'$ *closure* and *inclusion*, respectively.

Vacuity: Let $\alpha \notin \mathbf{K}$. By $\ominus$ *inclusion* and *vacuity* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha$. Thus, $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Therefore, $\mathbf{K} \subseteq \mathbf{K} \div \alpha$.

Success: Let $\nvdash \alpha$. We will prove by cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Thus $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Therefore, by $\ominus$ *closure*, it follows that $\mathbf{K} \div \alpha \nvdash \alpha$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \nsubseteq \mathbf{K} \ominus \alpha + \alpha$. Thus $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$. From which it follows by $\div'$ *success* that $\mathbf{K} \div \alpha \nvdash \alpha$.

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. It holds by $\ominus$ *extensionality* and *closure* that

$(\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha)$ or $\vdash \alpha$ if and only if

$(\beta \notin \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \beta + \beta)$ or $\vdash \beta$.

From which it follows, by definition of $\div$ and by $\ominus$ and $\div'$ *extensionality* that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.

Recovery: We will prove by cases:

case 1) $\vdash \alpha$. Hence, by definition of g and $\div$, $g(\alpha) = \alpha$ and $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Thus, by $\ominus$ *failure* it follows that $\mathbf{K} \div \alpha = \mathbf{K}$. Therefore, $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \alpha$.

case 2) $\nvdash \alpha$.

case 2.1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, by definition of $\div$, $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Thus $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \alpha$.

case 2.2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \nsubseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, by definition of $\div$, $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$. From which it follows, by $\div'$ *recovery*, that $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \alpha$.

(3) We will now prove that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$.

We will consider two cases:

case 1) $(\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha)$ or $\vdash \alpha$. By definition of g and $\div$ it follows, respectively that, $g(\alpha) = \alpha$ and $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Hence $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus \alpha$.

case 2) $(\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \nsubseteq \mathbf{K} \ominus \alpha + \alpha)$ and $\nvdash \alpha$. By definition of g it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \ominus g(\alpha)$. It also follows that $\mathbf{K} \subseteq \mathbf{K} \ominus g(\alpha) + g(\alpha)$ and $g(\alpha) \notin \mathbf{K} \ominus g(\alpha)$. From which it follows, by definition of $\div$, that $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus g(\alpha)$. Therefore, $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus \alpha$.

(2. to 1.)

This direction of the proof follows from Observation 11, having in mind Observation 2.

■

Proof of Theorem 16. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } (\alpha \notin \mathbf{K} \ominus \alpha \text{ and } \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha) \\ & \text{or } \vdash \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} & \\ \text{such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha, \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha), & \\ \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + r(\alpha), \vdash r(\alpha) \rightarrow \alpha & \\ \text{and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *proxy recovering success* and *extensionality*.

Let $\div$ be defined (for all α) as in Theorem 15.

We need to show that: (1) g satisfies the properties listed in Statement 2.; (2) $\div$ is a basic AGM contraction; and (3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

That g satisfies reverse implication follows by definition of g. The rest of the proof for this part is similar to the corresponding part of Theorem 15.

(2. to 1.)

This direction of the proof follows from Observation 11 having in mind Observation 2 and that non-validity preservation follows from reverse implication.

■

Proof of Theorem 17. **(1. to 2.)**

We will first define g and $\div$.

Let $g : \mathcal{L} \longrightarrow \mathcal{L}$ be such that:

$$g(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \text{ and } \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha \\ r(\alpha) & \text{otherwise} \\ \text{where } r \text{ is a function from } \mathcal{L} \text{ to } \mathcal{L} & \\ \text{such that} & \\ r(\alpha) \notin \mathbf{K} \ominus \alpha = \mathbf{K} \ominus r(\alpha), & \\ \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + r(\alpha), \vdash r(\alpha) \rightarrow \alpha & \\ \text{and if } \vdash \alpha \leftrightarrow \alpha' \text{ then } r(\alpha) = r(\alpha'). & \end{cases}$$

This definition is possible since $\ominus$ satisfies *strong proxy recovering success* and *extensionality*.

Let

$$\mathbf{K} \div \alpha = \begin{cases} \mathbf{K} \ominus \alpha & \text{if } \alpha \notin \mathbf{K} \ominus \alpha \text{ and } \mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha \\ \mathbf{K} \div' \alpha & \text{otherwise} \\ \text{where } \div' \text{ is any} & \\ \text{basic AGM contraction operator.} & \end{cases}$$

We need to show that:

(1) g satisfies the properties listed Statement 2.;

(2) $\div$ is a basic AGM contraction;

(3) $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$, for all α.

(1) That g satisfies reverse implication follows trivially by definition of g. The proof that g satisfies weak minimality is similar to the one presented in the proof of Theorem 15. The proof that g satisfies non-tautological choice is similar to the one presented in the proof of Theorem 14.

To show that g satisfies idempotence we will consider two cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence $g(\alpha) = \alpha$. Thus $g(g(\alpha)) = g(\alpha)$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \not\subseteq \mathbf{K} \ominus \alpha + \alpha$. By definition of g it follows that $g(\alpha) \notin \mathbf{K} \ominus g(\alpha)$ and $\mathbf{K} \subseteq \mathbf{K} \ominus g(\alpha) + g(\alpha)$.

Thus, by definition of g that $g(g(\alpha)) = g(\alpha)$.

Now we show that g satisfies extensionality. Let $\vdash \alpha \leftrightarrow \beta$. By $\ominus$ *closure* and *extensionality*, it holds that $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$ if and only if $\beta \notin \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \beta + \beta$. We will consider two cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. By definition of g it follows that $g(\alpha) = \alpha$ and $g(\beta) = \beta$. Thus $\vdash g(\alpha) \leftrightarrow g(\beta)$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \not\subseteq \mathbf{K} \ominus \alpha + \alpha$. By definition of g it follows that $\vdash g(\alpha) \leftrightarrow g(\beta)$.

(2) That $\div$ satisfies **closure**, **inclusion** and **vacuity** follows trivially by definition of $\div$ and by $\ominus$ and $\div'$ *closure*, *inclusion* and *vacuity*, respectively. The proof that $\div$ satisfies **success** follows as in the proof of this postulate presented in part (2) of Theorem 15.

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. By $\ominus$ *extensionality* and *closure* it follows that $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$ if and only if $\beta \notin \mathbf{K} \ominus \beta$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \beta + \beta$. From which it follows by definition of $\div$, and $\ominus$ and $\div'$ *extensionality* that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$.

Recovery: We will prove by cases:

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, by definition of $\div$, $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Therefore, $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \alpha$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ and $\mathbf{K} \not\subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence, by definition of $\div$, $\mathbf{K} \div \alpha = \mathbf{K} \div' \alpha$. From which it follows, by $\div'$ *recovery*, that $\mathbf{K} \subseteq \mathbf{K} \div \alpha + \alpha$.

(3) We will now prove that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$.

case 1) $\alpha \notin \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus \alpha + \alpha$. Hence $g(\alpha) = \alpha$ and $\mathbf{K} \div \alpha = \mathbf{K} \ominus \alpha$. Thus $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus \alpha$.

case 2) $\alpha \in \mathbf{K} \ominus \alpha$ or $\mathbf{K} \not\subseteq \mathbf{K} \ominus \alpha + \alpha$. By definition of g it holds that $g(\alpha) \notin \mathbf{K} \ominus g(\alpha) = \mathbf{K} \ominus \alpha$ and $\mathbf{K} \subseteq \mathbf{K} \ominus g(\alpha) + g(\alpha)$. Hence by definition of $\div$ it follows that $\mathbf{K} \div g(\alpha) = \mathbf{K} \ominus g(\alpha)$, from which it follows that $\mathbf{K} \ominus \alpha = \mathbf{K} \div g(\alpha)$.

(2. to 3.)

Follows trivially since non-tautological choice implies consistency recoverability.

(3. to 1.)

This direction of the proof follows from Observation 11. ■

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