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A Family of Ehrlich-type Accelerated Methods with King's Correction for the Simultaneous Approximation of Polynomial Complex Zeros

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Abstract

In this paper, we present a new family of accelerated iterative methods for the simultaneous approximation of simple complex zeros of a polynomial. These simultaneous methods are constructed on the basis of the third order Ehrlich iteration, accelerated by using the so-called Gauss–Seidel approach, and combined with a correction based on King's family of optimal fourth order iterative methods for solving nonlinear equations. Using King's correction, the R-order of convergence of the basic accelerated method is increased from at least 3 to at least 6. A numerical example is provided to illustrate the convergence and effectiveness of the proposed family of combined accelerated methods for the simultaneous approximation of simple polynomial zeros.

Keywords Polynomial zeros, simultaneous iterative methods, combined methods, accelerated convergence, Ehrlich method

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1. INTRODUCTION

Although there are a large number of numerical methods for determining polynomial roots, there is still a significant interest in the development of new and efficient iterative methods for their approximation. This interest stems from the enormous importance of polynomials in different branches of science and engineering.

Several of these methods, such as the well-known Jenkins–Traub's and Laguerre's algorithms [14], calculate only one real zero or a pair of complex conjugate zeros at a time, and the sequential approximation of all the zeros of a given polynomial by one of such methods involves repeated deflations, which can yield very inaccurate results due to error accumulation through successive deflation steps when using finite precision floating-point arithmetic.

In turn, the simultaneous root-finding algorithms, such as the methods of Ehrlich–Aberth and Durand–Kerner (see, e.g., [20, 23]), appeared in literature only in the 1960s. In addition to being inherently parallel, these iterative algorithms for finding all zeros of a polynomial simultaneously have the advantage of avoiding the unwanted polynomial deflation stages, although they require very good initial approximations for all the zeros in order to converge.

Trying to take advantage of the positive characteristics of the simultaneous zero-finding methods, we propose a new family of accelerated iterative methods for the simultaneous approximation of simple complex zeros of a polynomial, described and analysed in subsequent sections of this paper.

2. PRELIMINARY RESULTS

We start presenting some preliminary results we will need in the following sections.

Let P be a monic polynomial of degree n, with (real or complex) simple zeros $\zeta_1, \ldots, \zeta_n$,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n} (z - \zeta_j) \quad (a_i \in \mathbb{C},\ i \in \{0, \ldots, n-1\}). \tag{1}$$

By using the logarithmic derivative of the polynomial, Maehly [13], and later several other authors, including Börsch-Supan [4], Dochev and Byrnev [6], Ehrlich [7], Weißenhorn [24], Aberth [1] and Farmer and Loizou [8], obtained the following equation:

$$\hat{z}_i = z_i - \frac{1}{\dfrac{P'(z_i)}{P(z_i)} - \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - z_j}} \quad (i = 1, \dots, n), \tag{2}$$

where, for simplicity of notation, $\hat{z}$ represents a new approximation for a zero ζ of P.

Equation (2) is the well-known Ehrlich's method for the simultaneous approximation of simple polynomial zeros, also known in the literature as the Ehrlich–Aberth method (or sometimes as the Maehly–Ehrlich–Aberth method).

For simple zeros, the Ehrlich method converges cubically, as proved in [2] and [7].

The convergence rate of Ehrlich's method (2) can be accelerated by applying the so-called Gauss–Seidel approach [16], that is, using the most recent approximations as they become available in the current iteration.

In this way, we obtain (see, e.g., [22]) the accelerated (single-step) version of the total-step iterative method (2), given by

$$\hat{z}_i = z_i - \frac{1}{\dfrac{P'(z_i)}{P(z_i)} - \displaystyle\sum_{j=1}^{i-1} \frac{1}{z_i - \hat{z}_j} - \displaystyle\sum_{j=i+1}^{n} \frac{1}{z_i - z_j}} \quad (i = 1, \dots, n). \tag{3}$$

The R-order of convergence of the single-step method (3), in the sense of the definition introduced by Ortega and Rheinboldt [19, Ch. 9], is given by the following proposition, proved in [2]:

Proposition 1. *The R-order of convergence of the iterative process* (3) *is at least* $2 + \sigma_n\,(> 3)$, *where* $\sigma_n > 1$ *is the unique positive root of the polynomial equation* $p_n(\sigma) = \sigma^n - \sigma - 2 = 0$.

3. EHRLICH-TYPE ACCELERATED METHODS WITH KING'S CORRECTION

Aiming to improve the convergence rate and efficiency of the simultaneous method (2), we recently proposed a family of simultaneous iterative methods constructed on the basis of the third order Ehrlich iteration, combined with an iterative correction derived from the optimal fourth order two-step King's method [11] for solving single variable nonlinear equations.

This new family of combined iterative methods for the simultaneous approximation of polynomial complex zeros, presented in [12], is defined by

$$\hat{z}_i = z_i - \cfrac{1}{\cfrac{P'(z_i)}{P(z_i)} - \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \cfrac{1}{z_i - z_j + C_K(z_j)}} \quad (i = 1, \dots, n), \tag{4}$$

where the correction term $C_K(z_j)$, obtained from King's family of methods for nonlinear equations, is given by

$$C_K(z_j) = \frac{P(y(z_j))}{P'(z_j)} \frac{P(z_j) + \beta P(y(z_j))}{P(z_j) + (\beta - 2)P(y(z_j))}, \tag{5}$$

where

$$y(z_j) = z_j - \frac{P(z_j)}{P'(z_j)}, \tag{6}$$

and $\beta \in \mathbb{C}$ is a parameter.

In the same paper, we proved the following result on the rate of convergence of the proposed family of simultaneous iterative methods:

Proposition 2. *For initial approximations sufficiently close to the simple zeros of the polynomial P, the order of convergence of the one-parameter family of iterative methods defined in* (4) *is six.*

Thus, using the proposed King's iterative correction (5), the order of convergence is increased from three to six.

As it was previously done for the basic total-step Ehrlich's method, the rate of convergence of the iterative process (4) can be accelerated by using the Gauss–Seidel approach (i.e., calculating the new approximations $\hat{z}_i$ using the already calculated approximations $\hat{z}_1, \dots, \hat{z}_{i-1}$ as soon as they are available).

In this way, using King' approximation $z_j - C_K(z_j)$ in (3) instead of z_j, we obtain a new one-parameter family of Ehrlich-type accelerated simultaneous methods with King's correction, defined by

$$\hat{z}_i = z_i - \cfrac{1}{\cfrac{P'(z_i)}{P(z_i)} - \displaystyle\sum_{j=1}^{i-1} \cfrac{1}{z_i - \hat{z}_j} - \displaystyle\sum_{j=i+1}^{n} \cfrac{1}{z_i - z_j + C_K(z_j)}} \quad (i = 1, \dots, n). \tag{7}$$

where $C_K(z_j)$ is the correction that appears in (5).

For $i = 1$, the first sum is omitted, thus obtaining the iterative process (4), whose order of convergence is six. Then, when $i > 1$, the order of convergence of the family of simultaneous iterative methods described by (7) will be greater than six, as will be shown below.

4. ORDER OF CONVERGENCE OF THE NEW FAMILY OF ACCELERATED METHODS

In this section, we state the convergence theorem for the proposed family of combined iterative methods (7) for simultaneously approximating simple complex zeros of a polynomial.

Theorem 1. *Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be sufficiently close initial approximations to the simple zeros $\zeta_1, \ldots, \zeta_n$ of the polynomial P. Then, the R-order of convergence O_R of the one-parameter family of accelerated simultaneous methods defined in* (7) *is at least $\rho(n) = 2 + \tau_n$ (> 6), where τ_n is the only positive root of the polynomial equation $\tau^n - 4^{n-1}\tau - 2^{2n-1} = 0$ $(n \geq 2)$.*

Proof. Starting from (7), which defines the proposed family of Ehrlich-type accelerated methods with King's correction, and considering the logarithmic derivative of the polynomial $P(z)$, we find from (1)

$$\frac{d}{dz}\log P(z) = \frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - \zeta_j}, \tag{8}$$

and then

$$\hat{z}_i = z_i - \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} - \sum_{j=1}^{i-1} \frac{1}{z_i - \hat{z}_j} - \sum_{j=i+1}^{n} \frac{1}{z_i - z_j + C_K(z_j)}}. \tag{9}$$

Dividing the first sum into two parts, and rewriting the resulting equation, we have

$$\hat{z}_i = z_i - \left[\frac{1}{z_i - \zeta_i} + \sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - \zeta_j} - \sum_{j=1}^{i-1} \frac{1}{z_i - \hat{z}_j} - \sum_{j=i+1}^{n} \frac{1}{z_i - z_j + C_K(z_j)}\right]^{-1}. \tag{10}$$

The first sum in (10) can be written as a sum of two sums. With this we get

$$\hat{z}_i = z_i - \left[\frac{1}{z_i-\zeta_i} + \sum_{j=1}^{i-1}\frac{1}{z_i-\zeta_j} + \sum_{j=i+1}^{n}\frac{1}{z_i-\zeta_j} - \sum_{j=1}^{i-1}\frac{1}{z_i-\hat{z}_j} - \sum_{j=i+1}^{n}\frac{1}{z_i-z_j+C_K(z_j)}\right]^{-1}. \tag{11}$$

Combining the first and the third sums, and the second and the fourth ones, and simplifying the resulting expressions, we obtain

$$\hat{z}_i = z_i - \left[\frac{1}{z_i-\zeta_i} + \sum_{j=1}^{i-1}\frac{-\hat{z}_j+\zeta_j}{(z_i-\zeta_j)(z_i-\hat{z}_j)} + \sum_{j=i+1}^{n}\frac{-z_j+C_K(z_j)+\zeta_j}{(z_i-\zeta_j)(z_i-z_j+C_K(z_j))}\right]^{-1}. \tag{12}$$

Now, let ϵ_i and $\hat{\epsilon_i}$ be the approximation errors defined by

$$\epsilon_i = z_i - \zeta_i, \tag{13}$$

$$\hat{\epsilon_i} = \hat{z}_i - \zeta_i, \tag{14}$$

and, for conciseness, consider the abbreviations

$$\theta_i^{(1)} = \sum_{j=1}^{i-1}\frac{\hat{z}_j-\zeta_j}{(z_i-\zeta_j)(z_i-\hat{z}_j)}, \tag{15}$$

$$\theta_i^{(2)} = \sum_{j=i+1}^{n}\frac{z_j-C_K(z_j)-\zeta_j}{(z_i-\zeta_j)(z_i-z_j+C_K(z_j))}. \tag{16}$$

Rewriting (12), and considering (13) and the preceding abbreviations, the equation reduces to

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{\epsilon_i} - \left(\theta_i^{(1)}+\theta_i^{(2)}\right)}. \tag{17}$$

Taking (14) into account, and considering again (13), we obtain, after simplifying and rearranging,

$$\hat{\epsilon}_i = -\frac{\epsilon_i^2\left(\theta_i^{(1)}+\theta_i^{(2)}\right)}{1-\epsilon_i\left(\theta_i^{(1)}+\theta_i^{(2)}\right)} \quad (i=1,\ldots,n). \tag{18}$$

Under the assumption of the theorem, the approximation errors are sufficiently small in modulus, and thus we can assume that $\epsilon_i = \mathcal{O}_M(\epsilon_j)$ and $\hat{\epsilon}_i = \mathcal{O}_M(\hat{\epsilon}_j)$ for any pair

$i, j \in \{1, \dots, n\}$, where such notation involving the Landau symbol indicates that the moduli of each of the pairs of complex numbers involved are of the same order, i.e., $|\epsilon_i| = \mathcal{O}(|\epsilon_j|)$ and $|\hat{\epsilon}_i| = \mathcal{O}(|\hat{\epsilon}_j|)$.

Now let $\epsilon \in \{\epsilon_1, \dots, \epsilon_n\}$ and $\hat{\epsilon} \in \{\hat{\epsilon}_1, \dots, \hat{\epsilon}_n\}$ be the errors of maximum modulus, that is, $|\epsilon| = \max_{1 \le i \le n} |\epsilon_i|$ and $|\hat{\epsilon}| = \max_{1 \le i \le n} |\hat{\epsilon}_i|$, with $\epsilon_i = \mathcal{O}_M(\epsilon)$ and $\hat{\epsilon}_i = \mathcal{O}_M(\hat{\epsilon})$ for any $i \in \{1, \dots, n\}$.

Let also h_i and $\hat{h}_i$ be, respectively, multiples of the errors $|\epsilon_i| = |z_i - \zeta_i|$ and $|\hat{\epsilon}_i| = |\hat{z}_i - \zeta_i|$ $(i = 1, \dots, n)$.

We wish to prove that (7) converge faster than (4), that is, we need to determine a lower bound greater than six for the order of convergence of the sequence

$$h^{(k)} = \max_{1 \le i \le n} |h_i^{(k)}| \quad (k = 0, 1, \dots). \tag{19}$$

Following a procedure similar to that adopted in [21] (see also [2, 15]), and after a somewhat lengthy deduction, we can derive the following relations for a class of single-step iterative methods with limit point $\zeta = (\zeta_1, \dots, \zeta_n)$, where ζ_i $(i = 1, \dots, n)$ are the zeros of the polynomial P, which includes the proposed family of accelerated methods (7):

$$\hat{h}_i \le \frac{1}{n-1}(h_i)^p \left(\sum_{j=1}^{i-1} \hat{h}_i + \sum_{j=i+1}^{n} (h_i)^q \right) \quad (i = 1, \dots, n;\ p, q \in \mathbb{N}), \tag{20}$$

where, in our particular case, $p = 2$ and $q = 4$.

According to the premise of the theorem, and since $\lim_{k \to \infty} |h_i^{(k)}| = 0$ for $i = 1, \dots, n$, we may assume that the chosen initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ satisfy

$$h_i^{(0)} \le h = \max_{1 \le i \le n} h_i^{(0)} < 1. \tag{21}$$

Now, following a procedure already considered by Alefeld and Herzberger [2, 3], we can derive from the relations (20) and (21) that, from the first iteration to convergence,

$$\hat{h}_i \le h^{\hat{u}_i} \quad (i = 1, \dots, n), \tag{22}$$

where the vectors $\boldsymbol{u} = (u_1, \dots, u_n)^T$ can be calculated by the recurrence relation

$$\hat{\boldsymbol{u}} = \boldsymbol{A}_n(p, q)\, \boldsymbol{u}, \tag{23}$$

starting with the vector $\boldsymbol{u}^{(0)} = (1, \ldots, 1)^T$, where the $n \times n$ matrix $\boldsymbol{A}_n(p, q)$ in (23) is given by

$$\boldsymbol{A}_n(p,q) = \begin{bmatrix} p & q & & & & & & \\ & p & q & & & \mathbf{0} & & \\ & & p & q & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & \mathbf{0} & & & & \cdot & \cdot & \\ & & & & & & p & q \\ p & q & 0 & 0 & \cdots & & 0 & p \end{bmatrix} \quad (p, q \in \mathbb{N}).$$

The square nonnegative matrix $\boldsymbol{A}_n(p, q)$ is irreducible, since its associated directed graph $D(\boldsymbol{A}_n(p, q))$ is strongly connected (see, e.g., [17, Ch. 4]). Thus, according to the Perron–Frobenius theorem for irreducible matrices, $\boldsymbol{A}_n(p, q)$ has a real positive eigenvalue which is greater than or equal to any other of its eigenvalues [17, Ch. 1].

The characteristic polynomial of $\boldsymbol{A}_n(p, q)$ is

$$f_n(\lambda) = (\lambda - p)^n - (\lambda - p)q^{n-1} - pq^{n-1}, \tag{24}$$

and substituting $\tau = \lambda - p$, we obtain

$$f_n(\tau + p) = \tau^n - \tau q^{n-1} - pq^{n-1}. \tag{25}$$

Thus, in our specific case, with $p = 2$ and $q = 4$, we have

$$\phi_n(\tau) = f_n(\tau + 2) = \tau^n - 4^{n-1}\tau - 2^{2n-1}. \tag{26}$$

This polynomial has only one change of signs in the sequence of its coefficients and thus, according to Descartes' rule of signs [10, Ch. 6], $\phi_n(\tau)$ has exactly one positive zero, which we denote by $\tau_n(2, 4)$ or simply τ_n.

The corresponding maximal positive eigenvalue of $\boldsymbol{A}_n(2, 4)$ is $2+\tau_n(2, 4)$, which is the spectral radius of the matrix, $\rho(\boldsymbol{A}_n(2, 4))$. Therefore, we have

$$O_R((7), \zeta) \geq \rho(\boldsymbol{A}_n(2, 4)) = 2 + \tau_n, \tag{27}$$

where τ_n is the only positive zero of (26).

□

Same values of the lower bound for $O_R((7), \zeta)$ are given in Table 1 below.

Table 1: Lower bounds of the R-order of convergence of the iterative process (7).

n	$\rho(n) = 2 + \tau_n$	n	$\rho(n) = 2 + \tau_n$
2	7.46410	8	6.22759
3	6.76595	9	6.19957
4	6.51960	10	6.17769
5	6.39333	15	6.11479
6	6.31647	20	6.08479
7	6.26476	25	6.06722

5. NUMERICAL EXAMPLE

In this section, a numerical example is presented to illustrate the convergence behaviour and effectiveness of the proposed family of combined accelerated methods for the simultaneous approximation of simple polynomial zeros.

As an example, we consider the so-called scaled Wilkinson polynomial [25],

$$P(z) = \prod_{k=1}^{20} (z - \frac{k}{20}).$$

The starting approximations to the zeros of the chosen polynomial were determined by the Aberth's initialization procedure [1], and are equidistantly distributed along the circumference $|z + a_1/n| = r_0$, where $n = 20$ is the polynomial degree, $a_1 = -10.5$ is the coefficient of the second leading term of the polynomial, and $r_0 = 21$ is the radius of a disk centered at the origin of the complex plane containing all the zeros of $P(z)$, obtained by applying the Guggenheimer's upper bound [9] for the zeros of a polynomial.

We have adopted the value $\beta = -7/10$ for the parameter of the fourth order King's method, which was obtained by means of a computational parameter optimization, since not all parameter values lead to convergence (see, e.g., [5]) and only a few other convenient values of β have been reported in the literature (e.g., $\beta = -9/2$, $\beta = 3.9 + 0.1i$ [5], $\beta = -1/2$ [11], $\beta = 3 - 2\sqrt{2}i$ [18]).

In this example, the maximum number of iterations was 50, and the numerical tolerance was set to 10^{-12}. The numerical results were obtained using the standard double precision (64-bit) floating-point arithmetic, which corresponds to about 16 decimal digits of precision.

Figure 1 shows the trajectories of approximations for the scaled Wilkinson polynomial generated by the proposed accelerated method (7).

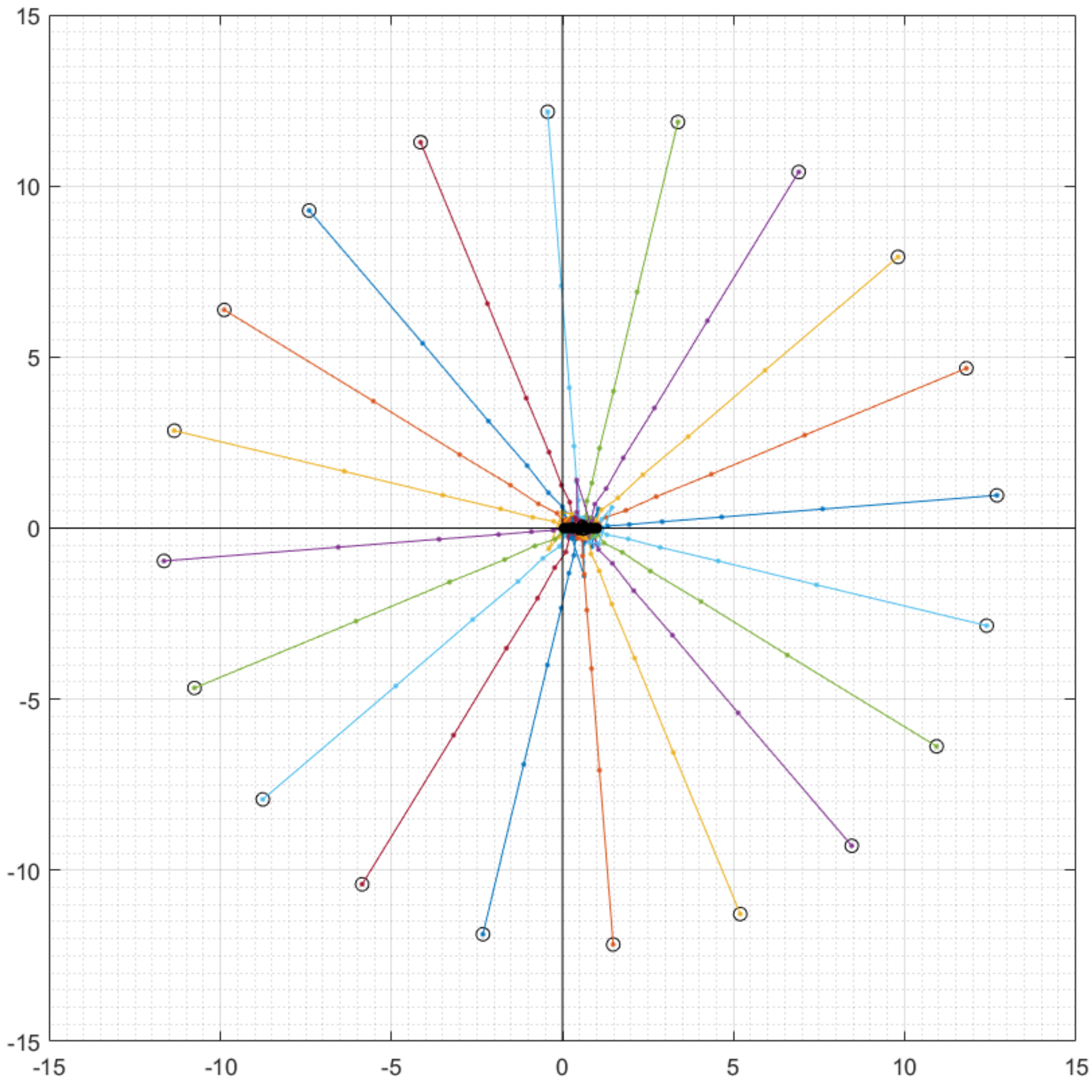

Figure 1: Trajectories of approximations for the scaled Wilkinson polynomial.

A more detailed view, showing the trajectories of approximations corresponding to the last 11 iterations produced by the proposed accelerated iterative method, is given in Figure 2.

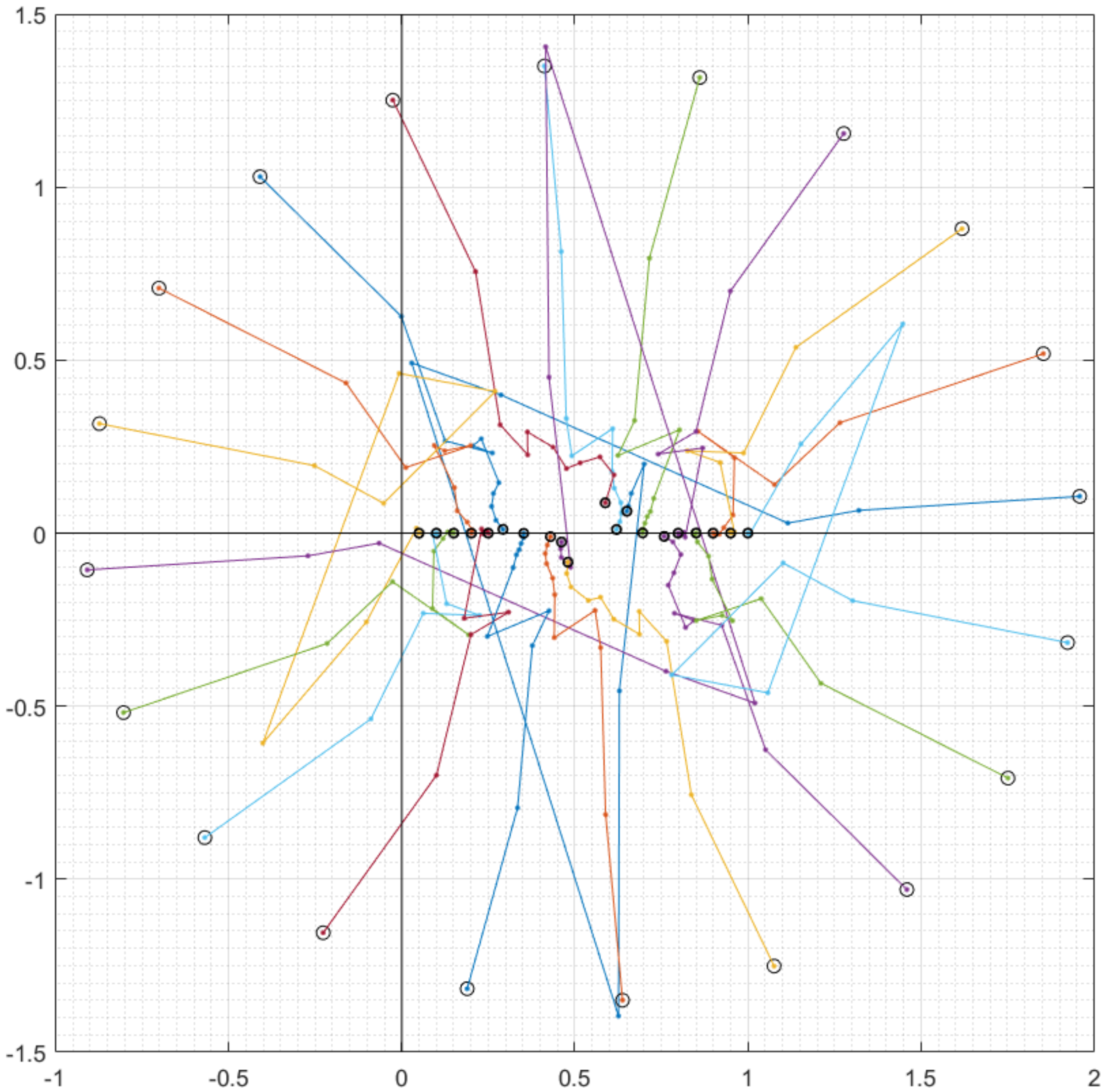

Figure 2: Detail of the trajectories of approximations for the example considered.

In this example, the well-known third order Ehrlich method (2) needed 45 iterations to reach the desired accuracy. In turn, method (4) with $\beta = -7/10$ required only 15 iterations, whereas the proposed Ehrlich-type accelerated method with King's correction (4), with the same value of parameter β, performed only 14 iterations to reach the requested accuracy.

6. CONCLUSION

The use of a correction term obtained from Kings's fourth-order family of methods for solving nonlinear equations allows to increase the order of convergence of Ehrlich's basic total-step method from 3 to 6.

The acceleration of the resulting family of simultaneous iterative methods (by using the so-called Gauss–Seidel approach), proposed in this paper, produces a class of accelerated iterative methods with R-order of convergence of at least six.

The numerical example presented illustrates the convergence and effectiveness of the proposed family of Ehrlich-like accelerated methods with King's correction for the simultaneous approximation of simple polynomial zeros.

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