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Compound Poisson processes: Potentials, Green measures and random times[☆]

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ABSTRACT

In this paper we study the existence of Green measures for Markov processes with a nonlocal jump generator. The non-singular jump kernel has no second moment and satisfies a suitable condition on its Fourier transform. We also study the same problem for certain classes of random time changes Markov processes with jump generator.

1. Introduction

Compound Poisson processes are well known in stochastic analysis as an important class of stochastic processes with independent increments, see Skorohod (1991). In mathematical physics these processes correspond to the random walks in the continuum with a given non-singular jump kernels. Note that random walks in physics are considered as diffusions in the wide sense contrary to the much more restricted notion in stochastic. That is why random walks are very useful in modeling several phenomena observed in concrete physical systems as, e.g., amorphous media, random impurity models, etc.

The central problem to be analyzed is how the properties of jump kernels will be reflected in the behavior of the random walk, precisely their time asymptotic. One of the many possibilities to analyze the long time behavior of general Markov processes is to study their potentials. The general theory of potentials is well elaborated, see, e.g., Blumenthal and Getoor (1968). But an application of this theory to particular examples of Markov processes is not a trivial work.

In the present paper we are concerned with the case of compound Poisson processes and its random time change. We have the following questions to address.

1. Under which conditions on the jump kernel do we have the existence of the corresponding potential?
2. How to obtain the integral representations for the potentials by means of measures?
3. Consider the random time changes in the compound Poisson processes. How to define the potentials and Green measures for these processes?

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Let us describe each of these questions in more details. Denote by X the compound Poisson process starting from $x \in \mathbb{R}^d$. The process X is associated to a generator L defined by a non-singular kernel $a : \mathbb{R}^d \longrightarrow \mathbb{R}$. Given a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ from a proper class, the potential of f is defined by

$$V(x, f) := \int_0^\infty \mathbb{E}^x[f(X(t))]\,\mathrm{d}t.$$

The class of functions f for which the potential $V(x, f)$ exists turns out to be the Banach space $CL(\mathbb{R}^d) := C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with its natural norm $\|f\|_{CL} := \|f\|_\infty + \|f\|_1$. The usual conditions on the kernel a are positive, continuous, bounded, symmetric and finite second moment. In this paper the finite second moment assumption is replaced by a rather weaker condition (see condition (A) in page 7) formulated in terms of its Fourier transform. More precisely, there is $A > 0$ and $0 < \alpha \leq 2$ such that

$$\hat{a}(k) = 1 - A|k|^\alpha + \mathrm{o}(|k|^\alpha) \quad \text{as} \quad k \to 0. \tag{1}$$

For example, the Cauchy distribution has no finite second moment but its Fourier transform admits and expansion as in (1) with $A = \alpha = 1$, see Example 2 below.

The measures which appear in question 2 above for the representations of potentials we call Green measures. The name is due to the fact that they are fundamental solutions to the Markov generators of the associated compound Poisson process. They are Green functions as generalized functions in the classical sense. These generalized function are realized as measures in our case. In addition, we would like to mention that our results are strongly dependent on the dimension of the location spaces. Actually, the Green measures may be defined only for the dimensions $d \geq \alpha$, see Theorem 3 below. For that reason, the study of asymptotic behaviors of additive functionals of classical Markov processes in dimensions $d < \alpha$ needs careful renormalization.

We also introduce the notion of random Green measures which is a more detailed characteristic of the stochastic process X. These measures rise from the representation of the corresponding random potentials of f. That this, given $f \in CL(\mathbb{R}^d)$ we define the random variable

$$Y^x(f) := \int_0^\infty f(X(t))\,\mathrm{d}t$$

and would like to obtain their integral representations as

$$Y^x(f)(\omega) = \int_{\mathbb{R}^d} f(y)\mathcal{G}(x, \mathrm{d}y, \omega), \quad \forall \omega \in \Omega.$$

It is clear that the Green measures for compound Poisson processes are the mathematical expectations of random Green measures, that is,

$$\mathcal{G}(x, \mathrm{d}y) = \mathbb{E}^x[\mathcal{G}(x, \mathrm{d}y, \cdot)].$$

Finally, on the third questions we are interested in the random time changes on the compound Poisson processes, that is, processes defined as $Y(t) := X(D(t))$, $t \geq 0$, where D is a random time change. As random times changes we have chosen a class of inverse of subordinators S satisfying assumption (H), see page 11 for details. Clearly, a random time change will destroy the Markov property of the stochastic process X. Nevertheless, we try to define the associated potentials and Green measures. It turns out that the definition of Green measure for this class of stochastic processes need to be renormalized. More precisely, the definition of the Green measures as introduced in Section 2 for this class of stochastic processes does not exist for any dimension d. Thus, we define a family of proper normalized Green measures such that the limit reduces to the Green measures of the initial stochastic processes X, see Theorem 7. As a consequence, we obtain an asymptotic result for the Green measures associated to the random time changes of the compound Poisson process, see Theorem 8.

The paper is organized as follows. In Section 2 we give the general definition of Green measure of a Markov process in terms of its transition probabilities, see Definition 1. In Section 3 we introduce the jump generator of a compound Poisson process given by a kernel a. We state the assumptions on the kernel a under which the Green measure for the associated process starting from $x \in \mathbb{R}^d$ exists, see assumption (A) on page 7. In addition, the class of functions f for which the potential $V(x, f)$ exists is described. In Section 4 we investigate the integral representation of the random potential $Y^x(f)(\omega)$. The resulting measure is called random Green measure. In Theorem 5 we show the existence of the random Green measure and identify the class of functions f which allows the integral representation. Finally, in Section 5 we study the Green measure of random time changes of the compound Poisson process. The class of subordinators determining the time changes is identified and the Green measures are obtained by a limit process, see Theorem 8.

2. Markov processes and Green measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a time-homogeneous Markov processes on $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$. That is, $X : [0, \infty) \times \Omega \longrightarrow \mathbb{R}^d$ with $X(0, w) = x$, for every $w \in \Omega$, see for example Blumenthal and Getoor (1968), Dynkin (1965), Meyer (1967) and Revuz and Yor (1999) for more details. The Markov process X may be defined in a standard way by its

transition probabilities $P_t(x, B)$. The functions $P_t(x, B)$ give the probability of the transition from the point $x \in \mathbb{R}^d$ at time $t = 0$ to the Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ at time $t > 0$. In certain cases, the transition probabilities $P_t(x, \cdot)$ may have densities $p_t(x, \cdot)$ with respect to the Lebesgue measure in $\mathbb{R}^d$, that is, for any Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$P_t(x, B) = \int_B p_t(x, y)\,\mathrm{d}y$$

and the density $p_t(x, \cdot)$ is also known as heat kernel.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel function and consider the, so-called potential of f, defined by

$$V(x, f) := \int_0^\infty \mathbb{E}^x[f(X(t))]\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} f(y)\,P_t(x, \mathrm{d}y)\,\mathrm{d}t, \tag{2}$$

In probability theory, the notion of potential is well known, see e.g., Blumenthal and Getoor (1968) and Revuz and Yor (1999). Nevertheless, to establish the existence of $V(x, f)$ is a very difficult problem and the class of admissible f have to be analyzed for each process X separately. Assume that for a certain class of functions f (e.g., $f \in C_0(\mathbb{R}^d)$ the space of continuous functions with compact support) the potential $V(x, f)$ is well defined, then $V(x, f)$ may be represented by a Radon measure on $\mathbb{R}^d$ as

$$V(x, f) = \int_{\mathbb{R}^d} f(y)\,\mathcal{G}(x, \mathrm{d}y).$$

The measure $\mathcal{G}(x, \cdot)$ we call the Green measure for the process X. Applying Fubini's theorem in (2) yields

$$\mathcal{G}(x, \mathrm{d}y) = \int_0^\infty P_t(x, \mathrm{d}y)\,\mathrm{d}t$$

assuming the existence of this object as a Radon measure on $\mathbb{R}^d$. That is, for any bounded Borel set $B \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$\mathcal{G}(x, B) = \int_0^\infty P_t(x, B)\,\mathrm{d}t < \infty.$$

In addition, if it happen that $P_t(x, \mathrm{d}y) = p_t(x, y)\,\mathrm{d}y$, then the function

$$g(x, y) := \int_0^\infty p_t(x, y)\,\mathrm{d}t, \quad \forall y \in \mathbb{R}^d,$$

is called the Green function of the Markov process X. In this case the Green measure is given by $\mathcal{G}(x, \mathrm{d}y) = g(x, y)\,\mathrm{d}y$. The existence of the Green function $g(x, y)$ for a given process X or transition probability $P_t(x, \cdot)$, even for simple classes of Markov processes, is not always guaranteed. As an example we consider $X = B$ the 1-dimensional Brownian motion (Bm for short), then for every $f \in L^1(\mathbb{R})$ and any $t > 0$ we have

$$\int_0^t f\big(B(s)\big)\,\mathrm{d}s = \int_{\mathbb{R}} f(y) L_t(y)\,\mathrm{d}y,$$

where $L_t(y)$ is the local time of Bm up to time t at the point y. For the definition and properties of local time of 1-dimensional Bm, see, e.g., Takács (1995b,a), Borodin (1984) and Borodin and Salminen (2002). Now, the asymptotic behavior as $t \to \infty$ of $\int_0^t f(B(s))\,\mathrm{d}s$, as the 1-dimensional Bm B is recurrent, we have $L_\infty(y) = \infty$ for all $y \in \mathbb{R}$, consequently, the integral functional $\int_0^\infty f(B(t))\,\mathrm{d}t$, roughly speaking, does not exists.

Nevertheless, Green functions for certain classes of Markov processes are well known in probability theory, see, e.g., Cao et al. (2021), Grigor'yan et al. (2018a) and references therein.

Thus, we give the definition of Green measure for a Markov process X.

Definition 1 (*Green Measure*)**.** Let $X(t)$, $t \geq 0$, be a Markov process starting from $x \in \mathbb{R}^d$ with transition probability $P_t(x, \cdot)$. The Green measure of X is defined by

$$\mathcal{G}(x, B) := \int_0^\infty P_t(x, B)\,\mathrm{d}t, \quad B \in \mathcal{B}_b(\mathbb{R}^d),$$

or

$$\int_{\mathbb{R}^d} f(y)\mathcal{G}(x, \mathrm{d}y) = \int_{\mathbb{R}^d} f(y) \int_0^\infty P_t(x, dy)\,\mathrm{d}t, \quad f \in C_0(\mathbb{R}^d)$$

whenever these integrals exist.

An equivalent definition of the Green measure $\mathcal{G}(x, \cdot)$ of X is as the fundamental solution of the corresponding generator L. We may start with a Markov semigroup $T(t)$, $t \geq 0$, that is, a family of linear operators on a Banach space E. As a Banach space E, we may use bounded measurable functions $B(\mathbb{R}^d)$, bounded continuous functions $C_b(\mathbb{R}^d)$ or the Lebesgue

spaces $L^p(\mathbb{R}^d)$, $p \geq 1$, depending on each particular case. The semigroup $T(t)$, $t \geq 0$, is associated with a Markov process $\{X(t), t \geq 0 \mid \mathbb{P}^x, x \in \mathbb{R}^d\}$ if for any $t \geq 0$ we have

$$(T(t)f)(x) = \mathbb{E}^x[f(X(t))] = \int_{\mathbb{R}^d} f(y)P_t(x, \mathrm{d}y), \quad f \in E.$$

The transition probabilities $P_t(x, \cdot)$ are constructed from the semigroup by choosing $f = \mathbb{1}_B$, $B \in \mathcal{B}(\mathbb{R}^d)$, that is,

$$P_t(x, B) = (T(t)\mathbb{1}_B)(x).$$

Let L be the generator of the semigroup $T(t)$, $t \geq 0$. From the relation between semigroup and generator we have

$$\begin{aligned}\int_0^\infty (T(t)f)(x)\,\mathrm{d}t &= \int_0^\infty \mathbb{E}^x[f(X(t))]\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} f(y)P_t(x, \mathrm{d}y)\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} f(y)\mathcal{G}(x, \mathrm{d}y) = -(L^{-1}f)(x).\end{aligned} \tag{3}$$

for every $f \in C_0(\mathbb{R}^d)$. Because any Radon measure defines a generalized function, then we may write

$$\mathcal{G}(x, \mathrm{d}y) = g(x, y)\,\mathrm{d}y,$$

where $g(x, \cdot) \in D'(\mathbb{R}^d)$ is a positive generalized function for all $x \in \mathbb{R}^d$. In view of (3) the Green measure is the fundamental solution corresponding to the operator L^{-1}. Note that the existence and regularity of this fundamental solution produces a description of admissible Markov processes for which the Green measure exists. We would like to emphasize that the existence of the inverse of the jump generator L is not always guaranteed and the characterization of its domain for a general Markov process is not an easy task.

3. Jump generators and Green measures

Let $a : \mathbb{R}^d \to \mathbb{R}$ be a fixed kernel function which is symmetric, positive, bounded, continuous, and integrable with $\int_{\mathbb{R}^d} a(x)\,\mathrm{d}x = 1$. Using the kernel a we define an operator $L = L_a$ on E (as stated above) by

$$(Lf)(x) := \int_{\mathbb{R}^d} a(x-y)[f(y) - f(x)]\,\mathrm{d}y = (a * f)(x) - f(x), \quad x \in \mathbb{R}^d. \tag{4}$$

We call this operator the jump generator with jump kernel a. The associated Markov process X is of a pure jump type and is known in stochastic as compound Poisson process (or continuous time random walk), see Skorohod (1991). If we denote $u(t, x) := \mathbb{E}^x[f(X(t))]$, $x \in \mathbb{R}^d$, $t > 0$, and $f \in E$, then $u(t, x)$ satisfies the Kolmogorov equation

$$\begin{cases}\partial_t u(t, x) &= Lu(t, x), \\ u(0, x) &= f(x).\end{cases} \tag{5}$$

If $u(t, x)$ denotes the density of a population at the position x at time t, and $a(x - y)$ is thought of as the probability distribution of jumping from location y to location x, then $(a * u)(t, x)$ gives the rate at which individuals arrive at the position x from other places.

Many analytic properties of the jump generator L were investigated in Grigor'yan et al. (2018b), Kondratiev et al. (2017) and Kondratiev et al. (2018). Here we recall some of these properties necessary in what follows.

The Fourier image of a is given by

$$\hat{a}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-i(k,y)}a(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \cos((k, y))a(y)\,\mathrm{d}y.$$

In addition, it is easy to see that $\hat{a}(0) = 1$, $|\hat{a}(k)| \leq 1$, $k \neq 0$, $\hat{a}(k) \to 0$ as $k \to \infty$. On the other hand, since L is a convolution operator, the Fourier image of L is the multiplication operator by (symbol of L)

$$\hat{L}(k) = \hat{a}(k) - 1.$$

We make the following additional assumption on the jump kernel a.

(A) The kernel a has an expansion of the form (with $A > 0$ and $0 < \alpha \leq 2$)

$$\hat{a}(k) = 1 - A|k|^\alpha + \mathrm{o}(|k|^\alpha) \quad \text{as} \quad |k| \to 0.$$

Example 2. The Gaussian kernel a satisfies Assumption (A) and we have:

$$a(x) = \frac{1}{2^d\pi^{d/2}}\mathrm{e}^{-|x|^2/4} \implies \hat{a}(k) = \mathrm{e}^{-|k|^2} = 1 - |k|^2 + \mathrm{o}(|k|^2), \quad |k| \to 0.$$

Although this kernel has second moment, in general it may fail and we would have a more general expansion: $\hat{a}(k) = 1 - A|k|^\alpha + o(|k|^\alpha)$ with $\alpha \in (0, 2)$. As an example in one dimension we mention the Cauchy distribution

$$a(x) = \frac{1}{1+x^2} \implies \hat{a}(k) = 1 - |k| + o(|k|), \quad k \to 0.$$

Another class of examples is given by the symmetric β-stable distributions, $0 < \beta < 2$. They satisfy Assumption (A) with $\alpha = \beta$ and have no second moment.

For $\lambda \in (0, \infty)$, let $R_\lambda(L) = (\lambda I - L)^{-1}$ be the resolvent of the operator L. Using the Neumann series we obtain

$$(\lambda I - L)^{-1} = \frac{1}{1+\lambda} I + \frac{1}{1+\lambda} A_\lambda, \tag{6}$$

where I is the identity operator and A_λ is defined by

$$A_\lambda := (1+\lambda)(\lambda I - L)^{-1} - I = \sum_{n=1}^{\infty} \frac{a^{*n}}{(1+\lambda)^n}, \tag{7}$$

where $a^{*n} = a * a * \cdots * a$ is the nth times convolution of a with itself. If $\mathcal{G}_\lambda(x, y)$, $x, y \in \mathbb{R}^d$, $\lambda \in (0, \infty)$, denotes the resolvent kernel of $R_\lambda(L)$, then it follows from (6) that the kernel $\mathcal{G}_\lambda(x, y)$ admits the decomposition (cf. Eq. (4) in Kondratiev et al. (2018))

$$\mathcal{G}_\lambda(x, y) = \frac{1}{1+\lambda}\big(\delta(x-y) + G_\lambda(x-y)\big), \tag{8}$$

where G_λ is the kernel of the operator A_λ. The kernel $G_\lambda(x)$ is given by

$$G_\lambda(x) = \sum_{n=1}^{\infty} \frac{a_n(x)}{(1+\lambda)^n}, \quad a_n(x) := a^{*n}(x). \tag{9}$$

The kernel $G_\lambda(x)$ is given by its inverse Fourier transform by

$$G_\lambda(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(k,x)} \frac{\hat{a}(k)}{1+\lambda-\hat{a}(k)}\, dk.$$

The resolvent kernel $\mathcal{G}_\lambda(x, y)$ has a singular part, $\delta(x-y)$ and a regular part $G_\lambda(x-y)$. The Green function, as a generalized function, has the form

$$\mathcal{G}_0(x) = \delta(x) + G_0(x).$$

In particular, the Fourier representation for G_0 has the form

$$G_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(k,x)} \frac{\hat{a}(k)}{1-\hat{a}(k)}\, dk. \tag{10}$$

Theorem 3. *Let a be a jump kernel as defined above which satisfies Assumption* (A)*. Then the Green measure for the random walk exists if $d > \alpha$.*

Proof. It is sufficient to show that the regular part of the resolvent kernel $G_0(x)$, given by (10), is finite for all $x \in \mathbb{R}^d$. It follows from the assumptions on a that the integral in (10) exists for all $x \in \mathbb{R}^d$ due to the integrability $(1 - \hat{a}(k))^{-1}$ at $k = 0$ by the assumption (A). □

In order to identify the class of integrable functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ for which the potential of f (see (2)) exits we define the following Banach space:

$$CL(\mathbb{R}^d) := C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \tag{11}$$

with the norm

$$\|f\|_{CL} := \|f\|_\infty + \|f\|_1 := \sup_{x\in\mathbb{R}^d} |f(x)| + \int_{\mathbb{R}^d} |f(x)|\, dx.$$

Corollary 4. *For any $f \in CL(\mathbb{R}^d)$ the potential $V(x, f)$ of f is well defined and we have*

$$V(x, f) = \int_0^\infty \mathbb{E}^x[f(X(t))]\, dt = \int_{\mathbb{R}^d} f(y)\mathcal{G}(x, dy).$$

Proof. It follows from (8) with $\lambda = 0$ that

$$V(x, f) = f(x) + \int_{\mathbb{R}^d} f(y) G_0(x-y)\, dy, \quad \forall x \in \mathbb{R}.$$

From Theorem 3 and (10) the function $G_0(x)$ is uniformly bounded, hence for any $f \in CL(\mathbb{R}^d)$ the potential $V(x, f)$ is well defined and the claim follows. □

4. Potentials and random Green measures

In this section we are interested in another object associated with the compound Poisson process X introduced in Section 3. Namely, having in mind that X starts from $x \in \mathbb{R}^d$, given a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define the random variables $Y^x(f)$ by

$$Y^x(f) := \int_0^\infty f(X(t))\,\mathrm{d}t. \tag{12}$$

We call $Y^x(f)$ the random potentials of f. The relation between $Y^x(f)$ and the Green measure $\mathcal{G}(x, \mathrm{d}y)$ introduced in the previous sections is clear:

$$\mathbb{E}^x[Y^x(f)] = \int_{\mathbb{R}^d} f(y)\mathcal{G}(x, \mathrm{d}y), \quad \forall f \in CL(\mathbb{R}^d).$$

Our aim is to show that the random variables $Y^x(f)$ admit the representation

$$Y^x(f)(\omega) = \int_{\mathbb{R}^d} f(y)\mathcal{G}(x, \mathrm{d}y, \omega), \quad \omega \in \Omega.$$

with a vector valued random measure $\mathcal{G}(x, \mathrm{d}y, \omega)$, called random Green measure from now on. These measures take values in $L^1(\mathbb{P}) := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and they are finite in all bounded Borel sets $B \in \mathcal{B}_b(\mathbb{R}^d)$.

The following theorem is the main result of this section on the existence of the random Green measure for the random potential (12).

Theorem 5. *For each $x \in \mathbb{R}^d$ the operator*

$$Y^x : CL(\mathbb{R}^d) \longrightarrow L^1(\mathbb{P})$$

has a unique representation given, for all $f \in CL(\mathbb{R}^d)$ and every $\omega \in \Omega$, by

$$Y^x(f)(\omega) = \int_{\mathbb{R}^d} f(y)\,\mathcal{G}(x, \mathrm{d}y, \omega) \tag{13}$$

with a vector valued σ-additive (in the strong topology of $L^1(\mathbb{P})$) Radon measure $\mathcal{G}(x, \mathrm{d}y, \omega)$ on $\mathcal{B}_b(\mathbb{R}^d)$.

Proof. The proof follows the same arguments as Theorem 2.1 in Kondratiev and da Silva (2022) to which we address the interested reader. □

Remark 6. For particular models, it is possible to show that $\mathbb{E}[\mathcal{G}(x, \mathbb{R}^d, \cdot)] < \infty$, that is $\mathcal{G}(x, \mathbb{R}^d, \omega) < \infty$ for $\mathbb{P}$-a.a. $\omega \in \Omega$. In addition, the random variables $Y^x(f)$ are not degenerated, that is, its variance is positive. An example is provided in Proposition 2.3 in Kondratiev and da Silva (2022).

5. Green measures for time changed Markov processes

Time change Markov processes have found many applications in probability theory, see Magdziarz and Schilling (2015) for a detailed discussion and several related references. In this section we investigate the existence of the Green measure $\mathcal{G}(x, \mathrm{d}y)$ for the time changes of the compound Poisson process X introduced before. As time change we choose a class $D(t)$, $t \geq 0$, of inverse subordinators to be specified later, see Section 5.1. Hence, we are interested in a new process $Y = Z(t)$, $t \geq 0$, which is defined by

$$Z(t) := X(D(t)), \quad t \geq 0. \tag{14}$$

The connection between the process X and Y were investigated first in the paper Meerschaert and Scheffler (2008), Mura et al. (2008) and later by other authors, see e.g., Toaldo (2015) and references therein. Define the function $u(t, x)$ by

$$u(t, x) := \mathbb{E}^x[f(X(t))], \ t > 0, \ x \in \mathbb{R}^d, \ f \in E.$$

The function $u(t, x)$ is the solution of the Kolmogorov Eq. (5) with L the jump generator defined in (4). We define a similar function for $Z(t)$:

$$v(t, x) = \mathbb{E}^x[f(Z(t))].$$

Then this function satisfies the following fractional Kolmogorov equation:

$$\mathbb{D}_t^{(k)} v(t, x) = Lv(t, x), \tag{15}$$

where L is the jump generator of X, $k \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a function defined in terms of the Lévy measure of the subordinator and $\mathbb{D}_t^{(k)}$ is the differential-convolution operator known as generalized fractional derivative, see Kochubei (2011) and references therein. Moreover, if the time change $D(t)$ admits a density ρ_t, $t > 0$, then the subordination formula holds:

$$v(t,x) = \int_0^\infty u(\tau,x)\rho_t(\tau)\,\mathrm{d}\tau. \tag{16}$$

In terms of the marginal distribution μ_t^x of $X(t)$ and ν_t^x of $Z(t)$ the subordination relations for these distributions is given by

$$\nu_t^x = \int_0^\infty \mu_\tau^x \rho_t(\tau)\,\mathrm{d}\tau. \tag{17}$$

Below we use these relations to study the renormalized Green measure associated to the stochastic process Z.

5.1. The class of times changes

Let $S = \{S(t),\ t \geq 0\}$ be a subordinator without drift starting from zero with Laplace exponent Φ and Lévy measure σ, see Bertoin (1996) for more details. The Lévy measure σ is supported in $[0,\infty)$ and satisfies $\int_0^\infty (1 \wedge \tau)\,\mathrm{d}\sigma(\tau) < \infty$. We define the kernel k by

$$k : (0,\infty) \longrightarrow (0,\infty),\ t \mapsto k(t) := \sigma\big((t,\infty)\big) \tag{18}$$

and denote its Laplace transform by $\mathcal{K}$, that is,

$$\mathcal{K}(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} k(t)\,\mathrm{d}t, \quad \forall \lambda \geq 0. \tag{19}$$

The following relation holds $\Phi(\lambda) = \lambda\mathcal{K}(\lambda)$, $\forall \lambda \geq 0$. We make the following assumption on the Laplace exponent Φ and $\mathcal{K}$ associated to the subordinator S.

(H) Φ is a complete Bernstein function, that is, σ has a completely monotone density with respect to the Lebesgue measure, and the functions Φ and $\mathcal{K}$ satisfy

$$\mathcal{K}(\lambda) \to \infty,\ \text{as } \lambda \to 0; \quad \mathcal{K}(\lambda) \to 0,\ \text{as } \lambda \to \infty; \tag{20}$$

$$\Phi(\lambda) \to 0,\ \text{as } \lambda \to 0; \quad \Phi(\lambda) \to \infty,\ \text{as } \lambda \to \infty. \tag{21}$$

Classical examples of subordinators satisfying assumption (H) are the α-stable process and the Gamma process. Denote by D the inverse process of the subordinator S, that is,

$$D(t) := \inf\{s \geq 0 \mid S(s) \geq t\}, \quad t \geq 0, \tag{22}$$

and its marginal density by ρ_t, for any $t \geq 0$.

As the class of admissible kernels k belong to the set $\mathbb{K}(\mathbb{R}_+) \subset L^1_{\mathrm{loc}}(\mathbb{R}_+)$ satisfying (H) and (A1) and (A2) in Definition 3.1 in Kochubei et al. (2020). In what follows, we use the notation $f \sim g$ as $t \to \infty$ and say that f and g are asymptotically equivalent when $t \to \infty$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

5.2. Renormalized Green measures

We are now ready to study the Green measure of the random time changes of the compound Poisson process X by the inverse subordinator $D(t)$, that is, the process Z Section 5. Below we distinguish between the Green measure $\mathcal{G}^X(x,\mathrm{d}y)$ of the Markov process X from that $\mathcal{G}^Z(x,\mathrm{d}y)$ of Z.

We would like to show that the following integral

$$\mathcal{G}^Z(x,\mathrm{d}y) := \int_0^\infty \nu_t^x(\mathrm{d}y)\,\mathrm{d}t = \int_0^\infty \int_0^\infty \mu_\tau^x(\mathrm{d}y)\rho_t(\tau)\,\mathrm{d}\tau\,\mathrm{d}t$$

exists and obtain the representation

$$\int_0^\infty v(t,x)\,\mathrm{d}t = \int_0^\infty \mathbb{E}^x[f(Z(t))]\,\mathrm{d}t = \int_{\mathbb{R}^d} f(y)\mathcal{G}^Z(x,\mathrm{d}y). \tag{23}$$

Formulated in this way, the Green measure of Z does not exists for the class of Markov processes X (in any dimension d) and subordinators introduced above, see Lemma 5 in Kondratiev and da Silva (2021). Thus, the representation (23) will

not be possible. To overcome this problem, we shall consider, instead, the renormalized Green measure. More precisely, we would like to find the following limit

$$\mathcal{G}_r^Z(x, \mathrm{d}y) := \lim_{T\to\infty} \frac{1}{N(T)} \int_0^T \nu_t^x(\mathrm{d}y)\,\mathrm{d}t. \tag{24}$$

The normalization $N(T)$ in (24) is chosen in a suitable way so that the renormalized Green measure $\mathcal{G}_r^Z(x, \mathrm{d}y)$ coincides with the Green measure $\mathcal{G}^X(x, \mathrm{d}y)$ of the initial Markov process X. More precisely, we have the following theorem borrowed from Kondratiev and da Silva (2021) and included here for completeness.

Theorem 7. *Assume that the Markov process X in $\mathbb{R}^d$, $d \geq \alpha$, has a Green measure $\mathcal{G}^X(x, \mathrm{d}y)$ and define*

$$N(T) := \int_0^T k(s)\,\mathrm{d}s, \quad T \geq 0. \tag{25}$$

Then the renormalized Green measure for Z exists and we have

$$\mathcal{G}_r^Z(x, \mathrm{d}y) = \mathcal{G}^X(x, \mathrm{d}y).$$

We are now ready to state the main result of this section.

Theorem 8. *Under the assumptions of Theorem 3 holds*

$$\frac{1}{N(t)} \int_0^t v(s, x)\,\mathrm{d}s \sim \int_{\mathbb{R}^d} f(y)\mathcal{G}^X(x, \mathrm{d}y), \quad t \to \infty.$$

Proof. Using the subordination formula (16) we obtain

$$\frac{1}{N(t)} \int_0^t v(s, x)\,\mathrm{d}s = \frac{1}{N(t)} \int_0^t \int_0^\infty u(\tau, x)\rho_s(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

The Fubini theorem, Thm. 3.1 in Kochubei et al. (2020) and (25) yields

$$\frac{1}{N(t)} \int_0^t v(s, x)\,\mathrm{d}s = \int_0^\infty u(\tau, x) \left(\frac{1}{N(t)} \int_0^t \rho_s(\tau)\,\mathrm{d}s \right) \mathrm{d}\tau \underset{t\to\infty}{\longrightarrow} \int_0^\infty u(\tau, x)\,\mathrm{d}\tau.$$

Then the result claim follows from Theorem 3. □

Data availability

No data was used for the research described in the article.

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