International Journal of Approximate Reasoning 162 (2023) 109005

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

On the logic of theory change iteration of KM-update

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ARTICLE INFO

Article history:
Received 29 January 2023
Received in revised form 24 July 2023
Accepted 2 August 2023
Available online 9 August 2023

Keywords:
Belief change
KM-update
Iteration
Possible world semantics

ABSTRACT

We present a model for iterating the Katsuno and Mendelzon Update (KM-Update) process, which is inspired by models developed for iterated belief revision in the AGM framework. Our model adapts the postulates of belief revision proposed by Darwiche and Pearl, including the independence postulate. We characterize two classes of iterated update operators based on natural and lexicographic revision. The semantics of our models are characterized using the framework of possible worlds.

1. Introduction

Belief change, or the process of modifying an agent's beliefs in response to new information, is a fundamental aspect of our daily lives, even if we may not always be aware of it. These changes are ubiquitous and ongoing, leading to a constant evolution of our beliefs. As a result, we may find ourselves accepting new beliefs, revising existing ones, and even discarding some altogether. However, formalizing this process is a complex challenge, as there is no method for univocally making these changes.

Research on belief change has been conducted in both philosophy and computer science since the late 1970s, with a significant number of contributions in the field of artificial intelligence [6,29,10,38,35] (some of these models are summarized in [20]). In 1983, Fagin, Ullman, and Vardi emphasized the importance of defining the dynamics of the update process in a database [13]. In the field of philosophy, early modern studies on belief change were conducted by scholars such as Levi [28], Harper [17] and von Wright [37], who proposed criteria of rationality for revisions of probability assignments and provided a significant portion of the basic formal framework for belief change.

The most widely recognized model of belief change is the AGM model, which was proposed in 1985 by Alchourrón, Gärdenfors, and Makinson in their seminal work [1]. The AGM model is a formal framework that characterizes the dynamics and state of beliefs of a rational agent. Within the AGM framework, beliefs are represented as belief sets, which are defined as deductively closed sets of sentences. A change in beliefs can occur by adding or removing a specific sentence from/to a belief set, which results in a new, consistent belief set.

Over the past 35 years, the AGM model has acquired the status of the standard model of belief change. This model has inspired numerous researchers to propose extensions, generalizations, and applications, as well as connections with other fields. For an overview of these proposals, see [16,15]. One of the key extensions is the concept of iterated change. One drawback of the AGM model's definition of revision is that the conditions for the iteration of the process are quite weak,

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https://doi.org/10.1016/j.ijar.2023.109005

which is due to the lack of expressive power of belief sets. To ensure good properties for the iteration of the revision process, a more complex structure is needed. To address this, Darwiche and Pearl proposed shifting from belief sets to epistemic states in [11] in order to accommodate iteration. An epistemic state encompasses more information than a belief set, allowing it to provide guidance for further changes. Additionally, the process of change should result in a complete representation of a new epistemic state, rather than just a new belief set. In this framework, it is possible to define interesting iterated revision operators. There are several ways to introduce iteration in the AGM operators, for an overview see [33].

In a model of iterated belief revision, there may be multiple ways to arrive at the same belief set. However, what happens with subsequent changes? Fermé and Hansson [16, Section 7.2] classified the operators into three categories based on their ability to remember and take into account the revision history:

(a) *Operators without memory:* In this case, each belief set is revised in a predetermined manner, regardless of how it was obtained. For example, if two belief states Ψ_1 and Ψ_2 have the same belief set, then $\Psi_1 * \alpha$ and $\Psi_2 * \alpha$ would also have the same belief set. This behavior can be represented by a binary revision operator denoted as $*$, defined for any theory K and any formula α.
(b) *Operators with full memory:* In this case, the complete history of changes is preserved, allowing for the possibility of rolling back previous revisions.
(c) *Operators with partial memory:* In this case, the process of obtaining a belief set can impact future revisions, but the remembered information is insufficient for identifying previous states. The majority of the proposed iterable revision operators fall into this category.

In 1992, Katsuno and Mendelzon introduced a type of belief change operator known as the update operator in [21]. While AGM operators are well-suited for capturing changes that reflect evolving knowledge about a static situation, the KM-update operators are intended to represent changes in beliefs that result from changes in the objects of belief. This difference was first pointed out by Keller and Winslett in [22] in the context of relational databases, and is captured in the following example [38]:

> Initially the agent knows that there is either a book on the table (α) or a magazine on the table (β), but not both.
>
> *Case 1:* The agent is told that there is a book on the table. She concludes that there is no magazine on the table. This is revision.
>
> *Case 2:* The agent is told that subsequently a book has been put on the table. In this case she should not conclude that there is no magazine on the table. This is update.

In this paper, we propose to analyze iteration in the context of update, which is inspired by the AGM revision model. It is important to note that an update operator $\diamond$ is defined for all possible belief sets φ. In this sense, iteration is not a problem since $(\varphi \diamond \alpha) \diamond \beta$ is well defined. However, if we wish to make changes to our preferences as a consequence of updating by α, we need to define a new operator that not only changes the belief set φ, but also results in changes to the operator $\diamond$. Similar to belief revision, we need to move from belief sets to more complex structures called epistemic states, which includes, in addition to φ, the strategy of belief update. We will define an epistemic state as a tuple $\langle \varphi, \diamond \rangle$, and a new operator $\blacklozenge$ such that $\langle \varphi, \diamond \rangle \blacklozenge \alpha = \langle \varphi', \diamond' \rangle$, where $\varphi \diamond \alpha = \varphi'$, but possibly $\diamond \neq \diamond'$. Therefore, $(\varphi \diamond \alpha) \diamond \beta$ and $\langle \varphi, \diamond \rangle \blacklozenge \alpha \blacklozenge \beta$ yield different belief sets.

We adapt the postulates of belief revision proposed by Darwiche and Pearl, as well as the *Independence* postulate proposed in [5,19], to Katsuno-Mendelzon's update. We present two families of iterated update operators, one based on natural revision [8], and the other based on lexicographic revision [30,32].

The paper is organized as follows: In Section 2, we introduce the necessary notations and background concepts. In Section 3, we present and characterize the model for iterative KM-update. In Section 4, we introduce two families of KM-update operators based on natural revision and lexicographic revision. In Section 5, we summarize the main contributions of the paper, discuss their relevance and future lines of research. Lastly, in Appendix A, we provide proofs for all original results presented in the paper.

A preliminary version of this paper was originally presented at the Gabriele Kern-Isberner Festschrift in Dortmund, and a previous version of this paper was published in [14]. The present version is a revised and extended version of the original paper, incorporating new analysis of the formal results and providing complete proofs. Notably, changes have been made to the notation to better specify all the requirements in the transition from belief sets to epistemic states. Specifically, we have made modifications to the notation in order to improve clarity and precision in expressing the requirements for this transition.

2. Background

2.1. Formal preliminaries

We denote by $\mathcal{L}$ the set of formulas of a propositional language built over a finite set of propositional variables $\mathcal{P}$. The elements of $\mathcal{L}$ are denoted by lower case Greek letters α, β, ... (possibly with subscripts). The set of valuation functions from the set of propositional variables into the boolean set $\{0, 1\}$ (false, true) is denoted $\mathcal{V}$. As usual, we write $\omega \models \alpha$ when a valuation $\omega \in \mathcal{V}$ satisfies a formula α, *i.e.* when ω is a model of α. The set of models (or possible worlds) of a formula α is denoted by $\|\alpha\|$. W is the set of all possible worlds. If M is a set of models we denote by α_M a formula such that $\|\alpha_M\| = M$. We often omit the braces, by writing, e.g., $\alpha_{\omega,\omega'}$ instead of $\alpha_{\{\omega,\omega'\}}$. If $\leq$ is a total preorder (a total, reflexive and transitive relation), then $\simeq$ is a notation for the associated equivalence relation ($a \simeq b$ iff $a \leq b$ and $b \leq a$), and $<$ is the notation for the associated strict order ($a < b$ iff $a \leq b$ and $b \nleq a$).

2.2. AGM model

In the AGM theory beliefs are represented by belief sets K, which is a set of sentences in a language $\mathcal{L}$ closed under logical consequence Cn, where Cn satisfies: $A \subseteq Cn(A)$, $Cn(Cn(A)) \subseteq Cn(A)$ and $Cn(A) \subseteq Cn(B)$ if $A \subseteq B$, as well as supraclassicality, deduction and compactness. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. We will denote the set of all belief sets by $\mathcal{K}$.

A change consists on adding or removing a sentence from a belief set in order to obtain a new belief set. AGM recognizes three change operations:

- Expansion: a sentence is added to the belief set and nothing is removed (represented as $K + \alpha$). Expansion is defined by $K + \alpha = Cn(K \cup \{\alpha\})$;

- Contraction: a sentence is removed from the belief set and nothing is added (represented as $K - \alpha$);

- Revision: a new sentence is added to the belief set and at the same time other sentences are removed if necessary to ensure the consistency of the revised set (represented as $K * \alpha$).

If the language $\mathcal{L}$ is finite, we can represent a belief set as a propositional sentence φ, such that $K = Cn(\varphi)$. Since we assumed $\mathcal{L}$ to be finite, in the rest of the paper we will denote a belief set by φ. We will say that a formula φ is *complete* if it is consistent and for any propositional formula α it follows that $\varphi \vdash \alpha$ or $\varphi \vdash \neg\alpha$. Alchourron, Gärdenfors, and Makinson have proposed a set of postulates to govern the process of belief revision [1]. Katsuno and Mendelzon rephrased these postulates for a finite language [20].

(R1) $\varphi * \alpha \vdash \alpha$
(R2) If $\varphi \wedge \alpha \nvdash \bot$ then $\varphi * \alpha \equiv \varphi \wedge \alpha$.
(R3) If $\alpha \nvdash \bot$ then $\varphi * \alpha \nvdash \bot$.
(R4) If $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$ then $\varphi_1 * \alpha_1 \equiv \varphi_2 * \alpha_2$.
(R5) $(\varphi * \alpha) \wedge \beta \vdash \varphi * (\alpha \wedge \beta)$
(R6) If $(\varphi * \alpha) \wedge \beta \nvdash \bot$ then $\varphi * (\alpha \wedge \beta) \vdash (\varphi * \alpha) \wedge \beta$.

Along with this definition Katsuno and Mendelzon provided a representation theorem that shows an equivalence between the postulates and a revision mechanism based on total preorders:

Definition 2.1. [20] Let W be the set of all worlds (or interpretations) of a propositional language $\mathcal{L}$. A function that maps each sentence φ in $\mathcal{L}$ to a total preorder $\leq_\varphi$ on worlds W is called a faithful assignment if and only if:

(1) $\omega_1, \omega_2 \models \varphi$ only if $\omega_1 =_\varphi \omega_2$.
(2) $\omega_1 \models \varphi$ and $\omega_2 \not\models \varphi$ only if $\omega_1 <_\varphi \omega_2$.
(3) $\varphi \equiv \phi$ only if $\leq_\varphi = \leq_\phi$.

The following observation characterizes a revision operator in terms of faithful assignments:

Observation 2.2. *[20]* $*$ is an operator that satisfies postulates (R1) - (R6) if and only if there exists a faithful assignment that maps each formula φ into a total preorder $\leq_\varphi$ such that:

$$\|\varphi * \alpha\| = min(\|\alpha\|, \leq_\varphi)$$

where $min(\|\alpha\|, \leq_\varphi)$ contains all worlds that are minimal in $\|\alpha\|$ according to the total preorder $\leq_\varphi$, i.e. all the worlds that include α and are closer to φ.

Fig. 1 provides a diagram of a belief revision function in the possible worlds approach.[1]

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[1] This type of representation diagram is due to Konieczny and Pino Pérez.

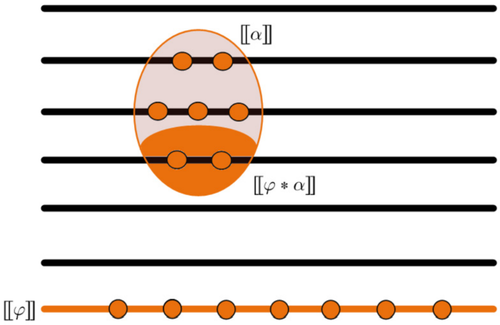

Fig. 1. Belief revision.

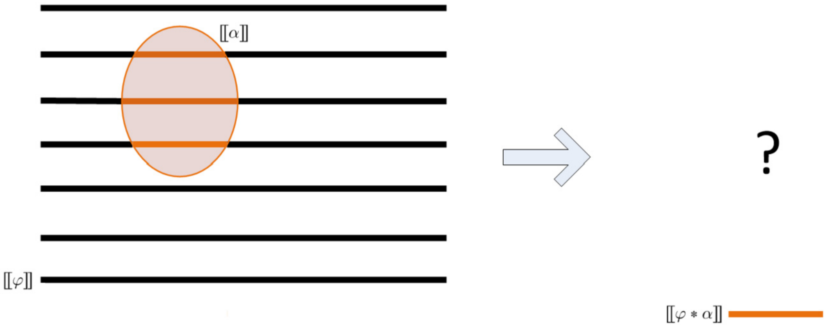

Fig. 2. AGM doesn't provide insight to iteration.

2.3. Iteration for belief revision

The AGM seminal paper was fundamental in advancing the field of belief revision. However, their proposed postulates were not adequate in ensuring the rational preservation of conditional beliefs. The AGM postulates fail to capture the dynamics of the structure used to encode one-step revision policies and, as a result, are not robust enough to regulate iterated belief revision, which refers to the sequential revision of beliefs in response to a string of propositions. This issue is illustrated in Fig. 2.

Darwiche and Pearl in their work [11] suggest that in order to properly define iteration, revision functions must operate on epistemic states instead of belief sets. While belief sets simply characterize the set of propositions that an agent holds, epistemic states also include more complex structures that govern the changes in beliefs. According to Darwiche and Pearl, an epistemic state is formally defined as:

Definition 2.3. [11] An epistemic state Ψ is an object to which we associate a consistent propositional formula $B(\Psi)$ that denotes the current beliefs of the agent in the epistemic state Ψ. We denote by $\mathcal{E}$ the set of epistemic states.

Darwiche and Pearl modified the (R1)-(R6) postulates to work in the more general framework of epistemic states.

(R*1) $B(\Psi * \alpha) \vdash \alpha$
(R*2) If $B(\Psi) \wedge \alpha \nvdash \bot$ then $B(\Psi * \alpha) \equiv B(\Psi) \wedge \alpha$.
(R*3) If $\alpha \nvdash \bot$ then $B(\Psi * \alpha) \nvdash \bot$.
(R*4) If $\Psi_1 = \Psi_2$ and $\alpha_1 \equiv \alpha_2$ then $B(\Psi_1 * \alpha_1) \equiv B(\Psi_2 * \alpha_2)$.
(R*5) $B(\Psi * \alpha) \wedge \beta \vdash B(\Psi * (\alpha \wedge \beta))$
(R*6) If $B(\Psi * \alpha) \wedge \beta \nvdash \bot$ then $B(\Psi * (\alpha \wedge \beta)) \vdash B(\Psi * \alpha) \wedge \beta$.

It is important to note that this modification is achieved through a modification of postulate (R4) to allow for belief revision to be based on an epistemic state. The revised postulate (R*4) requires that two epistemic states be identical for them to result in equivalent belief states when revised with equivalent input.

The four additional postulates proposed by these authors, and which are known as DP-postulates, are the following:

(C1) If $\alpha \vdash \mu$ then $B((\Psi * \mu) * \alpha) \equiv B(\Psi * \alpha)$.
(C2) If $\alpha \vdash \neg\mu$, then $B((\Psi * \mu) * \alpha) \equiv B(\Psi * \alpha)$.
(C3) If $B(\Psi * \alpha) \vdash \mu$, then $B((\Psi * \mu) * \alpha) \vdash \mu$.

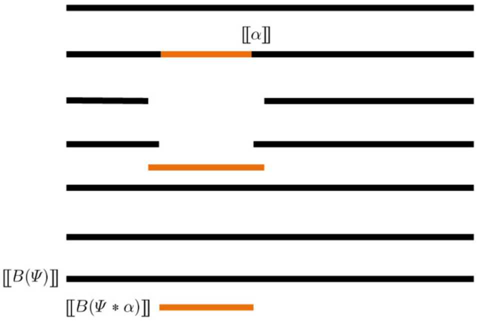

Fig. 3. An example of a belief revision function satisfying (CR1)-(CR4).

(C4) If $B(\Psi * \alpha) \nvdash \neg\mu$, then $B((\Psi * \mu) * \alpha) \nvdash \neg\mu$.

Postulate (C1) states that the later evidence α cannot discredit the previous evidence μ if α implies μ (thus, in a sense, making μ redundant). Postulate (C2) states that if α contradicts the previous evidence μ, then α completely eradicates the effect of μ on the belief set. Postulate (C3), on the other hand, ensures that μ is retained after accommodating the more recent evidence α, given that μ would be believed after revision by α. Lastly, postulate (C4) stipulates that if μ is not contradicted after revision by α, then it should not be contradicted after revision by first μ, then α.

Postulates (C1)-(C4) have a correspondence in terms of total preorders, where the definition of faithful assignment must be adapted to belief states (i.e., by using $\leq_\Psi$ instead of $\leq_\varphi$):

Definition 2.4. [20,11] Let Ψ, Ψ' be belief states. A total preorder $\leq_\Psi$ on possible worlds, with the strict part $<_\Psi$ and the symmetric part $\simeq_\Psi$, is a *global faithful assignment* associated with the belief state Ψ if and only if the following conditions hold:

1. If $\omega \models B(\Psi)$ and $\omega' \models B(\Psi)$, then $\omega \simeq_\Psi \omega'$.
2. If $\omega \models B(\Psi)$ and $\omega' \not\models B(\Psi)$, then $\omega <_\Psi \omega'$.
3. $\Psi = \Psi'$ only if $\leq_\Psi = \leq_{\Psi'}$.

Observation 2.5. [11, Theorem 13] Suppose that a revision operator satisfies postulates (R*1)-(R*6). The operator satisfies postulates (C1)-(C4) iff the operator and its corresponding faithful assignment satisfy:

(CR1) If $\omega_1 \models \mu$ and $\omega_2 \models \mu$, then $\omega_1 \leq_\Psi \omega_2$ iff $\omega_1 \leq_{\Psi*\mu} \omega_2$.
(CR2) If $\omega_1 \models \neg\mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leq_\Psi \omega_2$ iff $\omega_1 \leq_{\Psi*\mu} \omega_2$.
(CR3) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 <_\Psi \omega_2$ only if $\omega_1 <_{\Psi*\mu} \omega_2$.
(CR4) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leq_\Psi \omega_2$ only if $\omega_1 \leq_{\Psi*\mu} \omega_2$.

According to (CR1), the order among the μ-worlds remains unchanged after revision by μ. (CR2) states that the order among the $\neg\mu$-worlds remains unchanged after revision by μ. (CR3) dictates that if a μ-world is strictly preferred over a $\neg\mu$-world, that strict preference is maintained after revision by μ. (CR4) states that if a μ-world is weakly preferred over a $\neg\mu$-world, that weak preference is maintained after revision by μ. Fig. 3 shows an example of a belief revision function satisfying (CR1)-(CR4).

Postulates (C1)-(C4) have become the standard for iterated revision. New proposals are almost always compared to them. However, Booth and Meyer [5] and Jin and Thielscher [19] have noted that these postulates are too permissive since they do not rule out operators by which all newly acquired information is given up as soon as an agent learns a fact that contradicts some of its current beliefs. To address this, they proposed the following additional condition known as the *independence postulate*:

(Ind) If $B(\Psi) * \alpha \nvdash \neg\mu$, then $B((\Psi * \mu) * \alpha) \vdash \mu$.

It is evident that (Ind) is stronger than (C3) and (C4). (C1), (C2), and (Ind) are considered to be defining features of a class of revision operators referred to as *admissible revision operators* [5]. Fig. 4 shows an example of non-admissible revision operator. In total preorders, (Ind) translates to the following postulate:

(CRInd) For $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, if $\omega_1 \leq_\Psi \omega_2$ then $\omega_1 <_{\Psi*\mu} \omega_2$.

In the literature, two of the major types of iterable revision operators are:

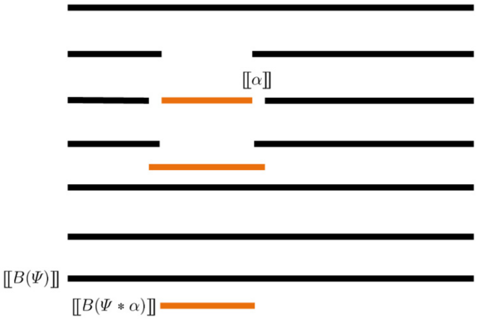

Fig. 4. An example of a non-admissible revision function.

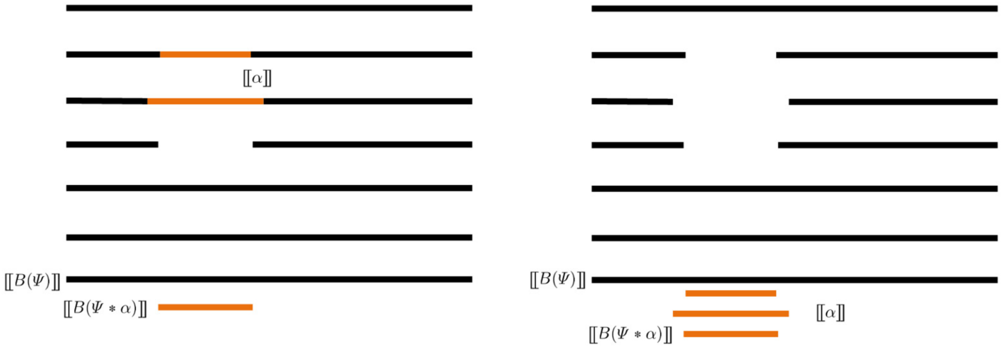

Fig. 5. Natural revision (left) and lexicographic revision (right).

Conservative revision, originally called natural revision, has been studied by Boutilier [7,8] and Rott [34]. This operation is conservative in the sense that it only makes the minimal changes of the preorder that are needed to accept the input. In revision by μ, the minimal μ-worlds are moved to the bottom of the preorder which is otherwise left unchanged. Fig. 5 (left) shows an example of a natural revision function. The main characteristic of this operator is:

(Nat) If $B(\Psi * \mu) \vdash \neg\alpha$, then $B((\Psi * \mu) * \alpha) = B(\Psi * \alpha)$.

Its semantic counterpart is as follows:

Observation 2.6. [11, Theorem 11] Suppose that a revision operator satisfies postulates (R*1)-(R*6). The operator satisfies postulate (Nat) iff the operator and its corresponding faithful assignment satisfy:

(CRNat) If $\omega_1 \notin \|\Psi * \mu\|$ and $\omega_2 \notin \|\Psi * \mu\|$, then $\omega_1 \leq_\Psi \omega_2 \Leftrightarrow \omega_1 \leq_{\Psi * \mu} \omega_2$.

Lexicographic revision was proposed by Nayak in [31] and deeply studied by Nayak, Pagnucco and Peppas [32]. When revising by μ, this operation rearranges the preorder by placing the μ-worlds at bottom (but preserving their relative order) and the $\neg\mu$-worlds at top (but preserving their relative order). Fig. 5 (right) shows an example of a lexicographic revision function. In addition to (C1)-(C4) it has the following property:

(Lex) If $\alpha \nvdash \neg\mu$, then $B((\Psi * \mu) * \alpha) \vdash \mu$.

Its semantic counterpart is as follows:

Observation 2.7. [32,4] Suppose that a revision operator satisfies postulates (R*1)-(R*6), (C1)-(C4). The operator satisfies postulate (Lex) iff the operator and its corresponding faithful assignment satisfy (CR1)-(CR4) and:

(CRLex) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 <_{\Psi * \mu} \omega_2$.

Fig. 5 shows an example of these revision operators.

2.4. KM-update

As previously mentioned, Katsuno and Mendelzon proposed an alternative model to depict the alteration of beliefs resulting from modifications in the objects of belief. The distinction between belief revision and update is clearer in the possible world approach, as stated in [22,38]. Instead of using a preorder and a global distance to worlds where the input

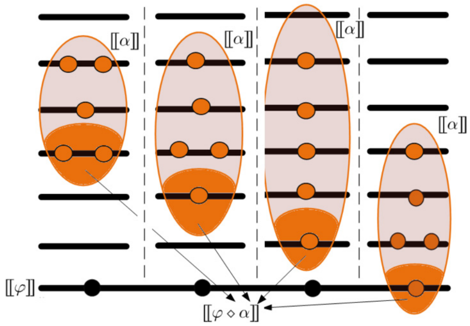

Fig. 6. An example of a KM-update operator.

is validated, Katsuno and Mendelzon emphasized that since an agent believes one of the current possible worlds to be the real world (even if they are unaware of which), then during a world change, the update function must consider each of the possible worlds and identify the minimal means of adjusting them to accommodate the input. In their formal presentation, Katsuno and Mendelzon assumed that $\mathcal{L}$ is finite. The postulates for update are the following:

Definition 2.8. Let φ and α be sentences of $\mathcal{L}$. Then $\varphi \diamond \alpha$ is the KM-update of φ by α if and only if it satisfies:

(U1) $\varphi \diamond \alpha \vdash \alpha$
(U2) If $\varphi \vdash \alpha$, then $\varphi \diamond \alpha \equiv \varphi$.
(U3) If $\varphi \nvdash \bot$ and $\alpha \nvdash \bot$, then $\varphi \diamond \alpha \nvdash \bot$.
(U4) If $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$ then $\varphi_1 \diamond \alpha_1 \equiv \varphi_2 \diamond \alpha_2$.
(U5) $(\varphi \diamond \alpha) \wedge \beta \vdash \varphi \diamond (\alpha \wedge \beta)$
(U6) If $\varphi \diamond \alpha \vdash \beta$ and $\varphi \diamond \beta \vdash \alpha$, then $\varphi \diamond \alpha \equiv \varphi \diamond \beta$.
(U7) If φ is a complete formula, then $(\varphi \diamond \alpha) \wedge (\varphi \diamond \beta) \vdash \varphi \diamond (\alpha \vee \beta)$.
(U8) $(\varphi \vee \phi) \diamond \alpha \equiv (\varphi \diamond \alpha) \vee (\phi \diamond \alpha)$

When comparing the postulates of KM-update with those of the AGM framework (R1)-(R6), it is evident that there exist notable formal differences. Of particular significance is the fact that postulate R2 (vacuity) does not hold for KM-update. The relationship between KM-update and revision has been subject to further analysis by Becher [3] and others. Katsuno and Mendelzon characterized KM-update in terms of possible worlds:

Definition 2.9. *[21]* Let W be the set of all worlds. A *faithful assignment* is a function mapping each world ω to a partial preorder $\leq_\omega$ such that if $\omega \not\equiv \omega'$, then $\omega <_\omega \omega'$.

Observation 2.10. *[21]* $\diamond$ is an KM-update operator if and only if there exists a faithful assignment that maps each possible world ω to a partial preorder $\leq_\omega$ such that:

$$\|\varphi \diamond \alpha\| = \bigcup_{\omega \models \varphi} \min(\|\alpha\|, \leq_\omega)$$

Fig. 6 provides an example of KM-update in terms of possible worlds.

3. Iteration for KM-update

As previously stated, KM-update operators have been defined for all potential belief sets φ. At first glance, iteration does not necessitate special consideration as $(\varphi \diamond \alpha) \diamond \beta$ is well-defined. However, if changes in preferences are desired as a result of updating by α, a new operator $\blacklozenge$ must be defined to reflect these changes. The operator $\blacklozenge$ not only modifies the belief set φ, but also impacts the preferences for future updates. Similar to belief revision, a transition from belief sets to more complex structures, called epistemic states, is necessary. Epistemic states contain not only the belief set φ, but also the strategy for belief updates. The next step is to define epistemic states for iterated belief update, and thus, Definition 2.3 needs to be modified accordingly:

Definition 3.1 *(adapted from [11]).* An epistemic state Ψ is a tuple $\Psi = \langle \varphi, \diamond \rangle$, where $B(\Psi) = \varphi$ is a propositional formula that denotes the current beliefs of the agent in the epistemic state Ψ and $O(\Psi) = \diamond$ is a belief update operator defined as in Definition 2.8.

The properties (U1)-(U8) of an update operator $\diamond$ need to be adapted for epistemic states. The translation process is generally straightforward, similar to belief revision, with the exception of (U8). The unique aspect of (U8) is that it refers to different belief states that are updated by the same operator $\diamond$.

Definition 3.2. Let φ and α sentences of $\mathcal{L}$. Let $\diamond$ be an update operator defined as in Definition 2.8. Let $\Psi = \langle \varphi, \diamond \rangle$ be a belief state defined as in Definition 3.1. $\blacklozenge$ is an *update operator* for Ψ if and only if satisfies the following properties:

(1) $B(\Psi \blacklozenge \alpha) = \varphi \diamond \alpha$
(2) If $O(\Psi_1) = O(\Psi_2)$ then $O(\Psi_1 \blacklozenge \alpha) = O(\Psi_2 \blacklozenge \alpha)$ for all $\alpha \in \mathcal{L}$.

The second property implies that the operator $\blacklozenge$ will modify $\diamond$ independently of the belief set. With the definition of $\blacklozenge$, it is now possible to adapt the KM-update postulates accordingly.

Observation 3.3. Let $\blacklozenge$ be an update operator defined as in Definition 3.2. Then $\blacklozenge$ satisfies:

(U♦1) $B(\Psi \blacklozenge \alpha) \vdash \alpha$
(U♦2) If $B(\Psi) \vdash \alpha$, then $B(\Psi \blacklozenge \alpha) \equiv B(\Psi)$.
(U♦3) If $B(\Psi) \nvdash \bot$ and $\alpha \nvdash \bot$, then $B(\Psi \blacklozenge \alpha) \nvdash \bot$.
(U♦4) If $B(\Psi_1) \equiv B(\Psi_2)$, $O(\Psi_1) = O(\Psi_2)$ and $\alpha_1 \equiv \alpha_2$ then $B(\Psi_1 \blacklozenge \alpha_1) \equiv B(\Psi_2 \blacklozenge \alpha_2)$.
(U♦5) $B(\Psi \blacklozenge \alpha) \wedge \beta \vdash B(\Psi \blacklozenge (\alpha \wedge \beta))$
(U♦6) If $B(\Psi \blacklozenge \alpha) \vdash \beta$ and $B(\Psi \blacklozenge \beta) \vdash \alpha$, then $B(\Psi \blacklozenge \alpha) \equiv B(\Psi \blacklozenge \beta)$.
(U♦7) If $B(\Psi)$ is a complete formula, then $B(\Psi \blacklozenge \alpha) \wedge B(\Psi \blacklozenge \beta) \vdash B(\Psi \blacklozenge (\alpha \vee \beta))$.
(U♦8) If $O(\Psi_1) = O(\Psi_2) = O(\Psi_3)$, and $B(\Psi_1) \equiv B(\Psi_2) \vee B(\Psi_3)$, then $B(\Psi_1 \blacklozenge \alpha) \equiv B(\Psi_2 \blacklozenge \alpha) \vee B(\Psi_3 \blacklozenge \alpha)$.

Apart from the obvious modification of applying update to an epistemic state instead of a belief set, there are two differences from the original KM postulates. Similar to belief revision, postulate (U♦4) now requires not only that $B(\Psi_1) \equiv B(\Psi_2)$, but also that $O(\Psi_1) = O(\Psi_2)$, i.e., the corresponding KM-update operators of both epistemic states must coincide. The same applies to Postulate (U♦8).

To define iteration of update we need to adapt the (C1)-(C4) postulates:

(CU1) If $\alpha \vdash \mu$ then $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \equiv B(\Psi \blacklozenge \alpha)$.
(CU2) If $\alpha \vdash \neg\mu$, then $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \equiv B(\Psi \blacklozenge \alpha)$.
(CU3) If $B(\Psi \blacklozenge \alpha) \vdash \mu$, then $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \vdash \mu$.
(CU4) If $B(\Psi \blacklozenge \alpha) \nvdash \neg\mu$, then $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \nvdash \neg\mu$.

In regards to faithful assignment, the (CR) properties must be adapted to belief states by utilizing $\leq_{\{\Psi,\omega\}}$ instead of $\leq_\omega$. Here, $\leq_{\{\Psi,\omega\}}$ represents the total preorder centered at ω for the epistemic state Ψ, and $\leq_{\{\Psi\blacklozenge\alpha,\omega\}}$ represents the total preorder centered at ω after updating Ψ by α with the use of $\blacklozenge$.

(CRU1) If $\omega_1 \models \mu$ and $\omega_2 \models \mu$, then $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2 \Leftrightarrow \omega_1 \leq_{\{\Psi\blacklozenge\mu,\omega\}} \omega_2$.
(CRU2) If $\omega_1 \models \neg\mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2 \Leftrightarrow \omega_1 \leq_{\{\Psi\blacklozenge\mu,\omega\}} \omega_2$.
(CRU3) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 <_{\{\Psi,\omega\}} \omega_2$ implies $\omega_1 <_{\{\Psi\blacklozenge\mu,\omega\}} \omega_2$.
(CRU4) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2$ implies $\omega_1 \leq_{\{\Psi\blacklozenge\mu,\omega\}} \omega_2$.

(CRU1)-(CRU2) require that the order among μ-worlds and the order among the $\neg\mu$-worlds remains unchanged after update by μ in all of the preorders defined for each ω_i. In the same way, (CRU3) says that if a μ-world is strictly preferred to a $\neg\mu$-world, then that strict preference is maintained after updating by μ in all of the preorders defined for each ω_i and finally (CRU4) says that if a μ-world is weakly preferred to a $\neg\mu$-world, then that weak preference is maintained after revision by μ in all of the preorders defined for each ω_i.

The next theorems characterize the iteration of update operators. By abuse of notation we will use $\|\Psi\|$ to denote $\|B(\Psi)\|$.

Theorem 3.4. *An operator $\blacklozenge$ satisfies (U♦1)-(U♦8) if and only if there exists a faithful assignment that maps each possible world ω to a partial preorder $\leq_{\{\Psi,\omega\}}$ such that:*

$$\|\Psi \blacklozenge \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}}).$$

Theorem 3.5. *Let $\blacklozenge$ be an update operator and let f be a faithful assignment that maps each possible world ω to a partial preorder $\leq_{\{\Psi,\omega\}}$. Then $\blacklozenge$*

1. *satisfies (CU1) iff $\leq_{\{\Psi,\omega\}}$ satisfies (CRU1).*

2. satisfies (CU2) iff $\leq_{\{\Psi,\omega\}}$ satisfies (CRU2).
3. satisfies (CU3) iff $\leq_{\{\Psi,\omega\}}$ satisfies (CRU3).
4. satisfies (CU4) iff $\leq_{\{\Psi,\omega\}}$ satisfies (CRU4).

The corresponding *independence postulate* for iterated update is

(U-Ind) If $B(\Psi)$ is a complete formula and $B(\Psi \blacklozenge \alpha) \nvdash \neg\mu$, then $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \vdash \mu$.

which corresponds, in terms of preorders of possible worlds, to the following property:

(CRUInd) If $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$, then $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2 \Rightarrow \omega_1 <_{\{\Psi\blacklozenge\mu,\omega\}} \omega_2$.

Formally:

Theorem 3.6. *Let $\blacklozenge$ be an update operator and let f be a faithful assignment that maps each possible world ω to a partial preorder $\leq_{\{\Psi,\omega\}}$. Then $\blacklozenge$ satisfies (U-Ind) iff $\leq_{\{\Psi,\omega\}}$ satisfies (CRUInd).*

3.1. Update vs. iterated update

In this subsection, we compare the original update operators with the new family of iterated update operators. We begin by noting that all original update operators are defined for all potential belief sets, meaning that any KM-update operator allows iteration, as demonstrated in the following property:

Observation 3.7. Let $\diamond$ be an update operator defined as in Definition 2.8. Then $\diamond$ satisfies[2]:

(CU1') If $\alpha \vdash \mu$ then $(\varphi \diamond \mu) \diamond \alpha \equiv \varphi \diamond \alpha$
(CU2') If $\alpha \vdash \neg\mu$, then $(\varphi \diamond \mu) \diamond \alpha \equiv \varphi \diamond \alpha$
(CU3') If $\varphi \diamond \alpha \vdash \mu$, then $(\varphi \diamond \mu) \diamond \alpha \vdash \mu$
(CU4') If $\varphi \diamond \alpha \nvdash \neg\mu$, then $(\varphi \diamond \mu) \diamond \alpha \nvdash \neg\mu$

However, the iterated update operators defined make modifications to the preorder associated with each possible world, thus $B((\Psi \blacklozenge \alpha) \blacklozenge \beta)$ in general differs from $(B(\Psi) \diamond \alpha) \diamond \beta$. We will provide an example to illustrate this point:

Example 3.8. Let $\mathcal{P} = \{\alpha, \beta\}$ and the correspondent possible worlds $W = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ such that

$\|\{\alpha, \beta\}\| = \{\omega_1\}$
$\|\{\alpha, \neg\beta\}\| = \{\omega_2\}$
$\|\{\neg\alpha, \beta\}\| = \{\omega_3\}$
$\|\{\neg\alpha, \neg\beta\}\| = \{\omega_4\}$

Let $\diamond$ be a KM-update operator for belief sets and $\blacklozenge$ be an update operator for epistemic states satisfying (CU1)-(CU4) and (U-Ind). Let $B(\Psi) \equiv \neg\beta$, i.e., $\|B(\Psi)\| = \{\omega_2, \omega_4\}$. Consider the total preorders represented in Fig. 7 (left side) associated to $\diamond$ and Ψ. Then:

$\|B(\Psi) \diamond \alpha\| = \{\omega_2\}$.
$\|\Psi \blacklozenge \alpha\| = \{\omega_2\}$.
$\|(B(\Psi) \diamond \alpha) \diamond \beta\| = \{\omega_1, \omega_3\}$.
$\|(\Psi \blacklozenge \alpha) \blacklozenge \beta\| = \{\omega_1\}$.

Explanation: Fig. 7 (left) shows the distribution of the possible worlds for $B(\Psi)$. In the first case (top right, marked with $\diamond$), $\|(B(\Psi) \diamond \alpha) \diamond \beta\|$ selects the closest β-worlds based on the order defined by $\diamond$ for $\|B(\Psi) \diamond \alpha\| = \{\omega_2\}$. Then $\|(B(\Psi) \diamond \alpha) \diamond \beta\| = \{\omega_1, \omega_3\}$. In the second case (bottom right, marked with $\blacklozenge$), ω_1 is improved in $\leq_{\{\Psi\blacklozenge\alpha,\omega_2\}}$, since it is an α-world. Thus, $\|(\Psi \blacklozenge \alpha) \blacklozenge \beta\| = \{\omega_1\}$.

4. Some iterated updated operators

In this section, we will define different iterated update operators, similar to belief revision. We will characterize natural and lexicographic update, as well as other desirable properties that can be implemented. First, we need to define and adapt

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[2] We only adapted (CU1)-(CU4) from epistemic states to belief sets.

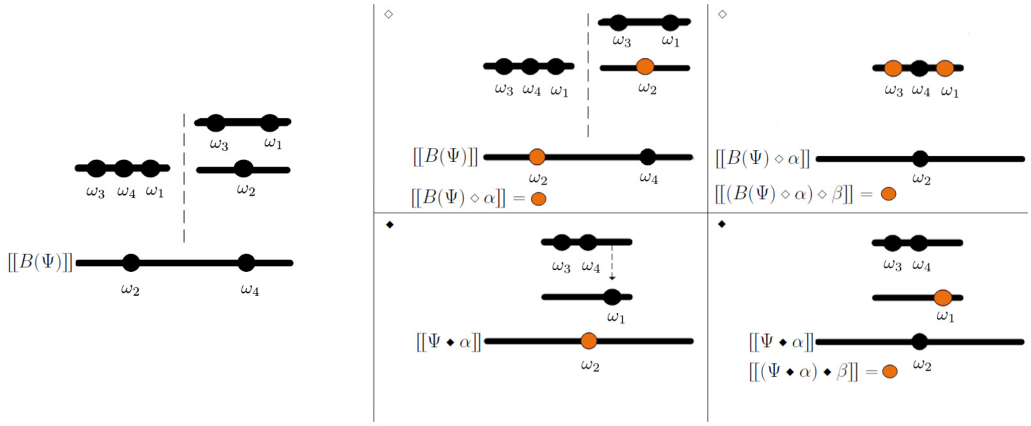

Fig. 7. Example 3.8.

the postulates, and then find the corresponding property in the semantic level. We will start by defining the *natural update* operator:

Definition 4.1. Let Ψ be a belief state defined as in Definition 3.1 and $\blacklozenge$ an update operator for Ψ defined as in Definition 3.2. Then $\blacklozenge$ is a *natural update* operator if also satisfies (CU1)-(CU4) and

(U-Nat) If φ is a complete formula and $\Psi\blacklozenge\mu\vdash\neg\alpha$, then $(\Psi\blacklozenge\mu)\blacklozenge\alpha\equiv\Psi\blacklozenge\alpha$.

The following theorem provides the semantic characterization of natural update:

Theorem 4.2. *Let $\blacklozenge$ be an update operator. Let f be its corresponding faithful assignment that maps each possible world ω to a partial preorder $\le_{\{\Psi,\omega\}}$ such that:*

$$\|\Psi\blacklozenge\alpha\|=\bigcup_{\omega\models B(\Psi)}\min(\|\alpha\|,\le_{\{\Psi,\omega\}}).$$

Then $\blacklozenge$ satisfies (U-Nat) iff the faithful assignment satisfies

(CRUNat) *If $\omega_1,\omega_2\models\neg B(\Psi\blacklozenge\mu)$, then $\omega_1\le_{\{\Psi,\omega\}}\omega_2\Leftrightarrow\omega_1\le_{\{\Psi\blacklozenge\mu,\omega\}}\omega_2$.*

This operation is conservative in the sense that it only makes the minimal changes of the preorder that are needed to accept the input, where the relative orders between the possible worlds that are not included in $\Psi\blacklozenge\alpha$ remain unchanged.

The second family of update operators that we characterize is *lexicographic update:*

Definition 4.3. Let Ψ be a belief state defined as in Definition 3.1 and $\blacklozenge$ an update operator for Ψ defined as in Definition 3.2. Then $\blacklozenge$ is a *lexicographic update* operator if also satisfies (CU1)-(CU4) and

(U-Lex) If $\varphi\vdash\neg\alpha$ and $\alpha\nvdash\neg\mu$, then $(\Psi\blacklozenge\mu)\blacklozenge\alpha\vdash\mu$.

The following theorem provides the semantic characterization of lexicographic update:

Theorem 4.4. *Let $\blacklozenge$ be an update operator. Let f be its corresponding faithful assignment that maps each possible world ω to a partial preorder $\le_{\{\Psi,\omega\}}$ such that:*

$$\|\Psi\blacklozenge\alpha\|=\bigcup_{\omega\models B(\Psi)}\min(\|\alpha\|,\le_{\{\Psi,\omega\}}).$$

Then $\blacklozenge$ satisfies (U-Lex) iff the faithful assignment satisfies

(CRULex) *If $\omega_1\models\mu$ and $\omega_2\models\neg\mu$, then $\omega_1<_{\{\Psi\blacklozenge\mu,\omega\}}\omega_2$ for all $\omega\neq\omega_2$.*

Lexicographic update induces more radical changes in the preorders compared to natural revision, as it is less conservative. It is based on the idea that the new beliefs are considered more important than the previous ones. When revising with respect to μ, this operation rearranges the preorder by placing the μ-worlds at the bottom (while preserving their relative order) and the $\neg\mu$-worlds at the top (while preserving their relative order) in all the $\le_{\{\Psi\blacklozenge\mu,\omega\}}$ relations.

Additional postulates can be added to specify new operators, particularly those that restrict the extent of changes. For instance, Konieczny and Pino Pérez introduced the concept of "soft update" in the context of improvement operators, as described in their work [27], which we can adapt for update operators.

(CRUSoft) If $\omega_1 \models \alpha, \omega_2 \models \neg\alpha$ then $\omega_2 <_{\{\Psi,\omega\}} \omega_1 \Rightarrow \omega_2 \leq_{\{\Psi\blacklozenge\alpha,\omega\}} \omega_1$.

This postulate ensures that the changes made in the preorders regarding the relative order between the μ-worlds and the $\neg\mu$-worlds are minimal. This property is desirable in many applications, as it ensures that the updated beliefs do not change dramatically from the original beliefs, unless there is strong evidence to support such a change.

5. Conclusions, discussion and future works

We have defined the basis for iteration of update. This approach leads to a different outcome compared to applying the original update operator repeatedly on the updated belief set, as showed in Example 3.8. The postulates (CRU1)-(CRU4), (CRUInd), (CRUNat) and (CRULex) establish a broad range of possible update operators. This opens up the possibility to define a range of update functions, each with different properties and behaviors, depending on the postulates used to define them. Further research can explore the properties of these different update functions and compare their behaviors.

There are other approaches for update (for a good overview see [18]) that were not explored in this paper. There are also similar works in the literature, in particular in the area of logic programming (sequences of updates) [2,12,36], and in iterative alignment of ontologies [9] and dynamics of ontologies [36]. Gabriele Kern-Isberner has several works in the area of iteration of conditionals in nonmonotonic reasoning and belief revision. In particular, in [23] revision operators have been proposed for multiple revision by sets of propositions and by sets of conditionals. The principle of conditional preservation proposed by Darwiche and Pearl is elaborated in detail as an invariance property, which is further developed in [24]. Iterated revision and update is studied in [25]. In [26], the framework of c-revisions is adapted to multiple propositional revision.

This paper focused on defining iteration for KM-update based on Darwiche and Pearl postulates. However, due to the differences between update and revision, further research is needed to explore new proper iterated update functions. A comparison between our proposal and related works in the areas of logic programming and ontology alignment will be studied in future works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

As mentioned in the introduction, an earlier draft of this paper was presented at the Gabriele Kern-Isberner Festschrift, where we received valuable feedback, suggestions, and corrections from several colleagues to enhance the paper. We will also like to thank the anonymous referees for their fruitful comments on the contents.

This paper was partially supported by FCT-Fundação para a Ciência e a Tecnologia, Portugal through project PTDC/CCI-COM/4464/2020. E.F. was partially supported by FCT through project UIDB/04516/2020 (NOVA LINCS).

Appendix A. Proofs

Proof of Observation 3.3. Due to property (1) of Definition 3.2 it follows that $B(\Psi\blacklozenge\alpha) = \varphi \diamond \alpha$ and therefore (U$\blacklozenge$1)- (U$\blacklozenge$3) and (U$\blacklozenge$5) - (U$\blacklozenge$7) follow trivially. For (U$\blacklozenge$4) and (U$\blacklozenge$8), we note that if $O(\Psi_1) = O(\Psi_2)$ then $O(\Psi_1 \blacklozenge \alpha) = O(\Psi_2 \blacklozenge \alpha)$ for all $\alpha \in \mathcal{L}$ and the rest is trivial. ■

Proof of Theorem 3.4. The proof follows the proofs provided in Theorem 3.1 in [21], Theorem 3.3 in [20] and Theorem 9 in [11].

($\Leftarrow$) It follows from Observation 3.3.

($\Rightarrow$) We will use the following convention: given a belief state Ψ, for all α, $B(\Psi_\alpha) \equiv \alpha$ and $O(\Psi_\alpha) = O(\Psi)$.
For any worlds ω' and ω'', we define a relation $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ as $\omega' \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$ if and only if $\omega' = \omega_i$ or $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega',\omega''\}}\| = \{\omega'\}$.
We need to show (1) that $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ is a partial preorder, (2) that $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ is faithful, (3) that $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$ and (4) that $\|\Psi \blacklozenge \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$.

(1) We first show that $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ is a partial preorder:

Reflexivity: By (U$\blacklozenge$1) and (U$\blacklozenge$3) it follows that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\omega'}\| = \{\omega'\}$.

Transitivity: Assume that $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega'$ and $\omega' \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$. We must show that $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$. The proof is trivial by definition of $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ in the case when $\omega = \omega_i$, so assume that $\omega \neq \omega_i$. Let us first prove that $\omega' \neq \omega_i$ and $\omega'' \neq \omega_i$. Assume toward a contradiction that $\omega' = \omega_i$. Since $\omega \neq \omega_i$ and $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega'$, we get that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \{\omega\}$ by definition of $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$. Yet from (U$\blacklozenge$2) and since $\omega' = \omega_i$, we have that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \|\Psi_{\alpha_{\omega'}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \{\omega'\} = \{\omega_i\}$. We got that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \{\omega\}$ and $\|\Psi_{\alpha_{\omega_i}}\| \blacklozenge \alpha_{\{\omega,\omega'\}} = \{\omega_i\}$, which contradicts $\omega \neq \omega_i$. Hence, $\omega' \neq \omega_i$. The same arguments can be used to prove that $\omega'' \neq \omega_i$.
At this point, we know that $\{\omega, \omega', \omega''\} \cap \{\omega_i\} = \emptyset$. When $\omega = \omega'$ or $\omega' = \omega''$, we trivially get that $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$; and when $\omega = \omega''$, we also get that $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$. Then, assume that ω, ω' and ω'' are pairwise different, i.e., $\omega \neq \omega'$, $\omega' \neq \omega''$ and $\omega \neq \omega''$.
By definition of $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ and since $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega'$ and $\omega' \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$, we know that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \{\omega\}$ and $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega',\omega''\}}\| = \{\omega'\}$. By (U$\blacklozenge$5), $B(\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}) \wedge \alpha_{\{\omega,\omega'\}} \vdash B(\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}})$. Yet $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}}\| = \{\omega\}$, which means that $\omega' \notin \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\|$. Similarly, by (U$\blacklozenge$5), $B(\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}) \wedge \alpha_{\{\omega',\omega''\}} \vdash B(\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega',\omega''\}})$. Yet $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega',\omega''\}}\| = \{\omega'\}$, which means that $\omega'' \notin \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\|$. Hence, by (U$\blacklozenge$1), we get that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\| \subseteq \{\omega\}$. Yet $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\omega}\| = \{\omega\}$ by (U$\blacklozenge$1) and (U$\blacklozenge$3).
By (U$\blacklozenge$6), since $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\| \subseteq \{\omega\}$ and $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\omega}\| \subseteq \{\omega, \omega', \omega''\}$, we get that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\| = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\omega}\| = \{\omega\}$.
By (U$\blacklozenge$6) again, since $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega''\}}\| \subseteq \{\omega, \omega', \omega''\}$ (by (U$\blacklozenge$1)) and $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\| \subseteq \{\omega, \omega''\}$, we get that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega''\}}\| = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega',\omega''\}}\| = \{\omega\}$.
This shows that $\omega \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega''$ and concludes the proof that $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ satisfies (transitivity).

(2) It follows by (U$\blacklozenge$2) that the assignment mapping each possible world ω_i to $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ is faithful.

(3) We now intend to show that for each formula α and each world $\omega_i \in W$, $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$. Assume first that $\omega_i \in \|\alpha\|$. Since $\omega_i <_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega^j$ for each $\omega^j \in \|\alpha\|$, we get that $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) = \{\omega_i\}$. And by (U$\blacklozenge$2), $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\| = \{\omega_i\}$. So, $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$. So assume now in the rest of the proof that $\omega_i \notin \|\alpha\|$.
Let us first show that $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) \subseteq \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$. So let $\omega \in \min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}})$, we must show that $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$.
Let us write the set of models of α as $\|\alpha\| = \{\omega^1, \ldots, \omega^k\}$ $(k \geq 1)$. Since $\omega \in \|\alpha\|$, we can also write $\|\alpha\| = \{\omega, \omega^1\} \cup \ldots \cup \{\omega, \omega^k\}$. Let $\omega^j \in \|\alpha\|$ and let us prove that $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\|$.
By (U$\blacklozenge$3) $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\| \neq \emptyset$ and in the case when $\omega^j \neq \omega$, since $\omega \in \|\alpha\|$, we know that $\omega^j \nleq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega$ (since by construction, $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$ is antisymmetric). Thus $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\| \neq \{\omega^j\}$ by definition of $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$.
We have by (U$\blacklozenge$1) that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\| \subseteq \{\omega, \omega^j\}$, that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\| \neq \emptyset$, and if $\omega^j \neq \omega$ that $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\| \neq \{\omega^j\}$. Hence, $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\|$.
We have proved that $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega^j\}}\|$ for each $\omega^j \in \|\alpha\|$. Thus $\omega \in \|(\alpha_{\omega_i} \blacklozenge \alpha_{\{\omega,\omega^1\}}) \wedge \ldots \wedge (\alpha_{\omega_i} \blacklozenge \alpha_{\{\omega,\omega^k\}})\|$. Using (U$\blacklozenge$7) multiple times, we get that $\omega \in \|(\alpha_{\omega_i} \blacklozenge (\alpha_{\{\omega,\omega^1\}} \vee \ldots \vee \alpha_{\{\omega,\omega^k\}}))\|$, that is, by (U$\blacklozenge$4), $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$.
This shows that $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) \subseteq \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$.
Let us now show the other inclusion, i.e., $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\| \subseteq \min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}})$. If $\alpha \vdash \bot$, then by (U$\blacklozenge$1) $\|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\omega^j}\| = \emptyset$. Let $\alpha \nvdash \bot$. Let $\omega \in \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$, and assume toward a contradiction that $\omega \notin \min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}})$. Then there exists a world $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}))$ such that $\omega' <_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega$. By (U$\blacklozenge$5), $(\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha) \wedge \alpha_{\{\omega,\omega'\}} \vdash \alpha_{\omega_i} \blacklozenge \alpha_{\{\omega,\omega'\}}$. Yet $\omega \nleq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}} \omega'$, so $\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha_{\{\omega,\omega'\}} = \{\omega'\}$ (by definition of $\leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}$). Hence, $\omega \notin \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$, which leads to a contradiction.

(4) We got for each formula α and each world $\omega_i \in W$, $\min(\|\alpha\|, \leq_{\{\Psi_{\alpha_{\omega_i}},\omega_i\}}) = \|\Psi_{\alpha_{\omega_i}} \blacklozenge \alpha\|$. Since $O(\Psi) = O(\Psi_{\alpha_{\omega_i}})$ for each world $\omega_i \in W$ and $B(\Psi) \equiv \bigvee_{\omega \models B(\Psi)} B(\Psi_{\alpha_\omega})$ we can use (U$\blacklozenge$8) to obtain

$$\|\Psi \blacklozenge \alpha\| = \|B(\Psi \blacklozenge \alpha)\| = \|\bigvee_{\omega \models B(\Psi)} B(\Psi_{\alpha_\omega} \blacklozenge \alpha)\| =$$

$$= \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi_{\alpha_\omega},\omega\}}) = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}}),$$

which concludes the proof. ■

Proof of Theorem 3.5.[3]

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[3] Adapted from [11, Proof of Theorem 13].

(CU1) ⇔ (CRU1)

($\Rightarrow$) Assume (CU1) holds and let $\omega_1 \models \mu$ and $\omega_2 \models \mu$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$ Then $\alpha \vdash \mu$ and due to (CU1) $B((\Psi \bullet \mu) \bullet \alpha) \equiv B(\Psi \bullet \alpha)$. Hence $\min(\{\omega_1,\omega_2\}, \leq_{\{\Psi,\omega\}}) = \min(\{\omega_1,\omega_2\}, \leq_{\{\Psi\bullet\mu,\omega\}})$ from which it follows that $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2 \Leftrightarrow \omega_1 \leq_{\{\Psi\bullet\mu,\omega\}} \omega_2$.
($\Leftarrow$) Assume (CRU1) holds and let $\alpha \vdash \mu$. We want to show that $B((\Psi \bullet \mu) \bullet \alpha) \equiv B(\Psi \bullet \alpha)$. Condition $(CRU1)$ implies that $\leq_{\{\Psi,\omega\}}$ and $\leq_{\{\Psi\bullet\mu,\omega\}}$ are equivalent for all $\omega' \in \|\alpha\|$ since $\|\alpha\| \subseteq \|\mu\|$. Hence:

$$\|\Psi \bullet \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$$

$$\|\Psi \bullet \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\omega\}})$$

$$\|\Psi \bullet \alpha\| = \|(\Psi \bullet \mu) \bullet \alpha)\|$$

(CU2) ⇔ (CRU2). The proof is symmetric with the one above.

(CU3) ⇔ (CRU3)

($\Rightarrow$) Assume (CU3) holds and let $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$ and $\omega_1 <_{\{\Psi,\omega\}} \omega_2$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$. Let Ψ' such that $B(\Psi') \equiv \alpha_\omega$ and $O(\Psi') = O(\Psi)$, from which it follows that $\leq_{\{\Psi',\omega\}} = \leq_{\{\Psi,\omega\}}$. $\|\Psi' \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi',\omega\}}) = \min(\|\alpha\|, \leq_{\{\Psi,\omega\}}) = \{\omega_1\}$, from which it follows that $B(\Psi' \bullet \alpha) \vdash \mu$. By (CU3) $B((\Psi' \bullet \mu) \bullet \alpha) \vdash \mu$, from which it follows that $\|(\Psi' \bullet \mu) \bullet \alpha)\| = \min(\|\alpha\|, \leq_{\{\Psi'\bullet\mu,\omega\}}) \subseteq \|\mu\|$, hence $\|(\Psi' \bullet \mu) \bullet \alpha)\| = \{\omega_1\}$. Then $\omega_1 <_{\{\Psi'\bullet\mu,\omega\}} \omega_2$, hence $\omega_1 <_{\{\Psi\bullet\mu,\omega\}} \omega_2$.
($\Leftarrow$) Assume (CRU3) holds and let $B(\Psi \bullet \alpha) \vdash \mu$. From $\|\Psi \bullet \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$ it follows that for all $\omega \models B(\Psi)$ if $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$ implies that $\omega' \models \alpha \wedge \mu$ and for all $\omega'' \models \alpha \wedge \neg\mu$ it follows that $\omega' <_{\{\Psi,\omega\}} \omega''$.(CRU3) yields $\omega' <_{\{\Psi\bullet\mu,\omega\}} \omega''$ for all $\omega'' \models \alpha \wedge \neg\mu$, hence $\omega'' \notin \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\omega\}})$. Since this is valid for all $\omega \models B(\Psi)$ we can conclude that $B((\Psi \bullet \mu) \bullet \alpha) \vdash \mu$.

(CU4) ⇔ (CRU4)

(CU4) If $B(\Psi \bullet \alpha) \nvdash \neg\mu$, then $B((\Psi \bullet \mu) \bullet \alpha) \nvdash \neg\mu$

($\Rightarrow$) Assume (CU4) holds and let $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$ and $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$. Let Ψ' such that $B(\Psi') \equiv \alpha_\omega$ and $O(\Psi') = O(\Psi)$, from which it follows that $\leq_{\{\Psi',\omega\}} = \leq_{\{\Psi,\omega\}}$. Then $\omega_1 \in \|\Psi' \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi',\omega\}}) = \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$, from which it follows that $B(\Psi' \bullet \alpha) \nvdash \neg\mu$. By (CU4) $B((\Psi' \bullet \mu) \bullet \alpha) \nvdash \neg\mu$, from which it follows that $\|(\Psi' \bullet \mu) \bullet \alpha\| \cap \|\mu\| \neq \emptyset$, i.e., $\min(\|\alpha\|, \leq_{\{\Psi'\bullet\mu,\omega\}}) \cap \|\mu\| \neq \emptyset$, hence $\omega_1 \in \|(\Psi' \bullet \mu) \bullet \alpha\|$, from which we can conclude that $\omega_1 \leq_{\{\Psi'\bullet\mu,\omega\}} \omega_2$ and hence that $\omega_1 \leq_{\{\Psi\bullet\mu,\omega\}} \omega_2$.
($\Leftarrow$) Assume (CRU4) holds and let $B(\Psi \bullet \alpha) \nvdash \neg\mu$. From $\|\Psi \bullet \alpha\| = \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$ it follows that there exists some υ such that $\upsilon \models B(\Psi)$ and for some $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi,\upsilon\}})$ it holds that $\omega' \models \alpha \wedge \mu$ and for all $\omega'' \models \alpha \wedge \neg\mu$ it follows that $\omega' \leq_{\{\Psi,\upsilon\}} \omega''$. (CRU4) yields $\omega' \leq_{\{\Psi\bullet\mu,\upsilon\}} \omega''$ for all $\omega'' \models \alpha \wedge \neg\mu$, hence $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\upsilon\}})$, from which it follows that $\omega' \in \bigcup_{\omega \models B(\Psi)} \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\omega\}})$. Hence $B((\Psi \bullet \mu) \bullet \alpha) \nvdash \neg\mu$. ■

Proof of Theorem 3.6. (U-Ind) ⇔ (CRUInd)

($\Rightarrow$) Assume (U-Ind) holds and let $\omega_1 \models \mu$, $\omega_2 \models \neg\mu$ and $\omega_1 \leq_{\{\Psi,\omega\}} \omega_2$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$. Let Ψ' such that $B(\Psi') \equiv \alpha_\omega$ and $O(\Psi') = O(\Psi)$, from which it follows that $\leq_{\{\Psi',\omega\}} = \leq_{\{\Psi,\omega\}}$. Then $\omega_1 \in \|\Psi' \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi',\omega\}}) = \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$, from which it follows that $B(\Psi' \bullet \alpha) \nvdash \neg\mu$. By (U-Ind) $B((\Psi' \bullet \mu) \bullet \alpha) \vdash \mu$, from which it follows that $\|(\Psi' \bullet \mu) \bullet \alpha)\| = \min(\|\alpha\|, \leq_{\{\Psi'\bullet\mu,\omega\}}) \subseteq \|\mu\|$, hence $\|(\Psi' \bullet \mu) \bullet \alpha)\| = \{\omega_1\}$. Then $\omega_1 <_{\{\Psi'\bullet\mu,\omega\}} \omega_2$, hence $\omega_1 <_{\{\Psi\bullet\mu,\omega\}} \omega_2$.
($\Leftarrow$) Assume (CRInd) holds and let $B(\Psi)$ be a complete formula such that $B(\Psi \bullet \alpha) \nvdash \neg\mu$. Since $B(\Psi)$ is complete, there exists some υ such that $B(\Psi) \equiv \alpha_\upsilon$. Then $\|\Psi \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi,\upsilon\}})$. Due to $B(\Psi \bullet \alpha) \nvdash \neg\mu$ it follows that there exists $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi,\omega\}})$ such that $\omega' \models \alpha \wedge \mu$ and for all ω'' such that $\omega'' \models \alpha \wedge \neg\mu$ it follows that $\omega' \leq_{\{\Psi,\upsilon\}} \omega''$. (CRUInd) yields $\omega' <_{\{\Psi\bullet\mu,\upsilon\}} \omega''$ for all $\omega'' \models \alpha \wedge \neg\mu$, hence $\omega' \in \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\upsilon\}})$ and there not exists ω'' such that $\omega'' \models \alpha \wedge \neg\mu$ and $\omega'' \in \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\upsilon\}})$. Hence $B((\Psi \bullet \mu) \bullet \alpha) \vdash \mu$. ■

Proof of Theorem 4.2. (U-Nat) ⇔ (CRUNat)

($\Rightarrow$) Assume (U-Nat) holds and let $\omega_1, \omega_2 \models \neg(B(\Psi \bullet \mu))$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$. Then $\alpha \vdash \neg B(\Psi \bullet \mu)$ from which it follows that $B(\Psi \bullet \mu) \vdash \neg\alpha$ and (by U-Nat) $B((\Psi \bullet \mu) \bullet \alpha) \equiv B(\Psi \bullet \alpha)$. Since $B(\Psi)$ is a complete formula, there exists υ such that $\alpha_\upsilon \equiv B(\Psi)$. Hence $\min(\{\omega_1,\omega_2\}, \leq_{\{\Psi,\upsilon\}}) = \min(\{\omega_1,\omega_2\}, \leq_{\{\Psi\bullet\mu,\upsilon\}})$ from which it follows that $\omega_1 \leq_{\{\Psi,\upsilon\}} \omega_2 \Leftrightarrow \omega_1 \leq_{\{\Psi\bullet\mu,\upsilon\}} \omega_2$.
($\Leftarrow$) Assume (CRUNat) holds and let $B(\Psi)$ be a complete formula such that $B(\Psi \bullet \mu) \vdash \neg\alpha$. Since $B(\Psi)$ is complete there exists υ such that $B(\Psi) \equiv \alpha_\upsilon$. Then $\|\Psi \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi,\upsilon\}})$. We have that $\|\alpha\| \models \neg(B(\Psi \bullet \mu))$. Condition (CRUNat) implies that $\leq_{\{\Psi,\upsilon\}}$ and $\leq_{\{\Psi\bullet\mu,\upsilon\}}$ are equivalent for all $\omega' \in \|\alpha\|$. Hence:

$$\|\Psi \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi,\upsilon\}})$$

$$\|\Psi \bullet \alpha\| = \min(\|\alpha\|, \leq_{\{\Psi\bullet\mu,\upsilon\}})$$

$$\|\Psi \bullet \alpha\| = \|(\Psi \bullet \mu) \bullet \alpha)\| \quad \blacksquare$$

Proof of Theorem 4.4. (U-Lex) $\Leftrightarrow$ (CRULex)

($\Rightarrow$) Assume (U-Lex) holds and let $\omega_1 \models \mu$ and $\omega_2 \models \neg\mu$. Let $\alpha \equiv \alpha_{\{\omega_1,\omega_2\}}$. Then $\alpha \nvdash \neg\mu$. Let $B(\Psi)$ be a complete formula such that $B(\Psi) \vdash \neg\alpha$. Then $B(\Psi) \not\equiv \alpha_{\omega_2}$. Then it follows by (U-Lex) that $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \vdash \mu$. Since $B(\Psi)$ is complete there exists $\upsilon \neq \omega_2$ such that $B(\Psi) \equiv \alpha_\upsilon$. Then $\|(\Psi \blacklozenge \mu) \blacklozenge \alpha)\| = \min(\|\alpha\|, \leq_{\{\Psi\blacklozenge\mu,\upsilon\}})$. Since $\|(\Psi \blacklozenge \mu) \blacklozenge \alpha)\| \subseteq \|\mu\|$ it follows that $\min(\|\alpha\|, \leq_{\{\Psi\blacklozenge\mu,\upsilon\}}) = \{\omega_1\}$. Hence $\omega_1 <_{\{\Psi\blacklozenge\mu,\upsilon\}} \omega_2$.
($\Leftarrow$) Assume (CRULex) holds and let $B(\Psi)$, α and μ such that $B(\Psi) \vdash \neg\alpha$ and $\alpha \nvdash \neg\mu$. $B(\Psi) \equiv \alpha_{\{\omega_1,\ldots,\omega_n\}}$, for $\omega_1, \ldots, \omega_n \in W$. Then $\{\omega_1, \ldots, \omega_n\} \subseteq \|\neg\alpha\|$. Then $\omega_i \notin \min(\|\alpha\|, \leq_{\{\Psi\blacklozenge\mu,\omega_i\}})$, for $i = 1, \ldots, n$. It follows from (CRULex) that $\omega_j <_{\{\Psi\blacklozenge\mu,\omega_i\}} \omega_k$ for all $\omega_j \models \mu$ and $\omega_k \models \neg\mu$, $k \neq i$ and $i = 1, \ldots, n$. Then $\min(\|\alpha\|, \leq_{\{\Psi\blacklozenge\mu,\omega_i\}}) \cap \|\neg\mu\| = \emptyset$. Hence $B((\Psi \blacklozenge \mu) \blacklozenge \alpha) \vdash \mu$. ■

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