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ABSTRACT

New first integral is found in the collision-free Tonks-Langmuir model. The integral has a clear physical interpretation: the weighted mean inverse kinetic energy of ions, evaluated in the quasineutral approximation, equals $(kT_e/2)^{-1}$ at all points in space. This feature is also present in the full (not relying on the assumption of quasineutrality) model: for small values of the Debye length, the weighted mean inverse kinetic energy is with good accuracy equal to $(kT_e/2)^{-1}$ in the entire region of quasineutral plasma, including in the vicinity of the space-charge sheath. These results constitute a mathematical proof of the kinetic Bohm criterion and provide a new look at the problem, which has been discussed for several decades. In particular, these results show that the much-debated problem of divergence for slow ions stems from a misinterpretation. Moreover, these results explain why no unique form of kinetic Bohm criterion, modified with the account of ionization and/or collisional and/or geometrical effects in the sheath, has emerged: it cannot be postulated in a nonarbitrary way since there is simply no definite value of the inverse mean kinetic energy with which the ions enter the sheath, if these effects are non-negligible.

I. INTRODUCTION

At very low pressures, there is a limiting regime in which ions created at different locations inside the discharge column fall without collisions to the wall.[1] A model describing this region, which was proposed in a seminal paper by Tonks and Langmuir,[2] is one of the classic models of gas discharge physics (see, for example, Refs. 1, 3, and 4).

Apart from being of considerable interest by itself, this model traditionally has served as a test case for studies of the kinetic Bohm criterion. Although the kinetic Bohm criterion was proposed long ago,[5] the debate about its interpretation and validity is still ongoing.[6–14] In particular, the authors[6,9,10] criticized the kinetic Bohm criterion on the grounds of the divergence for slow ions and proposed different versions of the kinetic Bohm criterion, which, in particular, take into account collisions and/or ionization and/or geometrical effects in the sheath.

In this work, a new first integral of the Tonks-Langmuir model is reported, and it is shown that under appropriate conditions, this integral is reduced to the classic kinetic Bohm criterion. This result constitutes a mathematical proof of the classic kinetic Bohm criterion for collision- and ionization-free one-dimensional space-charge sheaths and provides a new look at the problem. In particular, this result shows that the much-debated problem of divergence for slow ions stems from a misinterpretation. It also explains why no unique form of kinetic Bohm criterion, modified with the account of ionization and/or collisional and/or geometrical effects in the sheath, has emerged: it cannot be postulated in a nonarbitrary way since there is simply no definite value of the inverse mean kinetic energy with which the ions enter the sheath, if these effects are non-negligible.

The outline of this paper is as follows: the collision-free Tonks-Langmuir model is briefly described in Sec. II. The first integral of this model is derived and confirmed by numerical calculations in Sec. III. The connection of the results to the kinetic Bohm criterion is discussed in Sec. IV. Conclusions are formulated in Sec. V. In the Appendix, a derivation of the kinetic Bohm criterion is given that is based on the investigation of asymptotic behavior, at large distances from the surface, of solutions of the Poisson equation in the sheath, similar to the derivation of the fluid Bohm criterion given in the original paper.[15]

II. THE TONKS-LANGMUIR MODEL OF THE COLLISION-FREE PLANE GLOW DISCHARGE COLUMN

The collision-free Tonks-Langmuir kinetic model[2] is well known (see, for example, Refs. 1, 3, and 4) and may be briefly described as follows: let us consider a collisionless positive column of a plane glow discharge, enclosed by two parallel absorbing insulating walls, and introduce the y-axis directed from the plane of symmetry of the discharge to the wall. The distribution of the electrostatic potential φ in a cross section of the discharge is governed by the Poisson equation

$$\varepsilon_0 \frac{d^2\varphi}{dy^2} = -e(n_i - n_e), \tag{1}$$

where n_i and n_e are the ion and electron densities. The boundary conditions at the plane of symmetry are

$$y = 0: \quad \varphi = 0, \ \frac{d\varphi}{dy} = 0. \tag{2}$$

The distribution of the electrons is related to the electrostatic potential through the Boltzmann distribution

$$n_e = n_{e0} \exp \frac{e\varphi}{kT_e}, \tag{3}$$

where $n_{e0} = n_e(0)$ is the electron density at the plane of symmetry (a quantity that is governed by the discharge current and can therefore be treated as a known parameter) and T_e is the electron temperature (a known parameter).

It is assumed that the temperature of neutral atoms is much lower than T_e. Then, the thermal energy of the atoms may be neglected, and so the ions are created, through ionization by the electron impact, with zero speed, after which they move, without collisions, to the wall under the effect of the electric field. The total energy of an ion is conserved, and the ions generated at a point z, when they have reached a point y, will have the velocity

$$v(y,z) = \sqrt{\frac{2e}{m}\left[\varphi(z) - \varphi(y)\right]}, \tag{4}$$

where m is the mass of an ion.

The number of the ions generated in the layer $z \le y \le z + dz$ per unit time and unit area (i.e., the density of the ion flux generated in this layer) is $w(z)\,dz$, where w is the ionization rate. When the ions generated in this layer have reached point y, their density is $w(z)\,dz/v(y,z)$. The total density of the ions at a point y is

$$n_i(y) = \int_0^y \frac{w(z)}{v(y,z)}\,dz. \tag{5}$$

Note that this expression may be rewritten as

$$n_i(y) = \int_0^{v_{\max}} \frac{mw(z)}{eE_y(z)}\,dv, \tag{6}$$

where $E_y(y) = -d\varphi(y)/dy$ is the y-projection of the electric field and $v_{\max} = \sqrt{-2e\varphi(y)/m}$ is the maximum speed of the ions at a given position y. It follows that the velocity distribution function of the ions may be written as

$$f(v,y) = \frac{mw(z)}{eE_y(z)}. \tag{7}$$

The density of the ion and electron fluxes to the (floating) wall is equal, which gives the boundary condition

$$y = l: \ \int_0^y w(z)\,dz = \frac{1}{4}\sqrt{\frac{8kT_e}{\pi m_e}} n_{e0} \exp \frac{e\varphi(y)}{kT_e}, \tag{8}$$

where l is the half-width of the discharge column.

The ionization rate may be written as $w = \nu_i n_e$, where ν_i is the ionization frequency per electron (a parameter to be determined). Note that ν_i may be expressed in terms of k_i the ionization rate constant and n_a the density of neutral atoms: $\nu_i = k_i n_a$.

Equation (1), supplemented with expressions (3) and (5) and boundary conditions (2) and (8), represents a closed problem governing the function $\varphi(y)$ and unknown parameter ν_i. This problem may be written in a dimensionless form as

$$\alpha \frac{d^2\Phi}{d\xi^2} = e^{\Phi(\xi)} - \int_0^\xi \frac{e^{\Phi(\zeta)}}{\sqrt{\Phi(\zeta) - \Phi(\xi)}}\,d\zeta, \tag{9}$$

$$\xi = 0: \quad \Phi = 0, \ \frac{d\Phi}{d\xi} = 0, \tag{10}$$

$$\xi = \xi_w: \ \int_0^\xi e^{\Phi(\zeta)}\,d\zeta = \sqrt{\frac{m}{4\pi m_e}} e^{\Phi(\xi)}, \tag{11}$$

where

$$\xi = \frac{y}{l_1}, \quad \Phi = \frac{e\varphi}{kT_e}, \quad l_1 = \frac{\sqrt{2}v_s}{\nu_i}, \quad v_s = \sqrt{\frac{kT_e}{m}},$$
$$\alpha = \left(\frac{\lambda_D}{l_1}\right)^2, \quad \lambda_D = \sqrt{\frac{\varepsilon_0 kT_e}{n_{e0}e^2}}, \quad \xi_w = \frac{l}{l_1}. \tag{12}$$

Note that λ_D is the Debye length evaluated at the plane of symmetry and v_s is the Bohm speed.

The integrodifferential equation (9), governing the function $\Phi(\xi)$, has to be integrated starting from $\xi = 0$ with the initial conditions (10) in the direction of increasing ξ until the condition (11) has been satisfied. Knowing the value $\xi = \xi_w$ at which the latter has happened, one can find the parameter ν_i.

III. FIRST INTEGRAL OF THE TONKS-LANGMUIR MODEL

A. The quasineutral solution

We are concerned with the limiting case of small α. Except in a thin layer adjacent to the wall (the space-charge sheath), the plasma is quasineutral and is described, to the first approximation in α, by Eq. (9) with the lhs being dropped

$$e^{\Phi(\xi)} = \int_0^\xi \frac{e^{\Phi(\zeta)}}{\sqrt{\Phi(\zeta) - \Phi(\xi)}}\,d\zeta. \tag{13}$$

This equation has to be solved with the boundary conditions (10).

This problem may be solved analytically.[5] By means of introducing new integration variable θ defined by the equation $\Phi(\xi)\sin^2\theta = \Phi(\zeta)$, Eq. (13) may be transformed to

$$g(\eta) = \frac{2}{\pi}\int_0^{\pi/2} h(\eta \sin\theta)\,d\theta, \tag{14}$$

where

$$\eta = \sqrt{-\Phi(\xi)}, \quad g(\eta) = e^{-\eta^2}, \quad h(\eta) = \frac{\pi\eta e^{-\eta^2}}{E(\eta)}, \tag{15}$$

and $E = -d\Phi/d\xi$ is the normalized y-projection of the electric field.

Equation (14) is Schlömilch's integral equation, and its solution reads (Ref. 16, Sec. 11.81)

$$h(\eta) = g(0) + \eta \int_0^{\pi/2} g'(\eta \sin\theta)\, d\theta. \tag{16}$$

Setting $g(\eta) = e^{-\eta^2}$ on the rhs, one can express the integral in terms of Dawson's integral (Ref. 17, p. 298) $F(\eta) = e^{-\eta^2} \int_0^\eta e^{t^2} dt$, and relation (16) assumes the form

$$h(\eta) = 1 - 2\eta F(\eta). \tag{17}$$

Substituting into this relation, the expression (15) for h, one obtains

$$\frac{d\eta}{d\xi} = \frac{\pi}{2} \frac{e^{-\eta^2}}{1 - 2\eta F(\eta)}. \tag{18}$$

Thus, the integral equation (13) has been transformed to an ordinary differential equation (18), which has to be solved with the initial condition $\eta(0) = 0$ (and the solution in the form $\xi(\eta)$ may be immediately found in quadratures).

The function $1 - 2\eta F(\eta)$ is positive for $0 \le \eta < \eta_m$ and vanishes at $\eta = \eta_m$, where $\eta_m \approx 0.9241$. Note that the derivative of Dawson's integral is given by the formula $F'(\eta) = 1 - 2\eta F(\eta)$, and therefore, η_m is the point of maximum of Dawson's integral. It follows from Eq. (18) that the electric field, evaluated in the quasineutral approximation, has a singularity at $\eta = \eta_m$. A breakdown of a quasineutral solution signals the formation of a space-charge sheath (and the physical sense of the singularity in the quasineutral electric field is quite clear: the electric field in the sheath is much higher than that in the quasineutral plasma). Since the sheath thickness tends to zero as $\alpha \to 0$ and we are considering the first approximation in α, the breakdown must be assumed to occur at $\xi = \xi_w$. One concludes that $\eta = \eta_m$ at $\xi = \xi_w$ and the function $\Phi(\xi)$, described by the quasineutral solution, decreases from 0 for $\xi = 0$ to $-\eta_m^2$ for $\xi = \xi_w$.

Substituting the obtained results in Eq. (7), one finds the ion distribution function

$$f(v, y) = \frac{\sqrt{2} n_{e0}}{\pi v_s} \left[\frac{1}{\omega} - 2F(\omega) \right] \tag{19}$$

for $0 \le v \le v_{\max}$ and $f(v, y) = 0$ for $v < 0$ or $v > v_{\max}$ (here, $\omega = \sqrt{-\Phi(y) - mv^2/2kT_e}$).

The ion distribution function determined in the quasineutral approximation at point $y = l$, where the quasineutral solution breaks down, represents the distribution function of the ions leaving the quasineutral plasma and entering the sheath and will be designated $f_s(v)$: $f(v, l) = f_s(v)$. $f_s(v)$ is given by Eq. (19) with $\omega = \sqrt{\eta_m^2 - mv^2/2kT_e}$. Note that since $2\eta_m F(\eta_m) = 1$ as mentioned above, $f_s(0) = 0$.

B. The first integral

The analytical solution described in Sec. III A goes back to the work[5] and is well known. In this section, we derive a useful property of this solution, which, apparently, was not noticed. Let us consider the quantity

$$B_1 = \frac{1}{\pi} \exp\left(\eta_m^2\right) \int_0^{\eta_m} \frac{1 - 2\omega F(\omega)}{\left(\eta_m^2 - \omega^2\right)^{3/2}}\, d\omega. \tag{20}$$

Taking into account Eq. (19), one can check that B_1 has the meaning of the mean inverse kinetic energy of the ions leaving the quasineutral plasma and entering the sheath. Note that since $2\eta_m F(\eta_m) = 1$ (see Sec. III A), the integral converges at $\omega = \eta_m$.

Let us replace 1 in the numerator of the integrand in Eq. (20) by $2\eta_m F(\eta_m)$. One can replace η_m by η in the obtained equation, thus making it applicable also inside the quasineutral plasma. Thus, we consider B_1 as a function of ξ or, equivalently, η and replace Eq. (20) with

$$B_1 = B_1(\xi) = \frac{2}{\pi} e^{\eta^2} \int_0^\eta \frac{\eta F(\eta) - \omega F(\omega)}{\left(\eta^2 - \omega^2\right)^{3/2}}\, d\omega. \tag{21}$$

Let us now consider the auxiliary function $I = I(\beta)$ defined as

$$I(\beta) = \int_0^\beta \frac{F(\eta) - \sin\theta F(\eta \sin\theta)}{\cos^2\theta}\, d\theta. \tag{22}$$

We split the rhs of Eq. (22) into two integrals, one of which is proportional to $F(\eta)$ and may be readily evaluated. Performing integration by parts in the second integral and making use of the formula $F'(\eta) = 1 - 2\eta F(\eta)$, one finds

$$I(\beta) = \frac{F(\eta)\sin\beta - F(\eta \sin\beta)}{\cos\beta} + \eta \int_0^\beta [1 - 2\eta \sin\theta\, F(\eta \sin\theta)]\, d\theta. \tag{23}$$

We need to evaluate the function $I(\beta)$ for $\beta = \pi/2$. The first term on the rhs of Eq. (23) tends to zero as $\beta \to \pi/2$, which can be conveniently verified by means of L'Hôpital's rule. The integral in the second term for $\beta = \pi/2$ may be evaluated by substituting Eq. (17) into Eq. (14). One obtains

$$I\left(\frac{\pi}{2}\right) = \frac{\pi\eta}{2} e^{-\eta^2}. \tag{24}$$

Let us return to Eq. (21). After the substitution $\omega = \eta \sin\theta$, this equation may be rewritten as

$$B_1 = \frac{2}{\pi\eta} e^{\eta^2} I\left(\frac{\pi}{2}\right). \tag{25}$$

By virtue of Eq. (24), one finds that

$$B_1 = 1 \tag{26}$$

for all values of ξ.

Let us express $\eta F(\eta)$ and $\omega F(\omega)$ in the numerator of the integrand on the rhs of Eq. (21) in terms of $E(\eta)$ and $E(\omega)$ by means of Eq. (17) and introduce the integration variable ζ defined by the equation $\omega^2 = -\Phi(\zeta)$. The resulting expression may be written as

$$B_1 = \frac{1}{2N_i(\xi)} \int_0^\xi \left[e^{\Phi(\zeta)} - e^{\Phi(\xi)} \sqrt{\frac{\Phi(\xi)}{\Phi(\zeta)}} \frac{E(\zeta)}{E(\xi)} \right] \frac{d\zeta}{[\Phi(\zeta) - \Phi(\xi)]^{3/2}}, \tag{27}$$

where $N_i = n_i/n_{e0}$ is the normalized ion density.

We have proved that the function $B_1 = B_1(\xi)$ defined by Eq. (27) equals unity for all the values of the coordinate ξ, i.e., at all the points in space, if evaluated in terms of the potential distribution Φ determined in the quasineutral approximation; an interesting property of the Tonks-Langmuir model. On the other hand, the function B_1, defined by Eq. (27), may be evaluated in terms of the "exact" potential distribution, obtained by means of numerical solution of the full Tonks-Langmuir model (9)–(11), which includes the Poisson equation and does not rely on the assumption of quasineutrality. The results of such evaluation for $\alpha = 10^{-2}$, 10^{-4}, 10^{-6}, and 10^{-8} are shown in Fig. 1. Here, $x = l - y$, and hence, x/l is the distance from the wall normalized by the column half-width l, so that $x/l = 0$ and $x/l = 1$ correspond to, respectively, the wall and the plane of symmetry of the discharge column. The curves are marked by the corresponding values of the ratio λ_D/l, which is related to α as $\lambda_D/l = \sqrt{\alpha}/\xi_w$. (Note that $\xi_w = 0.9747$, 0.5073, 0.4190, and 0.4067 for $\alpha = 10^{-2}$, 10^{-4}, 10^{-6}, and 10^{-8}, respectively.) The ratio m/m_e, involved in the boundary condition (11) was set equal to that for argon.

The circles on the curves in Fig. 1 and subsequent figures represent points where the charge separation $(n_i - n_e)/n_i$ decreases, in the direction from the wall into the plasma, to 10%, 1%, and 0.1%. For $\lambda_D/l = 0.10$, the charge separation at the plane of symmetry is approximately 4%, and the only point shown is the one marking the charge separation of 10%. One can see that for λ_D/l equal to 0.02 and smaller the function B_1 is very close to unity in the region of quasineutral plasma, in agreement with the above analytical treatment. Thus, the property of the Tonks-Langmuir model, expressed by Eq. (26) and derived from the analytical treatment, is manifested in the results of exact numerical calculations as the approximate equality $B_1 \approx 1$, which holds in the quasineutral plasma in the case of small λ_D/l.

Expressing the lhs of Eq. (27) in terms of the velocity distribution function by means of Eq. (7), one can rewrite Eq. (26) as

$$\frac{1}{n_i(y)}\int_0^{v_{\max}} \frac{2}{mv^2}\left[f(v,y) - \sqrt{\frac{2e\varphi(y)}{mv^2 + 2e\varphi(y)}}f(0,y)\right]dv = \frac{2}{kT_e}. \quad (28)$$

Note that the second term in the square brackets is singular at $v = v_{\max}$; however, the singularity is integrable. There is no singularity in the integrand at $v = 0$, which can be readily verified taking into account that $\frac{\partial f}{\partial v}(0, y) = 0$ as follows from Eq. (7).

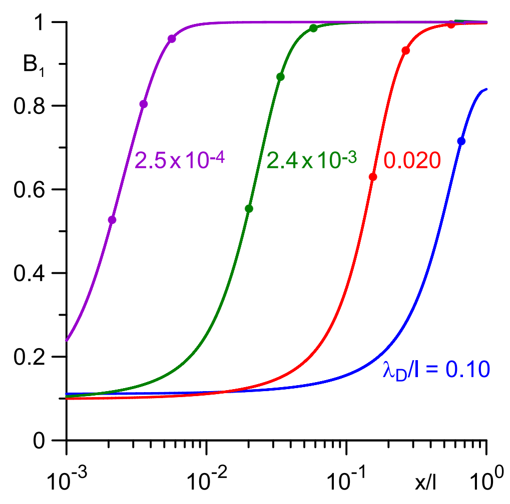

FIG. 1. Distribution of the normalized weighted inverse ion kinetic energy. Circles represent points where the charge separation equals 10%, 1%, and 0.1%.

The lhs of Eq. (28) has the meaning of a weighted mean inverse kinetic energy of the ions, so the physical meaning of Eq. (28) is clear: the weighted mean inverse kinetic energy of the ions, evaluated in the quasineutral approximation, equals $(kT_e/2)^{-1}$ at all points in space. When evaluated in terms of the exact numerical solution, the weighted mean inverse kinetic energy is equal, for small λ_D/l, to $(kT_e/2)^{-1}$ with good accuracy in the entire region of quasineutral plasma, including in the region adjacent to the space-charge sheath.

IV. CONNECTION TO THE KINETIC BOHM CRITERION

A. The kinetic Bohm criterion

The kinetic Bohm criterion[5] requires that the mean inverse kinetic energy of the ions entering the space-charge sheath from the quasineutral plasma be smaller than or equal to $(kT_e/2)^{-1}$,

$$\frac{1}{n_i}\int_0^{\infty} \frac{2}{mv^2} f_s(v)\,dv \le \frac{2}{kT_e}, \quad (29)$$

where $f_s = f_s(v)$ is the distribution function of the ions entering the sheath as above.

The kinetic Bohm criterion (29) may be derived by means of investigation of asymptotic behavior, at large distances from the surface, of solutions of the Poisson equation in the sheath, i.e., in the same way as the original Bohm criterion was derived in Ref. 15 for the case where all the ions entering the sheath have the same velocity. This was shown by the derivation in Ref. 18 (Sec. 3.2); a more straightforward version of this derivation is given in the Appendix. Similar to the original Bohm derivation, the kinetic derivations in Ref. 18 (Sec. 3.2) and the Appendix refer to the case where ion-neutral collisions, ionization, and geometrical effects in the sheath are negligible, i.e., $\lambda_D \ll \lambda_i, d, l$ (here, λ_i is the mean free path for the ion-neutral collisions, d is the ionization length, and l is a characteristic geometrical dimension), and are valid in the first approximation in the small parameter λ_D/L, where $L = \min(\lambda_i, d, l)$. On the other hand, the kinetic Bohm criterion (29) may be derived by means of investigation of asymptotic behavior of the first-approximation solution describing the quasineutral plasma; e.g., Appendix in Ref. 7.

Thus, the kinetic Bohm criterion (29) represents an inherent property of collision- and ionization-free one-dimensional space-charge sheaths. Nevertheless, the debate about its interpretation and validity is still ongoing.[6–14] The reason is that many researchers view the kinetic Bohm criterion as a real physical feature, while in reality, it is an asymptotic feature valid in the first approximation in the small parameter λ_D/L—"and only in this approximation."

The main argument made against the kinetic Bohm criterion is that the integral on the lhs diverges at $v = 0$ since the ion distribution function does not vanish at zero velocity. This argument originates from a failure to recognize the asymptotic nature of the Bohm criterion: the distribution function f_s of the ions leaving the quasineutral region and entering the sheath in the kinetic Bohm criterion (29) is the one evaluated in the first approximation in the small parameter λ_D/L, i.e., in the quasineutral approximation, rather than the exact one, and the former function does vanish at the zero velocity: $f_s(0) = 0$, as shown at the end of Sec. III A by the example of the Tonks-Langmuir problem. $f_s(0) = 0$ follows also from the analysis of the sheath equations: the equality $f_s(0) = 0$ is a necessary condition for the sheath equations to admit solutions with a monotonic (nonoscillating) potential distribution; see Appendix.

The appropriate mathematical tool for the investigation of space-charge sheaths is the method of matched asymptotic expansions (see, for example, Refs. 19–24). This is a powerful technique that automatically exploits all the simplifications that are justifiable and distinctly reveals governing physical mechanisms. The method goes back to Prandtl's viscous boundary layer theory and is widely used in different areas of applied mathematics, mechanics, and physics. The method of matched asymptotic expansions was applied to the case of a collision-dominated sheath in Refs. 25 and 26 and to the case of a collisionless sheath in Ref. 27; refined treatments were given in Refs. 28, 29, and 30, respectively. However, nowadays, the method of matched asymptotic expansions is not popular in gas discharge physics; labels such as a mathematical trick, a mathematical formality, and a purely mathematical construct are not uncommon. (Also not uncommon are claims that the Bohm criterion is based on the asymptotic limit $\lambda_D/L = 0$ as opposite to $\lambda_D/L \ll 1$ or that the asymptotic sheath edge may equivalently be characterized by a field singularity in the quasineutral plasma solution and is a necessary reference point to construct the intermediate layer connecting the plasma and the sheath, which attests to the confusion existing in the sheath physics-related literature about the important aspects of the method of matched asymptotic expansions.) Even when the validity of results obtained by the method is recognized, the asymptotic derivation of these results is plainly ignored and they are viewed as postulated *ad hoc*, with the conclusion that the good physical intuition of the author had fortuitously led him to the correct result(!).

The use of Eq. (28) makes it possible to study the mathematical meaning of the kinetic Bohm criterion in the Tonks-Langmuir model without invoking arguments based on the method of matched asymptotic expansions. This is done in Sec. IV B. One can hope that results obtained in this way will contribute to resolving the controversy.

B. Bohm criteria in the Tonks-Langmuir model

Figuring out the objective mathematical meaning of the Bohm criterion should be an important step in resolving the controversy around it. Suppose that exact distributions of plasma parameters in the near-surface region are known, obtained from a numerical solution of a full problem including the Poisson equation, computed without *a priori* dividing the computation domain into a quasineutral plasma and a space-charge sheath. Is there a nonarbitrary way to identify in the distribution of a particular parameter (the mean speed of ions, or the mean energy of ions, or the electric field, or…) a value separating the sheath from the quasineutral plasma?

This question was first considered in Ref. 31. Distribution of plasma parameters near a floating wall in collisional argon plasma was computed in the framework of the fluid model with the account of ion-atom collisions but without the account of ionization, under the conditions of weakly to strongly collisional sheaths with the ratio λ_D/λ_i varying over the range $[0.01, 10]$. No peculiarities at the Bohm speed in distributions of parameters have been found. Subsequent work[32] revealed, however, that a peculiarity does exists but for the smaller values of λ_D/λ_i, of the order of 10^{-3} and lower: a plateau, corresponding to the Bohm speed, is seen in the ion speed distribution in the intermediate region between the sheath and the adjacent quasineutral plasma (presheath), if this distribution is plotted on the logarithmic scale (which is appropriate in multiscale problems).

In Fig. 2, the distribution of the normalized ion speed, computed in the framework of the fluid version of the Tonks-Langmuir model, is shown. Here, v_i is speed of the ion fluid and v_s is the Bohm speed as above; note that the fluid version coincides with the kinetic Tonks-Langmuir model described in Sec. II except that Eqs. (4) and (5) are replaced with the equations[30]

$$\frac{d}{dy}(n_i v_i) = \nu_i n_e, \quad \frac{d}{dy}\left(m n_i v_i^2\right) = -e n_i \frac{d\varphi}{dy}. \tag{30}$$

The two lines in Fig. 2 referring to the cases $\lambda_D/l = 10^{-4}, 10^{-3}$ reveal a plateau, corresponding to the Bohm speed, $v_i =$ vs, in the intermediate region between the presheath (the region $x \gtrsim 0.1l$, where the two lines coincide) and the sheath: the reduction of the ion speed from v_s to $0.7v_s$ occurs as x increases by the factor of 34 for $\lambda_D/l = 10^{-4}$ and 6.2 for $\lambda_D/l = 10^{-3}$. The plateau becomes less pronounced with an increase in λ_D/l, i.e., as ionization in the sheath comes into play, and disappears for $\lambda_D/l \gtrsim 10^{-2}$.

Thus, if ionization in the sheath is weak enough, which means that the ratio λ_D/l should be of the order of 10^{-3} or smaller, the ion speed in the intermediate region between the sheath and the presheath varies little and is approximately equal to the Bohm speed; this is the speed with which the ions "reach the sheath," using Bohm's words.[15] This is the bottom mathematical meaning of the Bohm criterion in the fluid Tonks-Langmuir model. If the ionization in the sheath is non-negligible, $\lambda_D/l \gtrsim 10^{-2}$, then there is no plateau in the ion speed distribution in the intermediate region and, consequently, no sense to talk about a definite speed with which the ions enter the sheath. Hence, there are no mathematical grounds and no nonarbitrary way to modify the Bohm criterion for a sheath with ionization.

Figure 2 is remarkably similar to the corresponding figure of Ref. 32, although the presheath physics considered in Ref. 32 and this work is different: collisional plasma without ionization in Ref. 32 and collisionless plasma with ionization in this work. The reason for this similarity is that both figures reflect one of the general scenarios of asymptotic matching, namely, matching on a constant, as shown in Ref. 32.

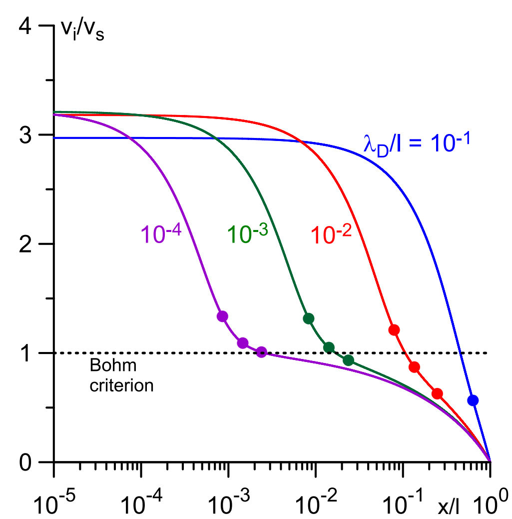

FIG. 2. Distribution of the normalized ion speed, fluid model.

Let us now consider the kinetic Bohm criterion (29). The validity of the kinetic Bohm criterion (29) in the collisionless Tonks-Langmuir model has been known for a long time[5,33] and can be conveniently illustrated by means of Eq. (28): if this equation is evaluated in terms of the distribution function determined in the quasineutral approximation and applied at $y = l$, then the second term in the square brackets vanishes [see Eq. (19) and the following text], and Eq. (28) assumes the form of the kinetic Bohm criterion (29) with the equality sign.

Thus, Eq. (28) is exactly equivalent to the classic kinetic Bohm criterion (29) with the equality sign, if the conditions ensuring the validity of the classic criterion are met (i.e., if the ion distribution function is determined in the first approximation in the small parameter λ_D/L or, in other words, under the assumption of quasineutrality). In this sense, Eq. (28) may be viewed as a new form of the kinetic Bohm criterion for the Tonks-Langmuir problem. On the other hand, Eq. (28) does not pose the problem of divergence for slow ions and its lhs may be evaluated for a wide class of distribution functions at any distance from the wall (and not only for the distribution function determined in the first approximation in the Debye length, taken at $y = 0$).

It is of interest to compare Eq. (28) with another form of the kinetic Bohm criterion, which was proposed in Ref. 13 and may be written, with the equality sign, as

$$\frac{1}{n_i(y)}\int_0^{v_{\max}} \frac{2}{mv^2}\left[f(v,y) - f(0,y)\right] dv \approx \frac{2}{kT_e}. \qquad (31)$$

The difference between Eqs. (31) and (28) is the presence of the square-root factor in the integrand in Eq. (28). Note that at $v = 0$, this factor equals unity, and the second term in the square brackets in Eq. (28) is identical to the corresponding term in Eq. (31).

An equivalent form of Eq. (31) is $B_2 \approx 1$, where

$$B_2 = \frac{1}{2N_i(\xi)}\int_0^{\xi}\left[e^{\Phi(\zeta)} - e^{\Phi(\xi)}\frac{E(\zeta)}{E(\xi)}\right]\frac{d\zeta}{\left[\Phi(\zeta) - \Phi(\xi)\right]^{3/2}}. \qquad (32)$$

The functions B_1 and B_2, evaluated with the use of numerical solution of the full Tonks-Langmuir model (9)–(11), are compared in Fig. 3. In the cases $\lambda_D/l = 2.5 \times 10^{-4}, 2.4 \times 10^{-3}$, the functions B_1 and B_2 coincide inside the sheath and are close to each other in the intermediate region between the sheath and the presheath, where $B_1 \approx 1$, while the function B_2 reveals a plateau with values close to 1. Note that the closeness of the functions B_1 and B_2 in the intermediate region means that the second term in the square brackets in Eqs. (28) and (31), which is where the difference between the two equations resides, is small. Hence, the left-hand side of each of these equations in the intermediate region to the first approximation represents the mean inverse ion kinetic energy, and both equations are close to the classic kinetic Bohm criterion (29), as they should.

Figure 3 clearly illustrates the bottom mathematical meaning of the kinetic Bohm criterion. The dashed lines in Fig. 3, depicting the function B_2, are remarkably similar to the lines in Fig. 2, which represent the fluid Bohm criterion, another manifestation of asymptotic matching on a constant. The function B_1 equals unity not only in the intermediate region but also in the presheath. In this aspect, the kinetic Bohm criterion in the Tonks-Langmuir problem expressed by Eq. (28) is different from the fluid Bohm criterion illustrated by Fig. 2 and the kinetic Bohm criterion expressed by Eq. (31). On the other hand, the kinetic Bohm criterion (28) is an exact property of the quasineutral solution of the Tonks-Langmuir problem, while the criterion (31) was hypothesized.

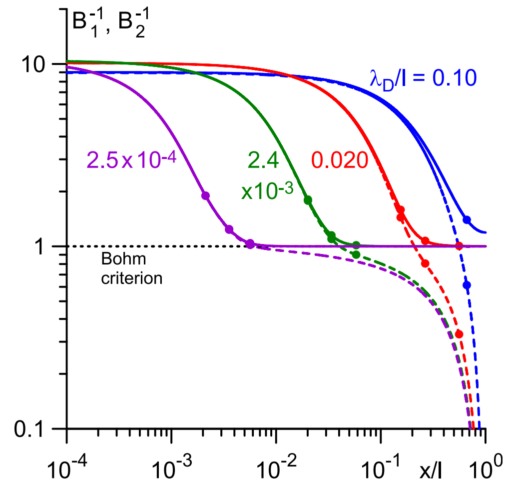

FIG. 3. Distribution of the inverse of the normalized weighted inverse ion kinetic energy. Solid: B_1^{-1}. Dashed: B_2^{-1}.

The logic that led to Eq. (31) was the following: the second term in the square brackets vanishes if f is evaluated in the quasineutral approximation at the point where this approximation breaks down, and hence, Eq. (31) is equivalent to the classic kinetic Bohm criterion (29) under the conditions of the validity of the latter; subtracting $f(0,y)$ is the simplest way to remove the nonintegrable singularity of the integrand at $v = 0$ (provided that $\partial f/\partial v = 0$ for $v = 0$, which is a necessary condition for the ion energy distribution to be analytic at low energies). The similarity between Eq. (31) and the (exact) Eq. (28) confirms this logic.

We have shown that if ionization in the sheath is weak enough, which means that the ratio λ_D/l should be of the order of 10^{-3} or smaller, then in the intermediate region between the sheath and the presheath, the left-hand side of each of Eqs. (28) and (31) to the first approximation represents the mean inverse ion kinetic energy, and this energy varies little and is approximately equal to $(kT_e/2)^{-1}$; this is the mean inverse kinetic energy with which the ions reach the sheath. In contrast, there is no way to identify a value of the inverse kinetic energy with which the ions reach the sheath if the ionization in the sheath is non-negligible, meaning that λ_D/l is of the order of 10^{-2} or higher. Hence, there are no mathematical grounds and no nonarbitrary way to modify the kinetic Bohm criterion for a sheath with ionization.

V. CONCLUSIONS

The weighted mean inverse kinetic energy of the ions in the collision-free Tonks-Langmuir model, defined by the lhs of Eq. (28), equals $(kT_e/2)^{-1}$ at all the points in space, if evaluated in terms of the quasineutral solution. If evaluated in terms of the numerical solution of the full model, the weighted mean inverse kinetic energy in the case of small λ_D/l is with good accuracy equal to $(kT_e/2)^{-1}$ in the entire region of quasineutral plasma, including in the vicinity of the space-charge sheath. These results constitute a mathematical proof of the kinetic Bohm criterion and show that is not just an assumption but rather a mathematical theorem.

The kinetic Bohm criterion represents an inherent property of collision- and ionization-free one-dimensional space-charge sheaths. The ongoing controversy about its validity stems from the fact that many researchers view the Bohm criterion, in both the fluid and kinetic formulations, as a real physical feature, while in reality, it is an asymptotic feature valid in the first approximation in the small parameter λ_D/L—and only in this approximation. The use of Eq. (28) makes it possible to study the mathematical meaning of the kinetic Bohm criterion in the collisionless Tonks-Langmuir model without using arguments based on the method of matched asymptotic expansion. In particular, it shows that the much-debated problem of divergence for slow ions stems from a misinterpretation. The results obtained, including Fig. 3, jointly with its fluid analog, Fig. 2, clearly reveal the objective mathematical meaning and the asymptotic nature of the Bohm criterion and should contribute to resolving the controversy.

In the results of numerical solution of the "full" problem (which includes the Poisson equation), shown in Figs. 2 and 3 above and Fig. 2 of Ref. 32, the Bohm criterion is manifested only for very low values of the ratio λ_D/L, of the order of 10^{-3} or smaller. Therefore, the Bohm criterion is hardly relevant for many conditions of practical interest. This explains the numerous attempts to modify it taking into account collisional and/or ionization and/or geometrical effects in the sheath. However, there is simply no definite value of speed or the inverse mean kinetic energy with which the ions enter the sheath if these effects are non-negligible, as seen from Figs. 2 and 3 for sheaths with ionization and in Fig. 2 of Ref. 32 for collisional sheaths. This explains why no universally accepted forms of fluid or kinetic Bohm criteria, modified with the account of collisions and/or ionization and/or geometrical effects in the sheath, have emerged: such criteria just cannot be postulated in a nonarbitrary way. Moreover, the term "Bohm criterion" is inappropriate in this context, even if adjectives "modified" or "generalized" are added: the Bohm criterion is a theorem valid for collision- and ionization-free one-dimensional space-charge sheaths, and there is no analogous theorem if collisional and/or ionization and/or geometrical effects in the sheath are taken into account. Using Brinkmann's words:[34] "there is nothing wrong with collisionally modified Bohm criteria; you only need to give a clear definition and choose a suitable name."

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APPENDIX: KINETIC BOHM CRITERION AS A COROLLARY OF THE SHEATH EQUATIONS

The aim of this Appendix is to derive the kinetic Bohm criterion by means of investigation of asymptotic behavior, at large distances from the surface, of solutions of the Poisson equation in the sheath, i.e., in the same way as the original (fluid) Bohm criterion was derived in Ref. 15 for the case where the ions entering the sheath have all the same velocity.

Following Ref. 15, let us consider a space-charge sheath separating a negative absorbing surface from an infinite uniform plasma with a constant potential. The sheath is formed by the ions, accelerated in the direction from the plasma to the surface by the sheath electric field, and the electrons, repelled from the surface back into the plasma by the sheath electric field. We consider the limiting case $\lambda_D \ll \lambda_i, d, l$ and restrict consideration to the first approximation in the small parameter λ_D/L, where $L = \min(\lambda_i, d, l)$.

In this approximation, one should consider a one-dimensional space-charge sheath where the ions move without collisions and no ionization occurs. The distribution of the electrostatic potential in the sheath, $\varphi(x)$, is governed by the Poisson equation (1) with coordinate y replaced by x the distance from the surface. The boundary conditions for large x are zero potential and electric field

$$\varphi(\infty) = \frac{d\varphi}{dx}(\infty) = 0. \quad \text{(A1)}$$

Let us designate by v the velocity of motion of an ion in the direction to the surface (i.e., the reversed x-component of the particle velocity of the ions) and by $f = f(v, x)$ a multiplier of the ion distribution function in the sheath describing its dependence on v (the dependence on velocity components parallel to the surface is supposed to be described by a normalized factor). Since ions cannot leave a one-dimensional, collisionless, and ionization-free sheath at an absorbing surface, the distribution function at large distances from the surface satisfies the condition $f(v,\infty) = 0$ for $v < 0$ and may be considered as a function of the kinetic energy of ion motion in the direction to the surface: $f(v,\infty) = f_s(mv^2/2)$, where f_s has the meaning of the distribution function of the ions entering the sheath.

Let us designate by u the total energy of an ion except for kinetic energy of motion in directions parallel to the surface: $u = mv^2/2 + e\varphi(x)$. As an ion moves across the sheath, the energy u remains unchanged. Therefore, the ion distribution function in the sheath after the transformation from the independent variables (v, x) to (u, x) becomes independent of x. (Alternatively, this may be derived from the Vlasov kinetic equation for the ions in the sheath: when written in the independent variables (u, x), it assumes the trivial form $\partial f/\partial x = 0$.) It follows that

$$f(v,x) = \begin{cases} f_s(u) & \text{for} \quad v \geq v_{\min}, \\ 0 & \text{for} \quad v < v_{\min}, \end{cases} \quad \text{(A2)}$$

where $v_{\min} = \sqrt{-2e\varphi(x)/m}$.

The ion density is

$$n_i(x) = \int_{v_{\min}}^{\infty} f_s(u)\, dv = \frac{1}{\sqrt{2m}} \int_0^{\infty} \frac{f_s(u)\, du}{\sqrt{u - e\varphi(x)}}. \quad \text{(A3)}$$

The electron density in the sheath is given by the Boltzmann distribution, Eq. (3), with n_{e0} replaced by n_0 the charged particle density in the uniform plasma, $n_0 = n_i(\infty) = n_e(\infty)$. Substituting this expression and Eq. (A3) into the Poisson equation Eq. (1), one obtains

$$\frac{\varepsilon_0}{e}\frac{d^2\varphi}{dx^2} = n_0 \exp\frac{e\varphi}{kT_e} - \frac{1}{\sqrt{2m}} \int_0^{\infty} \frac{f_s(u)\, du}{\sqrt{u - e\varphi(x)}}. \quad \text{(A4)}$$

This is the equation that governs, jointly with the boundary conditions (A1), the distribution of the electric field in the sheath.

In order to reduce the differential order of Eq. (A4), let us multiply it by $d\varphi/dx$, integrate over x from x to infinity, and change the order of integration in the double integral. Assuming that the function $\varphi(x)$ is monotonic, the integrals over x may be evaluated. Applying the second boundary condition (A1), one obtains the equation

$$\frac{\varepsilon_0}{2}\left(\frac{d\varphi}{dx}\right)^2 = \sqrt{\frac{2}{m}}\int_0^\infty f_s(u)\left(\sqrt{u-e\varphi}-\sqrt{u}\right)du - n_0 kT_e\left(1-\exp\frac{e\varphi}{kT_e}\right), \quad \text{(A5)}$$

which has to be solved with the first boundary condition (A1).

Let us investigate the asymptotic behavior of Eq. (A5) for $x\to\infty$ (or, equivalently, $\varphi\to 0$). To this end, the rhs of Eq. (A5) needs to be expanded in φ; terms of the order up to (including) φ^2 will be required. The second term on the rhs of Eq. (A5) may be expanded in a straightforward way. Let us designate the integral in the first term by I_1. One can write

$$I_1 = -e\varphi\int_0^\infty \frac{f_s(u)\,du}{\sqrt{u-e\varphi}+\sqrt{u}}. \quad \text{(A6)}$$

The integral on the rhs needs to be expanded up to terms of the order of φ. A straightforward procedure would be to expand the integrand in powers of φ and integrate term by term; however, this procedure is inapplicable since the term proportional to φ has a nonintegrable singularity at $u=0$ (unless $f_s(0)=0$). Therefore, we will represent

$$I_1 = -e\varphi(I_2 + C_1), \quad \text{(A7)}$$

$$I_2 = \int_0^\infty\left[\frac{f_s(u)}{\sqrt{u-e\varphi}+\sqrt{u}} - \frac{f_s(u)}{2\sqrt{u}}\right]du, \quad C_1 = \frac{1}{2}\int_0^\infty \frac{f_s(u)}{\sqrt{u}}du. \quad \text{(A8)}$$

It follows from Eq. (A3) that $C_1 = \sqrt{m/2}\,n_0$.

The integral I_2 may be represented as

$$I_2 = \frac{e\varphi}{2}\left[I_3 + f_s(0)I_4\right], \quad \text{(A9)}$$

where

$$I_3 = \int_0^\infty \frac{f_s(u)-f_s(0)}{\sqrt{u}\left(\sqrt{u-e\varphi}+\sqrt{u}\right)^2}du, \quad I_4 = \int_0^\infty \frac{du}{\sqrt{u}\left(\sqrt{u-e\varphi}+\sqrt{u}\right)^2}. \quad \text{(A10)}$$

The integral I_4 may be evaluated directly: $I_4 = \frac{4}{3\sqrt{-e\varphi}}$. For brevity, we confine ourselves to the case where the derivative $\frac{df_s}{du}(0)$ is finite. The one-term expansion of I_3 in powers of φ reads

$$I_3 = C_2 + \cdots, \quad C_2 = \frac{1}{4}\int_0^\infty\left[f_s(u)-f_s(0)\right]\frac{du}{u^{3/2}}. \quad \text{(A11)}$$

Now, Eq. (A5) for $x\to\infty$ assumes the form

$$\frac{\varepsilon_0}{2}\left(\frac{d\varphi}{dx}\right)^2 = -\frac{4f_s(0)}{3\sqrt{2m}}(-e\varphi)^{3/2} + \left(\frac{n_0}{2kT_e}-\frac{C_2}{\sqrt{2m}}\right)(e\varphi)^2 + \cdots. \quad \text{(A12)}$$

Since the lhs of Eq. (A12) is positive, the leading term of the expansion on the rhs must be non-negative. Hence, the first term, which is negative, must be excluded, which amounts to requiring

$$f_s(0) = 0. \quad \text{(A13)}$$

The second term on the rhs of Eq. (A12) must be non-negative, which amounts to requiring

$$\int_0^\infty \frac{f_s(u)}{u^{3/2}}du \le \frac{2\sqrt{2m}n_0}{kT_e}. \quad \text{(A14)}$$

Thus, Eq. (A13) and inequality (A14) represent the necessary conditions that have to be imposed on the distribution function of the ions coming into the sheath from the quasineutral plasma for the sheath equation to be solvable, or, more precisely, to admit solutions with a monotonic (nonoscillating) potential distribution. Of course, there is no need to explicitly mention in this context Eq. (A13), since if this equation is not satisfied, then the integral on the lhs of inequality (A14) diverges, and this inequality is not satisfied as well. It can be readily checked that inequality (A14) is exactly equivalent to the kinetic Bohm criterion (29).

One can conclude that the kinetic Bohm criterion (29) represents an inherent property of the first-approximation sheath equation Eq. (A5). It should be stressed that this equation is valid only in the first approximation in small parameter λ_D/L, which means that this equation—and, consequently, the kinetic Bohm criterion (29)—bears no account of collisions, ionization, and geometrical effects inside the sheath.

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