# On ensconcement and contraction

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## Abstract

In this article we present an axiomatic characterization for the ensconcement-based contractions. We study the interrelation among ensconcement-based contractions and brutal contractions, and we present two ways of defining an ensconcement relation by means of a base contraction operation. Finally, we study the interrelations among ensconcement-based contraction and epistemic entrenchment-based contractions and among brutal contractions and severe withdrawals.

*Keywords*: Belief change; belief bases; contraction; ensconcement; axiomatic characterization.

## 1 Introduction

The main goal underlying the research area of *belief change* consists in finding appropriate ways of modelling the belief state of a rational agent and, additionally, the changes which occur in such a state when the agent receives new information.

The one that is nowadays generally considered as the standard model of belief change was proposed by Alchourrón, Gärdenfors and Makinson in [1] and is commonly known as the AGM model. In that framework, the belief state of an agent is modelled by a *belief set*—i.e., a logically closed set of sentences — and *contractions* are seen as the main primitive kind of belief change, where by contraction it is meant the removal of one or several beliefs from the belief state of the agent.

In the mentioned paper, a set of conditions—nowadays commonly referred to as the *AGM postulates for contraction*—were identified as the properties that should be satisfied by a contraction function. Also in that reference, a constructive definition of contraction functions, which were named *partial meet contractions*, was presented and a representation theorem was obtained, attesting that the AGM postulates for contraction exactly characterize that class of functions. Subsequently, several other constructive definitions have been proposed which give rise to that same class of functions, namely *safe/kernel contraction* [2, 18], *system of spheres-based contraction* [13] and *epistemic entrenchment-based contraction* [10, 12]. In the last three decades, several authors have proposed extensions and alternatives to the AGM model.[1] Among them, we can mention:

*Belief set contraction without recovery:* One of the AGM postulates for contraction is *recovery*, which essentially states that if the result of contracting a belief set by a certain belief is (subsequently) expanded by that same belief then all the initial beliefs are recovered. Some

[1]For an overview see [5].


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alternative classes of contractions on belief sets which do not satisfy the *recovery* postulate are Levi contractions [20], severe withdrawals (or mild contractions or Rott contractions) [22, 25], semi-contractions [3, 9] and systematic withdrawal [21].

*Belief change operators for belief bases:* The AGM model only accounts for the case when the belief state of an agent is represented by a set of sentences that is closed under logical consequence (i.e. a *belief set* or *theory*). Several of the existing models of contraction for beliefs sets have been adapted to the case when belief states are represented by sets of sentences not (necessarily) closed under logical consequence — the so-called belief bases: the partial meet contractions for belief bases were presented in [15–17]; the kernel contractions — which can be seen as a generalization of safe contractions — were introduced in [18]; and, in [28], Mary-Anne Williams introduced the *ensconcement-based contractions* and the *brutal contractions* (of belief bases), which can be seen as adaptations to the case of belief bases of the *epistemic entrenchment-based contractions* and of the *severe withdrawals*, respectively. In fact, the definitions of the two classes of contraction functions proposed in [28] are based on the concept of ensconcement, which is a generalization to the case of belief bases of the concept of epistemic entrenchment introduced by Gärdenfors and Makinson in [10, 12].

In this article, we study the interrelation among the two kinds of belief bases contraction operators introduced by Mary-Anne Williams in [28], namely *brutal contractions* and *ensconcement-based contractions*. We start by presenting an axiomatic characterization for the *ensconcement-based contractions*. At this point, we must note that an axiomatic characterization for that class of functions was already presented in [6, Theorem 14]. However, as it was mentioned by Zhiqiang Zhuang in personal communications with the first and third authors of the present article, there are gaps in the proof of that theorem. In the end of Section 3, we indicate precisely which steps of the mentioned proof are not correctly justified. After that we compare this result with the axiomatic characterization for *brutal contractions* that was presented in [11] in the sense of identifying which postulates of each one of the mentioned axiomatic characterizations are (and which are not) satisfied by the other kind of operators. This comparison allows us to determine which ones of the postulates used in those representation theorems can be considered characteristic properties of each one of those two kinds of contraction functions. We also study the construction of ensconcement relations by means of each one of the two kinds of base contraction operations considered. Furthermore, we investigate the interrelation between ensconcement and epistemic entrenchment and provide some further evidence that ensconcement relations can be considered as a generalization of epistemic entrenchment relations. In particular, we present some results which clarify the interrelation among epistemic entrenchment-based contractions and ensconcement-based contractions, as well as the interrelation among severe withdrawals and brutal contractions.

This article is organized as follows: In Section 2, we provide the notation and background needed for the rest of the article. In Section 3, we propose some new postulates for belief base contraction and use them to obtain an axiomatic characterization for the ensconcement-based contractions. Then, in Section 4, we compare that axiomatic characterization for the ensconcement-based contractions with the axiomatic characterization for brutal contractions that was presented in [11] and, furthermore, we present two methods for constructing an ensconcement by means of a contraction operation. After that, in Section 5, we study the interrelation among those two classes of belief base contractions and the two above mentioned classes of belief set contractions whose definitions are based on epistemic entrenchment relations (namely, the classes of epistemic entrenchment-based contractions and of severe withdrawals). Finally, in Section 6 we briefly summarize the main contributions of the article. In the Appendix, we provide proofs for all the original results presented.

## 2 Background

### *2.1 Formal preliminaries*

Beliefs are expressed in a language $\mathcal{L}$ that is called the object language. We assume that the language contains the usual truth functional connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$) and equivalence ($\leftrightarrow$). We say that two sentences $\alpha$ and $\beta$ are logically independent if and only if all combinations of truth values are logically possible for them. We shall make use of a consequence operation $Cn$ that takes sets of sentences to sets of sentences and which satisfies the standard Tarskian properties, namely inclusion ($A \subseteq Cn(A)$), monotony (if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$), and iteration ($Cn(A) = Cn(Cn(A))$). We also assume it satisfies supraclassicality, compactness and deduction (if $\beta \in Cn(A \cup \{\alpha\})$, then $(\alpha \rightarrow \beta) \in Cn(A)$). $A \vdash \alpha$ will be used as an alternative notation for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$ and $Cn(\alpha)$ for $Cn(\{\alpha\})$. The letters $\alpha, \alpha_i, \beta, \ldots$ will be used to denote sentences of $\mathcal{L}$. $\top$ stands for an arbitrary tautology and $\bot$ for an arbitrary contradiction. $A, A', B, \ldots$ denote subsets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e. $\mathbf{K} = Cn(\mathbf{K})$)—such a set is called a *belief set* or *theory*. Given $A \subseteq \mathcal{L}$, we shall say that $-$ is a (contraction) operation (or function) on $A$ when $-$ is a function such that $- : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{L})$. Furthermore, in that context, we shall represent by $A - \alpha$ the image of a sentence $\alpha$ by $-$. Given $A \subseteq \mathcal{L}$, a binary relation $\leq$ on $A$ is total if and only if for all $\alpha, \beta \in A$ it holds that either $\alpha \leq \beta$ or $\beta \leq \alpha$. Given a binary relation $\leq$ on $\mathcal{L}$ and a set $S \subseteq \mathcal{L}$, we denote by $\leq|_S$ the binary relation on $S$ such that for all $\alpha, \beta \in S$, $\alpha \leq|_S \beta$ if and only if $\alpha \leq \beta$. Given a binary relation $\leq$ on $\mathcal{L}$ we shall write $\alpha < \beta$ to denote $\alpha \leq \beta$ and $\beta \not\leq \alpha$, and $\alpha =_{\leq} \beta$ to denote $\alpha \leq \beta$ and $\beta \leq \alpha$.

### *2.2 Epistemic entrenchment*

We start by recalling, in the following definition, the concept of *epistemic entrenchment relation*.

Definition 2.1 ([10, 12])
An ordering of epistemic entrenchment with respect to a belief set $\mathbf{K}$ is a binary relation $\leq$ on $\mathcal{L}$ which satisfies the following properties:

**(EE1)** For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \delta$ then $\alpha \leq \delta$. (Transitivity)
**(EE2)** For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \vdash \beta$ then $\alpha \leq \beta$. (Dominance)
**(EE3)** For all $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$. (Conjunctiveness)
**(EE4)** When $\mathbf{K} \not\vdash \bot$, $\alpha \notin \mathbf{K}$ iff $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$. (Minimality)
**(EE5)** If $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$, then $\vdash \alpha$. (Maximality)

Now we recall the definition of the *epistemic entrenchment-based contractions* which has been introduced in [10, 12].

Definition 2.2 ([10, 12])
Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$.
The $\leq$-*based contraction* on $\mathbf{K}$ is the contraction operation $\div_{\leq}$ defined, for any $\alpha \in \mathcal{L}$, by:

$$\mathbf{K} \div_{\leq} \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & ,\text{if } \not\vdash \alpha \\ \mathbf{K} & ,\text{if } \vdash \alpha. \end{cases} \qquad (\mathbf{C}_{\div_{\leq}})$$

An operation $\div$ on $\mathbf{K}$ is an *epistemic entrenchment-based contraction* on $\mathbf{K}$ if and only if there is an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} \div_{\leq} \alpha$.

In the next observation we recall the axiomatic characterization for the epistemic entrenchment-based contractions that was obtained by Gärdenfors and Makinsson [12].

OBSERVATION 2.3 ([12])
Let $\mathbf{K}$ be a belief set and $\div$ be a contraction function on $\mathbf{K}$. Then $\div$ is an epistemic entrenchment-based contraction if and only if it satisfies the following postulates:

($\div$1) $\mathbf{K}\div\alpha = Cn(\mathbf{K}\div\alpha)$ (i.e. $\mathbf{K}\div\alpha$ is a belief set).
($\div$2) $\mathbf{K}\div\alpha\subseteq\mathbf{K}$.
($\div$3) If $\alpha\notin\mathbf{K}$, then $\mathbf{K}\div\alpha=\mathbf{K}$.
($\div$4) If $\nvdash\alpha$, then $\alpha\notin\mathbf{K}\div\alpha$.
($\div$5) $\mathbf{K}\subseteq Cn((\mathbf{K}\div\alpha)\cup\{\alpha\})$.
($\div$6) If $\vdash\alpha\leftrightarrow\beta$, then $\mathbf{K}\div\alpha=\mathbf{K}\div\beta$.
($\div$7) $(\mathbf{K}\div\alpha)\cap(\mathbf{K}\div\beta)\subseteq\mathbf{K}\div\alpha\wedge\beta$.
($\div$8) $\mathbf{K}\div\alpha\wedge\beta\subseteq\mathbf{K}\div\alpha$ whenever $\alpha\notin\mathbf{K}\div\alpha\wedge\beta$.

Postulates ($\div$1)–($\div$6) are commonly known as *basic AGM postulates for contraction* [1].[2] Postulates ($\div$7) and ($\div$8) were introduced in [1] and are known as *supplementary AGM postulates for contraction*. It is convenient to recall here the following result from [1]:

OBSERVATION 2.4 ([1])
Let $\mathbf{K}$ be a belief set, and $\div$ be an operation on $\mathbf{K}$ that satisfies ($\div$1), ($\div$2), ($\div$3), ($\div$4), ($\div$5) and ($\div$6). Then $\div$ satisfies both ($\div$7) and ($\div$8) if and only if $\div$ satisfies the following postulate:
($\div V$) $\mathbf{K}\div\alpha\wedge\beta=\mathbf{K}\div\alpha$ or $\mathbf{K}\div\alpha\wedge\beta=\mathbf{K}\div\beta$ or $\mathbf{K}\div\alpha\wedge\beta=\mathbf{K}\div\alpha\cap\mathbf{K}\div\beta$.

We note here also that, apart from presenting a way of defining a contraction operation based on an epistemic entrenchment relation (by means of condition ($\mathbf{C}_{\div_\leq}$)), Gärdenfors and Makinsson [10, 12] have also exposed a way of proceeding to the converse construction. More precisely, in [12, Theorem 5], it is stated that if $\div$ is a contraction function on $\mathbf{K}$ that satisfies both the basic and the supplementary AGM postulates for belief set contraction, then the binary relation $\leq$ on $\mathcal{L}$ defined by the following condition:

$$\alpha\leq\beta \text{ iff } \alpha\notin\mathbf{K}\div\alpha\wedge\beta \text{ or } \vdash\alpha\wedge\beta, \qquad (\mathbf{C}_{\leq})$$

is an epistemic entrenchment relation with respect to $\mathbf{K}$ and, furthermore, it holds that $\mathbf{K}\div\alpha=\mathbf{K}\div_{\leq}\alpha$.

Next we recall the definition of the *severe withdrawals* (also known as mild contractions or Rott's contractions) which was introduced by Rott in [22] and consists of an intuitively appealing simplification of the definition of *epistemic entrenchment-based contractions*.

DEFINITION 2.5 ([22])
Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. The $\leq$-based severe withdrawal on $\mathbf{K}$ is the operation $\div^S_\leq$ defined, for any $\alpha\in\mathcal{L}$, by:

$$\mathbf{K}\div^S_\leq\alpha=\begin{cases}\{\beta\in\mathbf{K}:\alpha<\beta\} & ,\text{if } \nvdash\alpha\\ \mathbf{K} & ,\text{if } \vdash\alpha.\end{cases} \qquad (\mathbf{R}_{\div_\leq})$$

An operation $\div$ on $\mathbf{K}$ is a severe withdrawal if and only if there is an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ such that, for all sentences $\alpha\in\mathcal{L}$, $\mathbf{K}\div\alpha=\mathbf{K}\div^S_\leq\alpha$.

[2] Postulate ($\div$5) is commonly designated by *recovery*.

*Severe withdrawals* were axiomatically characterized independently by Rott and Pagnucco in [25] and by Fermé and Rodriguez in [8].

OBSERVATION 2.6 ([25])
Let $\mathbf{K}$ be a belief set and $\div$ be a contraction function on $\mathbf{K}$. Then $\div$ is a severe withdrawal if and only if it satisfies the following postulates:[3] $(\div\mathbf{1})$, $(\div\mathbf{2})$, $(\div\mathbf{3})$, $(\div\mathbf{4})$,

$(\div 3')$ If $\vdash \alpha$, then $\mathbf{K} - \alpha = \mathbf{K}$.
$(\div 9)$ If $\alpha \notin \mathbf{K} \div \beta$, then $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$.

## 2.3 Ensconcement and its interrelation with entrenchment

In [27–29], Mary-Anne Williams defines an *ensconcement* as a pair $(A, \preceq)$ where $A$ is a belief base and $\preceq$ is a transitive and total relation on $A$ that satisfies the following three conditions:

$(\preceq\mathbf{1})$ If $\beta \in A \setminus Cn(\emptyset)$, then $\{\alpha \in A : \beta \prec \alpha\} \nvdash \beta$.
$(\preceq\mathbf{2})$ If $\nvdash \alpha$ and $\vdash \beta$, then $\alpha \prec \beta$, for all $\alpha, \beta \in A$.
$(\preceq\mathbf{3})$ If $\vdash \alpha$ and $\vdash \beta$, then $\alpha \preceq \beta$, for all $\alpha, \beta \in A$.

Informally an ensconcement relation establishes an order over the beliefs of an agent.[4] $(\preceq 1)$ says that, for any non-tautological $\beta$, the formulae that are strictly more ensconced than $\beta$ do not (even conjointly) imply $\beta$. Conditions $(\preceq 2)$ and $(\preceq 3)$ say that tautologies are the most ensconced formulae.

The concept of ensconcement relation can be seen as a generalization of the notion of epistemic entrenchment to the context of belief bases (i.e. sets of sentences not necessarily closed under logical consequence). In what follows we expose more formally the interrelation between these two kinds of binary relations.

The following result implies that the restriction of an epistemic entrenchment relation to a belief set is an ensconcement relation.

OBSERVATION 2.7
Let $\mathbf{K}$ be a belief set and $\leq$ a relation on $\mathcal{L}$ that satisfies (EE1), (EE2), (EE3) and (EE5). Then $(\mathbf{K}, \leq|_{\mathbf{K}})$ is an ensconcement.

Since on the one hand an epistemic entrenchment is defined over the set of all $\mathcal{L}$-sentences while, on the other hand, an ensconcement relation is defined only over a set of sentences (which does not even need to be closed under logical consequence), we can immediately conclude that not every ensconcement relation is an epistemic entrenchment. Nevertheless, in [28], Mary-Anne Williams proposed a method for extending an ensconcement relation to the set of all $\mathcal{L}$-sentences which is such that the resulting relation is an epistemic entrenchment.

Let $(A, \preceq)$ be an ensconcement. In what follows we recall Williams' definition of an epistemic entrenchment $\leq_{\preceq}$ related to $Cn(A)$ such that for all $\alpha, \beta \in A$, it holds that $\alpha \leq_{\preceq} \beta$ if and only if $\alpha \preceq \beta$.

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[3] We note that the axiomatization of severe withdrawals presented in [25] consists of the postulates $(\div\mathbf{1})$, $(\div 2)$, $(\div\mathbf{4})$, $(\div\mathbf{6})$, $(\div\mathbf{9})$, and $(\dot{-}3)$ If $\alpha \notin \mathbf{K}$ or $\vdash \alpha$, then $\mathbf{K} \subseteq \mathbf{K} - \alpha$.
However, in the presence of $(\div 2)$, the postulate $(\dot{-}3)$ is equivalent to the postulates $(\div 3)$ and $(\div 3')$ (taken together). On the other hand, in [11], it was shown that $(\div\mathbf{6})$ follows from $(\div\mathbf{1})$, $(\div\mathbf{3'})$, $(\div\mathbf{4})$ and $(\div\mathbf{9})$.

[4] When forced to give up some of its beliefs an agent is more willing to remove the less ensconced ones.

First we recall the notion of *cut* which was defined, in [28]. For any sentence $\alpha \in Cn(A)$ the cut of $\alpha$, denoted $cut_{\preceq}(\alpha)$ is the following subset of $A$:

$$cut_{\preceq}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha\}.$$

In the following observation we present the above mentioned definition of an epistemic entrenchment $\leq_{\preceq}$ from the ensconcement relation $\preceq$.

OBSERVATION 2.8 ([28])
Let $(A, \preceq)$ be an ensconcement and let $\leq_{\preceq}$ be the binary relation on $\mathcal{L}$ defined by: $\alpha \leq_{\preceq} \beta$ if and only if either

(i) $\alpha \notin Cn(A)$, or
(ii) $\alpha, \beta \in Cn(A)$ and $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$.

Then $\leq_{\preceq}$ is an epistemic entrenchment related to $Cn(A)$.

## 2.4 *Base contraction functions*

In this subsection, we recall the two kinds of base contraction functions defined by means of ensconcement relations that were proposed by Mary-Anne Williams in [28].

Both of the mentioned definition are based on the *proper cut* operator, which is defined as follows: Given an *ensconcement* $(A, \preceq)$, for any sentence $\alpha \in \mathcal{L}$ the *proper cut* of $\alpha$, denoted $cut_{\prec}(\alpha)$ is the subset of $A$ defined by:

$$cut_{\prec}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \preceq \gamma\} \nvdash \alpha\}.$$

The following observation states that when $\alpha$ is an explicit belief, its *proper cut* is the subset formed by the sentences of $A$ which are strictly more ensconced than $\alpha$.

OBSERVATION 2.9 ([28])
If $\alpha \in A$, $cut_{\prec}(\alpha) = \{\beta \in A : \alpha \prec \beta\}$.

Next we recall the definition of the so-called *ensconcement-based contractions*.

DEFINITION 2.10 ([28])
Let $(A, \preceq)$ be an ensconcement. The $\preceq$-based contraction on $A$ is the operation $-_{\preceq}$ such that:

$$A -_{\preceq} \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}. \qquad \textbf{(EBC)}$$

An operation $-$ on $A$ is an ensconcement-based contraction if and only if there is an ensconcement $(A, \preceq)$ such that for all sentences $\alpha$: $A - \alpha = A -_{\preceq} \alpha$.

Note that if $\alpha \in Cn(\emptyset)$ and $-$ is an ensconcement-based contraction, then $A - \alpha = A$.

The other kind of base contraction functions introduced in [28], results of an intuitively appealing change in condition (**EBC**), used to define the ensconcement-based contractions. In fact, according to this condition in order for $\beta$ to be preserved when contracting $A$ by $\alpha$ it is necessary that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. However, it seems more intuitive to require only that $\beta \in cut_{\prec}(\alpha)$ instead. Below we recall the definition of the class of contraction functions based on this simpler condition.

Definition 2.11 ([28])
Let $(A, \preceq)$ be an ensconcement. The $\preceq$-based brutal contraction on $A$ is the operation $-^B_{\preceq}$ such that:

$$A -^B_{\preceq} \alpha = \begin{cases} cut_{\prec}(\alpha) & \text{if } \nvdash \alpha \\ A & \text{otherwise.} \end{cases} \qquad \textbf{(BC)}$$

An operation $-$ on $A$ is a brutal contraction if and only if there is an ensconcement $(A, \preceq)$ such that for all sentences $\alpha$: $A - \alpha = A -^B_{\preceq} \alpha$.

In the following observation, we recall the axiomatic characterization of brutal contraction functions that was presented in [11].

Observation 2.12 ([11])
Let $A$ be a belief base. An operator $-$ on $A$ is a brutal contraction if and only if it satisfies the following postulates:

**(Relative Closure)** $A \cap Cn(A - \alpha) \subseteq A - \alpha$.
**(Inclusion)** $A - \alpha \subseteq A$.
**(Vacuity)** If $A \nvdash \alpha$, then $A \subseteq A - \alpha$.
**(Failure)** If $\vdash \alpha$, then $A - \alpha = A$.
**(Success)** If $\nvdash \alpha$, then $A - \alpha \nvdash \alpha$.
**(Strong Inclusion)** If $A - \beta \nvdash \alpha$, then $A - \beta \subseteq A - \alpha$.
**(Uniform Behaviour)** If $\beta \in A, A \vdash \alpha$ and $A - \alpha = A - \beta$, then $\alpha \in Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$.

Having in mind the interrelation between epistemic entrenchment and ensconcement relations that we have exposed in Subsection 2.3, at this point it is worth to highlight that ensconcement-based contraction can be seen as the analogue, in the context of belief bases, of the epistemic entrenchment-based (belief set) contraction, whereas brutal contraction is related to severe withdrawal in a similar way. In Section 5, we study more thoroughly the interrelations among these classes of functions.

## *2.5 Postulates for belief base contraction*

In this subsection we recall some well-known base contraction postulates.

We start by mentioning the three following postulates, whose statements result of adapting the statements of postulates $(\div 6)$, $(\div 8)$, $(\div V)$ to obtain similar properties suitable for belief base contractions (rather than only for belief set contractions).

**Extensionality** If $\vdash \alpha \leftrightarrow \beta$, then $A - \alpha = A - \beta$.

*Extensionality* states that contracting a set by logically equivalent sentences produces the same output.

**Conjunctive inclusion** If $A - \alpha \wedge \beta \nvdash \alpha$, then $A - \alpha \wedge \beta \subseteq A - \alpha$.

*Conjunctive inclusion* states that if $\alpha$ is not implied by the contraction of A by the conjunction of $\alpha$ and $\beta$, then anything that is removed when contracting $A$ by $\alpha$ is also removed when contracting $A$ by $\alpha \wedge \beta$.

**Conjunctive factoring** $A - \alpha \wedge \beta = \begin{cases} A - \alpha \text{ or} \\ A - \beta \text{ or} \\ A - \alpha \cap A - \beta. \end{cases}$

*Conjunctive factoring*, states that the output of contracting a set $A$ by the conjunction of $\alpha$ and $\beta$ is identical to either the output of contracting $A$ by $\alpha$ or to the output of contracting $A$ by $\beta$ or to the intersection of these two outputs.

Other classical base contraction postulates are as follows:

**Disjunctive Elimination** If $\beta \in A$ and $\beta \notin A - \alpha$ then $A - \alpha \nvdash \alpha \vee \beta$.

*Disjunctive elimination* was proposed in [6] and states that if a sentence $\beta$ is removed in the process of contracting $A$ by another sentence $\alpha$ then the disjunction of $\alpha$ and $\beta$ is not deducible from the outcome of that contraction.

**Relevance** If $\beta \in A$ and $\beta \notin A - \alpha$, then there is a set $A'$ such that $A - \alpha \subseteq A' \subseteq A$ and $A' \nvdash \alpha$ but $A' \cup \{\beta\} \vdash \alpha$.

The *relevance* postulate [14, 16] states that if a sentence $\beta$ is removed from $A$ when contracting it by $\alpha$, then there exists a subset $A'$ of $A$ that contains $A - \alpha$ such that $A'$ does not imply $\alpha$ but the set obtained by adding $\beta$ to $A'$ implies $\alpha$.

**Logical Relevance** If $\beta \in A$ and $\beta \notin A - \alpha$, then there is a set $A'$ such that $A - \alpha \subseteq A' \subseteq Cn(A)$ and $A' \nvdash \alpha$ but $A' \cup \{\beta\} \vdash \alpha$.

This postulate, that was presented in [26], is a weaker version of the *relevance* postulate. Instead of requiring inclusion of $A'$ on $A$, it only requires logical inclusion.

**Linearity** $A - \alpha \subseteq A - \beta$ or $A - \beta \subseteq A - \alpha$.

*Linearity* [8, 25] states that the outputs of contraction of a set $A$ by a belief $\alpha$ and by a belief $\beta$ are connected by means of the set inclusion relation.

**Expulsiveness** If $\nvdash \alpha, \nvdash \beta$ then $A - \beta \nvdash \alpha$ or $A - \alpha \nvdash \beta$.

This postulate, states that for every non-tautological sentences $\alpha$ and $\beta$, either $\alpha$ is not implied by the contraction of $A$ by $\beta$ or $\beta$ is not implied by the contraction of $A$ by $\alpha$. *Expulsiveness* was first presented in [19, page 102] and, as it is mentioned there and also in [24, 25], it is a highly implausible property of belief contraction, since according to it two unrelated sentences influence the result of the contraction by each other.

**Decomposition** $A - \alpha \wedge \beta = A - \alpha$ or $A - \alpha \wedge \beta = A - \beta$.

According to *decomposition* [1] (which is also known as *linear hierarchical ordering*) the output of contracting a set $A$ by a conjunction of two beliefs is equal to the contraction of $A$ by one of those two beliefs.

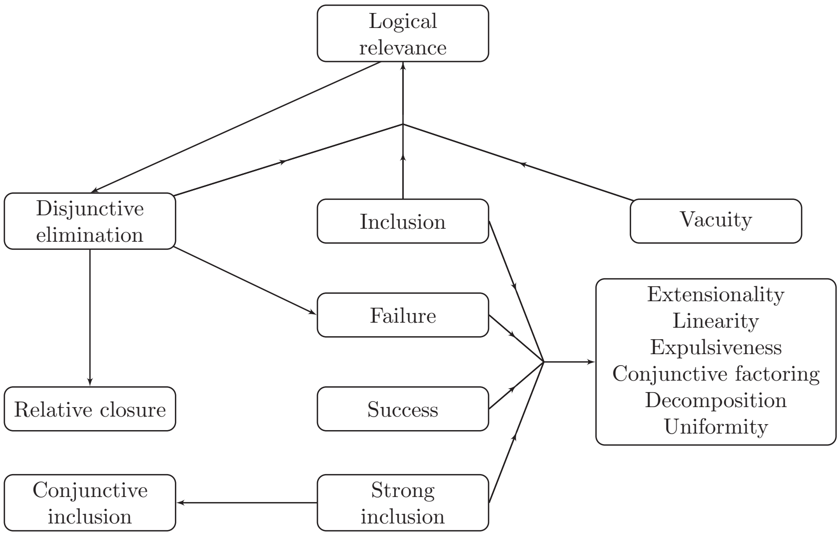


FIGURE 1. Diagram of the interrelations among postulates stated in Observation 2.13.

**Uniformity** If it holds for all subsets $A'$ of $A$ that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$ then $A-\alpha = A-\beta$.

This postulate, which was presented in [16], states that when $\alpha$ and $\beta$ are two sentences implied by exactly the same subsets of $A$, then the result of contracting $A$ by $\alpha$ is identical to the outcome of contracting $A$ by $\beta$.

The following observation presents some relations among the postulates listed above:

OBSERVATION 2.13
Let $A$ be a belief base and $-$ be an operator on $A$. Then:

(a) [6] If $-$ satisfies disjunctive elimination, then it satisfies relative closure and failure.
(b) If $-$ satisfies logical relevance, then it satisfies disjunctive elimination.
(c) If $-$ satisfies inclusion, vacuity and disjunctive elimination, then it satisfies logical relevance.
(d) [11] If $-$ satisfies inclusion, failure, success and strong inclusion, then it satisfies extensionality, linearity and expulsiveness.
(e) [25] If $-$ satisfies strong inclusion, then it satisfies conjunctive inclusion.
(f) If $-$ satisfies inclusion, failure, success and strong inclusion, then it satisfies conjunctive factoring, decomposition and uniformity.

In Figure 1, we present a diagram that summarizes all the interrelations among postulates that were stated in the above observation.

## 3 Axiomatic characterization of ensconcement-based contraction

Our main goal in the present section is to obtain an axiomatic characterization for the ensconcement-based contractions. We will start by introducing, in Subsection 3.1 some postulates that will be necessary for achieving that goal. Then, in Subsection 3.2, we present a representation theorem for the ensconcement-based contraction functions.

### 3.1 Some new postulates for belief base contraction

Now we introduce some new postulates which will be useful afterwards in the process of obtaining an axiomatic characterization of the ensconcement-based contractions.

**Transitivity** If $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$, then $\alpha \notin A - \alpha \wedge \delta$.

The intuition behind this postulate (for the principal case) is the following: Provided that $\alpha, \beta \in A$ and $\nvdash \alpha \wedge \beta$, at least one of the sentences $\alpha$ or $\beta$ is removed when contracting $A$ by $\alpha \wedge \beta$. Thus, $\alpha \notin A - \alpha \wedge \beta$ can be interpreted as meaning that $\beta$ is 'at least as good as $\alpha$'. Having this interpretation in mind, the postulate of *transitivity* essentially states that if $\beta$ is at least as good as $\alpha$ and $\delta$ is at least as good as $\beta$, then $\delta$ is at least as good as $\alpha$.

Still considering the above reasoning, we notice that $\alpha \in A - \alpha \wedge \beta$ can be considered as meaning that '$\alpha$ is (strictly) better than $\beta$'. Taking this into account two other postulates (similar to *transitivity*) which are natural to expect to be fulfilled by a contraction function are the following:

**ST** If $\delta \in A$, $\beta \in A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$, then $\delta \in A - \alpha \wedge \delta$.

**SST** If $\alpha \in A$, $\beta \in A - \alpha \wedge \beta$ and $\delta \in A - \beta \wedge \delta$, then $\delta \in A - \alpha \wedge \delta$.

The *ST* postulate can be interpreted as follows: if $\beta$ is (strictly) better than $\alpha$ and $\delta$ is at least as good as $\beta$, then $\delta$ is (strictly) better than $\alpha$. While the *SST* postulate can be interpreted as: if $\beta$ is (strictly) better than $\alpha$ and $\delta$ is (strictly) better than $\beta$, then $\delta$ is (strictly) better than $\alpha$.

**EB1** If $\beta \in A$ and $\{\gamma \in A : \beta \notin A - \beta \wedge \gamma\} \nvdash \alpha$, then $\beta \in A - \alpha$.

The condition $\beta \notin A - \beta \wedge \gamma$ when $\beta, \gamma \in A$ can be seen as 'it is at least as easy to give up the belief $\beta$ as it is to give up $\gamma$'. Therefore the set $\{\gamma \in A : \beta \notin A - \beta \wedge \gamma\}$, for a non-tautological $\beta$ can be seen as the set of formulae that are at least as 'good' as $\beta$. Having this interpretation in mind, postulate *EB1* essentially states that if the subset of $A$ formed by the sentences which are at least as 'good' as $\beta$ does not imply $\alpha$, then $\beta$ is kept when contracting $A$ by $\alpha$.

**EB2** If $\beta \in A - \alpha$ then $\{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\} \vdash \alpha \vee \beta$.

We note that $\{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\}$ is the set of formulae of $A$ that are retained when a contraction by its conjunction with $\alpha$ occurs. To give up $\gamma \wedge \alpha$, either $\gamma$ or $\alpha$ (or both) must be removed. If $\gamma$ is kept, during the removal of $\gamma \wedge \alpha$ from $A$ this means that $\gamma$ is in some sense 'better' than $\alpha$. Hence, *EB2* can be read as follows: if $\beta$ is kept when contracting $A$ by $\alpha$, then the set formed by the formulae that are 'better' than $\alpha$ implies $\alpha \vee \beta$.

The following observation clarifies that in the presence of extensionality, *transitivity* is equivalent to *ST* and, furthermore, that *SST* follows from *transitivity* provided that some other postulates also hold.

Observation 3.1
Let $A$ be a belief base and $-$ be an operator on $A$. Then:

(a) If $-$ satisfies extensionality, then $-$ satisfies transitivity if and only if $-$ satisfies ST.
(b) If $-$ satisfies success, inclusion, extensionality, relative closure and transitivity, then it satisfies SST.

## 3.2 *Representation theorem*

Our main goal in the present section is to obtain an axiomatic characterization for the ensconcement-based contractions and our first step in that direction is to introduce the following condition which defines a binary relation on $A$ by means of a contraction function $-$ on $A$:

$$\alpha \preceq \beta \text{ if and only if } \alpha \notin A - \alpha \wedge \beta \text{ or } \vdash \alpha \wedge \beta. \qquad (\mathbf{C_{EB}} \preceq)$$

We note that this construction is similar to condition ($\mathbf{C}_{\leq}$) proposed by Gärdenfors and Makinson [12] to define an epistemic entrenchment relation by means of a given operator of belief set contraction.

The following theorem exposes that, provided that the contraction $-$ satisfies some of the postulates presented above, it holds that the binary relation $\preceq$ defined by condition ($\mathbf{C_{EB}} \preceq$) is an ensconcement relation on $A$ and, furthermore, that the operation $-$ is the $\preceq$-based contraction on $A$.

Theorem 3.2
Let $A$ be a belief base and $-$ an operator on $A$. If $-$ satisfies inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1 and EB2, then the binary relation $\preceq$ on $A$ defined by ($\mathbf{C_{EB}} \preceq$) is an ensconcement relation and $-$ satisfies (**EBC**).

The next result attests that an ensconcement-based contraction satisfies all the postulates mentioned in the previous theorem and that condition ($\mathbf{C_{EB}} \preceq$) holds whenever $(A, \preceq)$ is an ensconcement and $-$ is the $\preceq$-based contraction.

Theorem 3.3
Let $(A, \preceq)$ be an ensconcement and $-$ be the $\preceq$-based contraction on $A$. Then $-$ satisfies inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1 and EB2 as well as the condition ($\mathbf{C_{EB}} \preceq$).

We note that it follows immediately from the two previous theorems that the ensconcement-based contraction is axiomatically characterized by the postulates of *inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*.

We highlight that, since it follows from Observation 2.13 (b) and (c) that, in the presence of *inclusion* and *vacuity*, *disjunctive elimination* is equivalent to *logical relevance*, we can conclude that if *disjunctive elimination* is replaced by *logical relevance* in the axiomatization highlighted in the above paragraph, the resulting list of postulates consists also of an (alternative) axiomatic characterization of the ensconcement-based contraction.

At this point we must remark that in [6, Theorem 14] an axiomatic characterization for the ensconcement-based contractions was presented which consisted only of the postulates of *success*, *inclusion*, *vacuity*, *extensionality*, *conjunctive factoring* and *disjunctive elimination*. However, in personal communications, Zhiqiang Zhuang pointed out two gaps in the *From postulates to Ensconcement-based contraction* part of the proof of the mentioned theorem.[5] Taking Zhuang's comments into consideration we concluded that the axiomatic characterization exposed in [6, Theorem 14] was incomplete. In the subsequent search for a (correct) axiomatization of that class of functions we identified three postulates—namely, the postulates of *transitivity*, *EB1* and *EB2*—which, as attested in the above results, when added to the ones that were already included in the mentioned result of [6], give rise to a (complete) axiomatic characterization for the ensconcement-based contractions. Nevertheless, we notice that in Section 5 we will show that these three new postulates are redundant in the context of belief set contraction and, therefore, the postulates included in [6, Theorem 14] are enough to axiomatically characterize ensconcement-based contractions on belief sets.

## 4 Ensconcement-based versus brutal contraction

In this section, we will compare the two kinds of base contraction functions presented in [28], namely brutal contractions and ensconcement-based contractions. More precisely we will check which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by the brutal contractions functions and which are not and vice versa. We end this section by comparing two methods for constructing an ensconcement from a contraction operator. One of the methods is the one used in the previous section to define an ensconcement from an ensconcement-based contraction (condition ($\mathbf{C_{EB}}\preceq$)) and we will establish that it works also when the operation under consideration is a brutal contraction. The other method works if the operation is a brutal contraction but does not work if it is an ensconcement-based contraction.

We start by presenting an example that clarifies the difference between the definitions of brutal contraction and ensconcement-based contraction.

EXAMPLE 4.1
Let $\alpha$ and $\beta$ be logically independent sentences. Let $(A,\preceq)$ be an ensconcement where $A=\{\alpha\vee\beta,\beta,\beta\to\alpha\}$ and $\preceq$ be the three-level ensconcement relation on $A$ defined by: $\beta\prec\beta\to\alpha\prec\alpha\vee\beta$. Let $-$ be the $\preceq$-based contraction and $-^B$ be the $\preceq$-based brutal contraction on $A$. Hence $cut_{\prec}(\alpha)=\{\alpha\vee\beta\}$. Therefore, $A-^B\alpha=\{\alpha\vee\beta\}$ and $A-\alpha=\{\alpha\vee\beta,\beta\}$.

It is clear that given an ensconcement $(A,\preceq)$, if $-$ is the $\preceq$-based contraction and $-^B$ is the $\preceq$-based brutal contraction, then for all sentences $\alpha$: $A-^B\alpha\subseteq A-\alpha$.

[5]The gaps in that part of the mentioned proof that were identified by Zhiqiang Zhuang are the following:

- It is not shown there that the relation $\preceq$ defined from the contraction $-$ is transitive and total.
- In case 2.2.1 of the proof for **(ebc)** (which appears in [6, end of page 751]), after obtaining that $cut_{\prec}(\alpha)=\{\delta\in A: A-\alpha\wedge\delta\nvdash\alpha$ and $A-\alpha\wedge\delta\vdash\delta\}$ it is said that it follows (from this equality), by *conjunctive factoring* and *success*, that $cut_{\prec}(\alpha)=\{\delta\in A: A-\alpha\vdash\delta\}$; however this statement is not valid. It is true that it follows from $A-\alpha\wedge\delta\nvdash\alpha$ and $A-\alpha\wedge\delta\vdash\delta$, by *conjunctive factoring*, that $A-\alpha\wedge\delta=A-\alpha$ and, consequently, $\{\delta\in A: A-\alpha\wedge\delta\nvdash\alpha$ and $A-\alpha\wedge\delta\vdash\delta\}\subseteq\{\delta\in A: A-\alpha\vdash\delta\}$. Nevertheless, the converse inclusion does not hold in general. Indeed, consider the following counter-example (which was provided by Zhiqiang Zhuang): Let $A=\{\alpha\wedge\beta,\alpha,\beta,\alpha\vee\beta\}$, where $\alpha$ and $\beta$ are logically independent sentences. Consider the three-level ensconcement relation $\preceq$ on $A$ defined by: $\alpha\wedge\beta=_{\preceq}\beta\prec\alpha\prec\alpha\vee\beta$ and let $-$ be the $\preceq$-based contraction on $A$. Then, according to Definition 2.10, $A-\alpha=\{\beta,\alpha\vee\beta\}$ and $A-\alpha\wedge\beta=\{\alpha,\alpha\vee\beta\}$. Hence $\beta\in A$ and $A-\alpha\vdash\beta$, but $A-\alpha\wedge\beta\nvdash\beta$.

In the two following observations we expose which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by brutal contractions functions and which are not and vice versa.

OBSERVATION 4.2
Let $A$ be a belief base. If $-$ is a brutal contraction on $A$, then:

**(a)** $-$ satisfies success, inclusion, vacuity, extensionality, conjunctive factoring, transitivity, EB1 and EB2;
**(b)** $-$ in general does not satisfy disjunctive elimination.

OBSERVATION 4.3
Let $A$ be a belief base. If $-$ is an ensconcement-based contraction on $A$, then:

**(a)** $-$satisfies success, inclusion, vacuity, failure and relative closure;
**(b)** $-$ in general does not satisfy strong inclusion nor uniform behaviour.

We will now show how an ensconcement ordering on a belief base $A$, can be obtained from an operator of brutal contraction. More precisely, we will show that a slightly simplified version of condition ($\mathbf{C_{EB}}\preceq$) can be used for that purpose when the contraction $-$ under consideration is a brutal contraction.

OBSERVATION 4.4
If $-$ is a brutal contraction on a belief base $A$, then for $\alpha,\beta\in A$:

**(a)** $(\alpha\notin A-\alpha\wedge\beta$ or $\vdash\alpha\wedge\beta)$ if and only if $(\alpha\notin A-\beta$ or $\vdash\beta)$.
**(b)** $\alpha\notin A-\beta$ if and only if $(A-\beta\subseteq A-\alpha$ and $\nvdash\alpha)$.
**(c)** Condition ($\mathbf{C_{EB}}\preceq$) is equivalent to each one of the two following conditions:

$$\alpha\preceq\beta \text{ if and only if } \alpha\notin A-\beta \text{ or } \vdash\beta. \qquad (\mathbf{C_{BR}}\preceq)$$

$$\alpha\preceq\beta \text{ if and only if } (A-\beta\subseteq A-\alpha \text{ and } \nvdash\alpha) \text{ or } \vdash\beta. \qquad (\mathbf{C'_{BR}}\preceq)$$

The following observation is a natural consequence of combining the proof of Observation 2.12 (presented in [11]) with the previous result (this is explained with more detail in the proof for this observation that is provided in the Appendix).

OBSERVATION 4.5
Let $A$ be a belief base and $-$ an operator on $A$. If $-$ satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion and uniform behaviour, then the binary relation $\preceq$ on $A$ defined by ($\mathbf{C_{BR}}\preceq$) is an ensconcement relation and $-$ satisfies (**BC**).

The following observation states that if $-$ is a brutal contraction, then condition ($\mathbf{C_{EB}}\preceq$) is equivalent to condition ($\mathbf{C_{BR}}\preceq$) and defines an ensconcement relation on $A$ whereas, if $-$ is an ensconcement-based contraction on $A$, then in general condition ($\mathbf{C_{BR}}\preceq$) does not define an ensconcement relation on $A$.

OBSERVATION 4.6
Let $A$ be a belief base. Then:

**(a)** If an operation $-$ on $A$ is a brutal contraction, then condition ($\mathbf{C_{EB}}\preceq$) is equivalent to condition ($\mathbf{C_{BR}}\preceq$) and defines an ensconcement relation on $A$.

**(b)** If an operation $-$ on $A$ is an ensconcement-based contraction, then condition ($\mathbf{C_{BR}} \preceq$) in general does not define an ensconcement relation on $A$.

To finish this section we briefly summarize the main results here presented. Hence, according to Observations 4.2 and 4.3, the postulates of success, inclusion, vacuity, failure, relative closure, extensionality, conjunctive factoring, transitivity, EB1 and EB2 are satisfied by both ensconcement-based contractions and brutal contractions. Furthermore, those two results allow us to conclude that the properties that distinguish those two kinds of base contractions are disjunctive elimination, strong inclusion and uniform behaviour. More precisely, disjunctive elimination (or, equivalently, logical relevance) can be considered the main characteristic property of ensconcement-based contractions since it is the only postulate included in the axiomatic characterization that is not satisfied by the related operation of brutal contraction. Analogously, the postulates that can be considered characteristic properties of brutal contractions (in the sense that they are not satisfied by ensconcement-based contractions) are strong inclusion and uniform behaviour. On the other hand, the remaining results of the present section allow us to conclude that, while the binary relation defined by condition ($\mathbf{C_{EB}} \preceq$) is an ensconcement relation on $A$ whether the belief contraction $-$ there considered is an ensconcement-based contraction or a brutal contraction, conditions ($\mathbf{C_{BR}} \preceq$) and ($\mathbf{C'_{BR}} \preceq$) give rise to an ensconcement relation on $A$ when $-$ is a brutal contraction, but in general do not define an ensconcement relation if $-$ is an ensconcement-based contraction.

## 5 Connections between base contraction and belief set contraction

In this section, we study the interrelations among ensconcement-based contraction and epistemic entrenchment-based contractions and among brutal contractions and severe withdrawals.

We start by highlighting, in the two following observations, some interrelations among the postulates included in the axiomatizations of those four classes of contraction functions. The first of these observations exposes which of the belief set contraction postulates are enough to assure the fulfilment of some of the belief base contraction postulates, while the second one highlights which belief base contraction are enough to imply certain belief set contraction postulates.

Observation 5.1
Let $\mathbf{K}$ be a belief set and $\div$ be an operator on $\mathbf{K}$.

(a) If $\div$ satisfies ($\div$1), then it satisfies relative closure.
(b) If $\div$ satisfies ($\div$3), then it satisfies vacuity.
(c) If $\div$ satisfies ($\div$1) and ($\div$4), then it satisfies success.
(d) If $\div$ satisfies ($\div$1) and ($\div$9), then it satisfies strong inclusion.
(e) If $\div$ satisfies ($\div$1), ($\div$2), ($\div$3) and ($\div$5), then $\div$ satisfies disjunctive elimination.
(f) If $\div$ satisfies ($\div$1), ($\div$2), ($\div$3), ($\div$4), ($\div$5), ($\div$6), and ($\div V$), then $\div$ satisfies transitivity.
(g) If $\div$ satisfies ($\div$1), ($\div$3), ($\div$4) and ($\div V$), then $\div$ satisfies EB1.
(h) If $\div$ satisfies ($\div$1), ($\div$2) and ($\div$6), then $\div$ satisfies EB2.
(i) $\div$ satisfies uniform behaviour.

Observation 5.2
Let $A$ be a belief base and $-$ be an operator on $A$.

(a) If $-$ satisfies success, then it satisfies ($\div$4).
(b) If $A$ is logically closed and $-$ satisfies inclusion and vacuity, then it satisfies ($\div$3).
(c) If $A$ is logically closed and $-$ satisfies inclusion and relative closure, then $-$ satisfies ($\div$1).

(d) If $A$ is logically closed and $-$ satisfies inclusion and disjunctive elimination, then $-$ satisfies $(\div 1)$.
(e) If $A$ is logically closed and $-$ satisfies inclusion, vacuity and disjunctive elimination, then $-$ satisfies $(\div 5)$.
(f) If $A$ is logically closed and $-$ satisfies inclusion, relative closure and strong inclusion, then $-$ satisfies $(\div 9)$.

Using the results presented above, we can easily identify which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by epistemic entrenchment-based contractions and which are not and vice–versa, as we expose below.

The most significant conclusion that follows from the two previous observations is that, in the context of belief set contraction, the class of epistemic entrenchment contractions coincides with the class of ensconcement-based contractions. This fact is formally stated in the following theorem, which highlights also some new axiomatic characterizations for the epistemic entrenchment-based contractions and for the ensconcement-based contractions on belief sets.

THEOREM 5.3
Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$. The following statements are equivalent:

(1) $\div$ is an epistemic entrenchment-based contraction on $\mathbf{K}$.
(2) $\div$ is an ensconcement-based contraction on $\mathbf{K}$.
(3) $\div$ satisfies the postulates $(\div 1)$–$(\div 6)$ and $(\div V)$.
(4) $\div$ satisfies the postulates of *inclusion, vacuity, success, extensionality, conjunctive factoring* and *disjunctive elimination*.

We notice that it follows from the above theorem that every epistemic entrenchment-based contraction satisfies all the postulates included in the axiomatic characterization of ensconcement-based contractions obtained in Subsection 3.2 and, moreover, that the postulates of *transitivity, EB1* and *EB2* are redundant in that axiomatic characterization if the set on which the contraction is defined is logically closed. Conversely, according to the above result it holds also that every ensconcement-based contraction on a belief set satisfies the postulates $(\div 1)$–$(\div 6)$ and $(\div V)$. Nevertheless, this is not the case regarding ensconcement-based contraction on a belief base that is not logically closed, as we clarify in the following observation.

OBSERVATION 5.4
Let $A$ be a belief base and $-$ be an ensconcement-based contraction on $A$. Then

(a) $-$ satisfies postulates $(\div 2)$, $(\div 4)$, $(\div 6)$ and $(\div V)$.
(b) If $A$ is not logically closed, then $-$ in general does not satisfy $(\div 1)$ nor $(\div 3)$ nor $(\div 5)$.

Above we have concluded that the class of epistemic entrenchment-based contractions and the class of ensconcement-based contractions on belief sets are identical, by means of a comparison among axiomatic characterizations. However, this fact can be proven directly as we expose in the two following observations.

From the following observation we can conclude in a direct way that every epistemic entrenchment-based contraction is an ensconcement-based contraction. We notice that this result is essentially based on the fact that, according to Observation 2.7, the restriction of an epistemic entrenchment relation to a belief set is an ensconcement relation. Therefore, any given epistemic entrenchment relation can be used to define both an epistemic entrenchment-based contraction and an ensconcement-based contraction.

Observation 5.5
Let $\mathbf{K}$ be a belief set. Let $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and $\div$ be the $\leq$-based contraction on $\mathbf{K}$ defined by condition ($\mathbf{C}_{\div_{\leq}}$). Let $-$ be the ensconcement-based contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$ by means of condition (**EBC**). Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.

Conversely, the following observation allows us to conclude in a direct way that every ensconcement-based contraction on a belief set is an epistemic entrenchment-based contraction.[6]

Observation 5.6
Let $\mathbf{K}$ be a belief set. Let $(\mathbf{K}, \preceq)$ be an ensconcement and $-$ be the $\preceq$-based contraction on $\mathbf{K}$ defined by condition (**EBC**). Let $\leq_{\preceq}$ be the epistemic entrenchment related to $Cn(A)$ defined from $\preceq$ as exposed in Observation 2.8 and $\div$ be the epistemic entrenchment-based contraction on $\mathbf{K}$ defined from $\leq_{\preceq}$ by means of condition ($\mathbf{C}_{\div_{\leq}}$). Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.

A comparison between brutal contractions and severe withdrawals analogous to the one that was presented above regarding ensconcement-based contractions and epistemic entrenchment-based contractions is in order.

The main conclusion, concerning the interrelation among brutal contractions and severe withdrawals, that follows from Observations 5.1 and 5.2 is that the class of severe withdrawals coincides with the class of brutal contractions on belief sets. This fact as well as some axiomatic characterizations for the above mentioned class of contractions are formally presented in the following theorem.

Theorem 5.7
Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$. The following statements are equivalent:

(1) $\div$ is a severe withdrawal on $\mathbf{K}$.
(2) $\div$ is a brutal contraction on $\mathbf{K}$.
(3) $\div$ satisfies the postulates ($\div$1), ($\div$2), ($\div$3), ($\div 3'$), ($\div$4) and ($\div$9).
(4) $\div$ satisfies the postulates of *relative closure, inclusion, vacuity, failure, success* and *strong inclusion*.

It follows immediately from the above theorem (and also from Observation 5.1) that severe withdrawals satisfy all the postulates included in the axiomatic characterization of brutal contractions presented in Observation 2.12 (including the postulate of *uniform behaviour*, which is not necessary for the axiomatic characterization of brutal contractions on belief sets). Conversely, every brutal contraction on a belief set satisfies all the postulates included in the axiomatic characterization of severe withdrawals presented in Observation 2.6. However, brutal contractions on sets that are not logically closed, in general do not satisfy all those postulates, as we clarify in the following observation.

Observation 5.8
Let $A$ be a belief base and $-$ be a brutal contraction on $A$.

(a) $-$ satisfies postulates ($\div$2), ($\div 3'$) and ($\div$4).
(b) If $A$ is not logically closed, then $-$ in general does not satisfy ($\div$1) nor ($\div$3) nor ($\div$9).

The two following observations show in a direct way (rather than by means of a comparison among axiomatizations) that every brutal contraction on a belief set is a severe withdrawal and

[6] We notice that, however this result follows immediately from [28, Theorem 14], we provide a direct proof for it in the Appendix.

vice versa, by means of a procedure analogous to the one that used above to prove explicitly that a contraction function on a belief set is an epistemic entrenchment-based contraction if and only if it is an ensconcement-based contraction.

OBSERVATION 5.9
Let $\mathbf{K}$ be a belief set. Let $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and $\div$ be the $\leq$-based severe withdrawal on $\mathbf{K}$ defined by condition ($\mathbf{R}_{\div_{\leq}}$). Let $-$ be the brutal contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$ by means of condition (**BC**). Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.

OBSERVATION 5.10
Let $\mathbf{K}$ be a belief set. Let $(\mathbf{K}, \preceq)$ be an ensconcement and $-$ be the $\preceq$-based brutal contraction on $\mathbf{K}$ defined by condition (**BC**). Let $\leq_{\preceq}$ be the epistemic entrenchment related to $Cn(A)$ defined from $\preceq$ as exposed in Observation 2.8 and $\div$ be the severe withdrawal on $\mathbf{K}$ defined from $\leq_{\preceq}$ by means of condition ($\mathbf{R}_{\div_{\leq}}$). Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.

To close this section, we remark that the main conclusion that raises from the results presented here is the fact that in the belief sets context the class of ensconcement-based contractions coincides with the class of epistemic entrenchment-based contractions as well as the class of brutal contractions is identical to the class of severe withdrawals. These facts, together with the results of Subsection 2.3, contribute to assert that Williams' concept of ensconcement relation is an adequate generalization to the context of belief bases of Gärdenfors and Makinson's notion of epistemic entrenchment relation (on a belief set).

## 6 Conclusion

We have presented, in Section 3 an axiomatic characterization of the ensconcement-based contractions. An axiomatic characterization for this class of functions had already been presented in [6, Theorem 14], however, in personal communications with the first and third authors of this article, Zhiqiang Zhuang pointed out a couple of gaps in the proof of that theorem. The investigation reported in this article led to the addition of three axioms to the ones included in the mentioned theorem to obtain a complete axiomatization for the ensconcement-based contractions. However, we have also shown that those three postulates are redundant when considering only ensconcement-based contractions on belief sets. The representation theorem obtained in this article opens up the possibility of comparing the ensconcement-based contractions with some other well-know contraction operations through an axiomatic perspective. In particular, in Section 4, we have made such a comparison among the ensconcement-based contractions and the brutal contractions. That comparison allows us to conclude, in particular, that neither one of those two classes of functions contains the other. We have also presented a method for defining an ensconcement relation by means of either an ensconcement-based or a brutal contraction operation and we have show that, given a contraction of either one of those two types, the ensconcement thus obtained coincides exactly with the ensconcement on which the operation under consideration is based. Furthermore, we have also proven that, when considering only brutal contractions, the mentioned construction of an ensconcement relation from a contraction operation can be significantly simplified. Finally, in Section 5, we have presented some results relating base contraction postulates and belief set contraction postulates and we have investigated the connections between the ensconcement-based contractions and the epistemic entrenchment-based contractions as well as between the brutal contractions and the severe withdrawals. The main conclusion that we have achieved in this regard was that a contraction function on a belief set is an ensconcement-based contraction (respectively, a brutal contraction) if and only if it is an epistemic entrenchment-based contraction (respectively, a severe withdrawal).


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## A Appendix: Proofs

### *A.1 Previous lemmas*

LEMMA 1 ([6])

- **(a)** If $\vdash \alpha$, then $cut_{\prec}(\alpha)=\emptyset$.
- **(b)** If $\nvdash \alpha, cut_{\prec}(\alpha)\nvdash \alpha$.
- **(c)** If $A\nvdash \alpha, cut_{\prec}(\alpha)=A$.
- **(d)** If $\beta\vdash \alpha$, then $cut_{\prec}(\alpha)\subseteq cut_{\prec}(\beta)$.
- **(e)** If $\vdash \alpha\leftrightarrow\beta$, then $cut_{\prec}(\alpha)=cut_{\prec}(\beta)$.
- **(f)** If $\alpha\preceq\beta$, then $cut_{\prec}(\beta)\subseteq cut_{\prec}(\alpha)$.
- **(g)** If $\alpha\prec\beta$, then $cut_{\prec}(\alpha)\vdash\beta$ and $cut_{\prec}(\beta)\nvdash\alpha$.
- **(h)** If $\alpha\prec\beta$, then $cut_{\prec}(\alpha\wedge\beta)=cut_{\prec}(\alpha)$.
- **(i)** If $\beta =_{\preceq} \alpha$, then $cut_{\prec}(\alpha\wedge\beta)=cut_{\prec}(\alpha)=cut_{\prec}(\beta)$.
- **(j)** If $cut_{\prec}(\alpha)\vdash\beta$, then $cut_{\prec}(\alpha\wedge\beta)=cut_{\prec}(\alpha)$.

**(k)** If $cut_{\prec}(\alpha) \nvdash \beta$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$.

LEMMA 2 ([11])
Let $A$ be a belief base and $-$ an operator on $A$ that satisfies success and strong inclusion. Then $-$ satisfies: If $\nvdash \alpha$, then $A - \alpha \subseteq A - (\alpha \wedge \beta)$.

LEMMA 3
Let $(A, \preceq)$ be an ensconcement. Let $-$ be the $\preceq$-based contraction on $A$. Then:

- **(a)** $cut_{\prec}(\alpha) \subseteq A - \alpha$.
- **(b)** If $A - \alpha \nvdash \beta$, then $cut_{\prec}(\alpha) \nvdash \beta$.
- **(c)** If $A - \alpha \vdash \beta$, then $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$.
- **(d)** If $\beta \in cut_{\prec}(\alpha)$, then $\beta \in A - \alpha \wedge \beta$.

PROOF.

- **(a)** Let $\beta \in cut_{\prec}(\alpha)$. It follows that $\beta \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Thus, by definition of $\preceq$-based contraction, $\beta \in A - \alpha$.
- **(b)** It follows trivially from (a).
- **(c)** It is trivial if $\vdash \beta$ or $\vdash \alpha$. Assume now that $\nvdash \beta$ and $\nvdash \alpha$. From $A - \alpha \vdash \beta$ by compactness there exists a finite subset of $A - \alpha$, $H = \{\gamma_1, \ldots, \gamma_k\}$, such that $H \vdash \beta$. It follows, by definition of $\preceq$-based contraction, that for each $\gamma_i \in H$, $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma_i$. Hence $cut_{\prec}(\alpha) \vdash (\alpha \vee \gamma_1) \wedge \ldots \wedge (\alpha \vee \gamma_k)$. Therefore, $cut_{\prec}(\alpha) \vdash \alpha \vee (\gamma_1 \wedge \ldots \wedge \gamma_k)$, from which it follows that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$.
- **(d)** Let $\beta \in cut_{\prec}(\alpha)$. If $\vdash \beta$, then $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \beta$. Hence, $\beta \in A - \alpha \wedge \beta$. Consider now that $\nvdash \beta$. Assume by *reduction ad absurdum* that $\beta \notin A - \alpha \wedge \beta$. Hence, by Definition 2.10 it follows that $cut_{\prec}(\alpha \wedge \beta) \nvdash \beta$ and by Lemma 1 (k) and (e) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$. We will consider two cases:

  Case 1) $A - \alpha \wedge \beta \vdash \alpha$. Hence, by (c), $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha$. Thus $cut_{\prec}(\beta) \vdash \alpha$. Therefore, by Observation 2.9, $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$. Hence $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$, from which it follows that $\beta \notin cut_{\prec}(\alpha)$. Contradiction.
  Case 2) $A - \alpha \wedge \beta \nvdash \alpha$. It follows from (b) that $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$. Hence, by Lemma 1 (k) and (e), $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. Therefore $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$. Hence $\beta \in cut_{\prec}(\beta)$ which contradicts Lemma 1 (b). ■

LEMMA 4
Let $A$ be a belief base and $-$ an operator on $A$ that satisfies success, inclusion, vacuity, failure, relative closure and strong inclusion. Then $-$ satisfies: If $\alpha \in A - \beta$ and $\nvdash \beta$, then $\beta \notin A - \alpha \wedge \beta$.

PROOF. Let $-$ be an operator on $A$ that satisfies *success, inclusion, vacuity, failure, relative closure* and *strong inclusion*. Then by Observation 2.13 (d) $-$ satisfies *extensionality* and *expulsiveness* and by Observation 2.13 (f) $-$ satisfies *decomposition*. Let $\alpha \in A - \beta$ and $\nvdash \beta$. If $\vdash \alpha$, then $A - \alpha \wedge \beta = A - \beta$, by *extensionality*. Hence, by *success* $\beta \notin A - \alpha \wedge \beta$. Consider now that $\nvdash \alpha$ and assume by *reduction ad absurdum* that $\beta \in A - \alpha \wedge \beta$. By *decomposition* and *success*, it follows that $A - \alpha \wedge \beta = A - \alpha$. Thus $\beta \in A - \alpha$. Contradiction, by *expulsiveness*. ■

LEMMA 5 ([4, 19])
Let $\mathbf{K}$ be a belief set and $\div$ be an operator on $\mathbf{K}$. Then:

(a) If $\div$ satisfies relevance, then it satisfies ($\div$5).
(b) If $\div$ satisfies ($\div$1), ($\div$2), ($\div$3), and ($\div$5), then it satisfies relevance.

LEMMA 6 ([23])
Let $\mathbf{K}$ be a belief set, and $\div$ be an operation on $\mathbf{K}$ that satisfies ($\div$1), ($\div$2), ($\div$3), ($\div$4), ($\div$5) and ($\div$6). Then $\div$ satisfies ($\div$7) if and only if $\div$ satisfies the following postulate:
**(Conjunctive trisection)** If $\alpha \in \mathbf{K} \div (\alpha \wedge \beta)$, then $\alpha \in \mathbf{K} \div (\alpha \wedge \beta \wedge \delta)$.

LEMMA 7 ([6])
Let $\mathbf{K}$ be a belief set and $-$ be an operator on $\mathbf{K}$ that satisfies inclusion, vacuity and disjunctive elimination. Then $-$ satisfies relevance.

LEMMA 8 ([12, Lemma 3 - (i)])
If the relation $\leq$ satisfies (EE1), (EE2) and (EE3), then it is a total relation i.e., either $\alpha \leq \beta$ or $\beta \leq \alpha$.

LEMMA 9 ([19, Observation 2.49$-$1])
If $\mathbf{K}$ is a consistent belief set and $\leq$ is a total relation on $\mathbf{K}$ that satisfies (EE1) and (EE4), then: If $\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}$, then $\alpha < \beta$.

LEMMA 10 ([7])
Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE5). If $\alpha \in Cn(\emptyset)$, then for all $\beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\beta < \alpha$.

LEMMA 11
Let $(A, \preceq)$ be an ensconcement. If $\alpha \in A \setminus Cn(\emptyset)$, then $cut_{\preceq}(\alpha) = \{\gamma \in A : \alpha \preceq \gamma\}$.

PROOF. Let $\beta \in cut_{\preceq}(\alpha)$. Hence $\{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha$. Since $\alpha, \beta \in A$ and $\preceq$ is a total relation on $A$, it follows that, $\alpha \preceq \beta$ or $\beta \prec \alpha$. In the latter case, it follows that $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$. Hence $\alpha \preceq \beta$.
Let $\beta \in \{\gamma \in A : \alpha \preceq \gamma\}$. Hence $\alpha \preceq \beta$. Since $(A, \preceq)$ is an ensconcement, by ($\preceq$1) it follows that $\{\gamma \in A : \alpha \prec \gamma\} \nvdash \alpha$. Hence $\{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha$. Therefore $\beta \in cut_{\preceq}(\alpha)$. ■

## *A.2 Proofs for the theorems and observations*

PROOF OF OBSERVATION 2.7. To prove that $(\mathbf{K}, \leq|_{\mathbf{K}})$ is an ensconcement we must show that $\leq|_{\mathbf{K}}$ is a transitive and total relation on $\mathbf{K}$ that satisfies ($\preceq$1), ($\preceq$2) and ($\preceq$3).

According to (EE1) $\leq$ is a transitive relation and it follows from (EE1), (EE2) and (EE3) that $\leq$ is a total relation (Lemma 8). Thus $\leq|_{\mathbf{K}}$ is a transitive and total relation on $\mathbf{K}$.
($\preceq$1) Let $\beta \in \mathbf{K} \setminus Cn(\emptyset)$. Assume by *reduction ad absurdum* that $\{\alpha \in \mathbf{K} : \beta <|_{\mathbf{K}} \alpha\} \vdash \beta$. Hence $\{\alpha \in \mathbf{K} : \beta < \alpha\} \vdash \beta$. By compactness, and since $\nvdash \beta$, it follows that there exists a non-empty finite subset $A' = \{\alpha_1, \ldots, \alpha_n\}$ of $\{\alpha \in \mathbf{K} : \beta < \alpha\}$ such that $A' \vdash \beta$. Hence $\alpha_1 \wedge \ldots \wedge \alpha_n \vdash \beta$, from which it follows by (EE2) that $\alpha_1 \wedge \ldots \wedge \alpha_n \leq \beta$. It follows from (EE3) and (EE1) that there exists $\alpha_i \in A'$ such that $\alpha_i \leq \alpha_1 \wedge \ldots \wedge \alpha_n$. Thus, from (EE1) $\alpha_i \leq \beta$. Contradiction.
($\preceq$2) Let $\alpha, \beta \in \mathbf{K}$ be such that $\nvdash \alpha$ and $\vdash \beta$. It follows from (EE2) that $\alpha \leq \beta$. Thus $\alpha \leq|_{\mathbf{K}} \beta$. Assume, by *reduction ad absurdum*, that $\beta \leq|_{\mathbf{K}} \alpha$. Hence $\beta \leq \alpha$. Let $\theta \in \mathcal{L}$, then by (EE2) $\theta \leq \beta$. Hence by (EE1) $\theta \leq \alpha$. Hence for all $\delta \in \mathcal{L}$, $\delta \leq \alpha$. Therefore, by (EE5), it holds that $\vdash \alpha$. Contradiction. Thus $\alpha <|_{\mathbf{K}} \beta$.
($\preceq$3) Let $\alpha, \beta \in \mathbf{K}$ be such that $\vdash \alpha$ and $\vdash \beta$. Hence, by (EE2), $\alpha \leq \beta$. Therefore $\alpha \leq|_{\mathbf{K}} \beta$. ■

PROOF OF OBSERVATION 2.13.

**(b)** Let $\beta \in A$ and $\beta \notin A - \alpha$. Then, by *logical relevance*, there is some set $A'$ such that $A - \alpha \subseteq A' \subseteq Cn(A)$ and $A' \nvdash \alpha$ but $A' \cup \{\beta\} \vdash \alpha$. From $A' \cup \{\beta\} \vdash \alpha$ it follows, by deduction, that $A' \vdash \beta \rightarrow \alpha$.

Hence $A' \vdash \neg\beta \vee \alpha$. If it was the case that $A' \vdash \alpha \vee \beta$ it would follow that $A' \vdash \alpha$. Hence $A' \nvdash \alpha \vee \beta$. Therefore, by monotony, $A - \alpha \nvdash \alpha \vee \beta$.

**(c)** Let $\beta \in A$, $\beta \notin A - \alpha$ and consider $A' = A - \alpha \cup \{\neg\beta \vee \alpha\}$. From $\beta \in A$ and $\beta \notin A - \alpha$ it follows, by *vacuity*, that $A \vdash \alpha$. Hence $\neg\beta \vee \alpha \in Cn(A)$. By *inclusion*, $A - \alpha \subseteq A \subseteq Cn(A)$. Therefore $A' \subseteq Cn(A)$. On the other hand, since $\neg\beta \vee \alpha \in A'$, it follows that $A' \cup \{\beta\} \vdash \alpha$. It remains to prove that $A' \nvdash \alpha$. Assume by *reduction ad absurdum* that $A' \vdash \alpha$. Hence, by deduction, it follows that $A - \alpha \vdash (\neg\beta \vee \alpha) \rightarrow \alpha$. Since $(\neg\beta \vee \alpha) \rightarrow \alpha$ is logically equivalent to $\alpha \vee \beta$, it follows that $A - \alpha \vdash \alpha \vee \beta$ which contradicts *disjunctive elimination*. Hence $A' \nvdash \alpha$.

**(f)** **Decomposition:** We will prove by cases:

Case 1) $\vdash \alpha \wedge \beta$. Follows trivially from *failure*.
Case 2) $\vdash \alpha$ and $\nvdash \beta$. It follows from *success* that $A - (\alpha \wedge \beta) \nvdash \beta$. Then, by Lemma 2 and *conjunctive inclusion* (Observation 2.13 (e)) it follows that $A - \beta = A - (\alpha \wedge \beta)$.
Case 3) $\nvdash \alpha$ and $\vdash \beta$. Due to the symmetry of the case, it follows that $A - \alpha = A - (\alpha \wedge \beta)$.

Case 4) $\nvdash \alpha$ and $\nvdash \beta$. It follows from Lemma 2 that $A - \alpha \subseteq A - (\alpha \wedge \beta)$ and $A - \beta \subseteq A - (\alpha \wedge \beta)$. On the other hand, by *success*, it follows that $A - (\alpha \wedge \beta) \nvdash \alpha$ or $A - (\alpha \wedge \beta) \nvdash \beta$. Then by *conjunctive inclusion*, $A - (\alpha \wedge \beta) \subseteq A - \alpha$ or $A - (\alpha \wedge \beta) \subseteq A - \beta$. Hence, $A - (\alpha \wedge \beta) = A - \alpha$ or $A - (\alpha \wedge \beta) = A - \beta$.

**Conjunctive factoring:** Follows trivially by *decomposition*.
**Uniformity:** Let $\alpha$ and $\beta$ be two sentences such that it holds for all subsets $A'$ of $A$ that $A' \vdash \alpha$ if and only if $A' \vdash \beta$.

Case 1) $\vdash \alpha$ and $\vdash \beta$. It follows trivially from *failure* that $A - \alpha = A - \beta$.
Case 2) $\nvdash \alpha$ and $\nvdash \beta$. It follows from *inclusion* that $A - \alpha \subseteq A$ and $A - \beta \subseteq A$. By *success* it follows that $A - \alpha \nvdash \alpha$. Then, by hypothesis, $A - \alpha \nvdash \beta$. By symmetry of the case it follows that $A - \beta \nvdash \alpha$. Therefore from *strong inclusion* it follows that $A - \alpha = A - \beta$. ■

PROOF OF OBSERVATION 3.1.

**(a)** Assume that $-$ is an operator on $A$ that satisfies *transitivity* we will show that $-$ satisfies *ST*. Let $\delta \in A$, $\beta \in A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Assume by *reduction ad absurdum* that $\delta \notin A - \alpha \wedge \delta$, then by *transitivity* $\beta \notin A - \alpha \wedge \beta$. Contradiction. Hence $\delta \in A - \alpha \wedge \delta$.
Assume now that $-$ is an operator on $A$ that satisfies *ST* we will show that $-$ satisfies *transitivity*. Let $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Assume by *reduction ad absurdum* that $\alpha \in A - \alpha \wedge \delta$, then by *ST* $\beta \in A - \beta \wedge \delta$. Contradiction. Hence $\alpha \notin A - \alpha \wedge \delta$.

**(b)** Let $\alpha \in A$, $\beta \in A - \alpha \wedge \beta$ and $\delta \in A - \beta \wedge \delta$. From *inclusion* it follows that $\beta, \delta \in A$ and from (a) it follows that $-$ satisfies *ST*. We will prove by cases:

Case 1) $\vdash \alpha$. From $\beta \in A - \alpha \wedge \beta$ it follows, by *extensionality*, that $\beta \in A - \beta$. Thus, by *success*, $\vdash \beta$. Therefore, by *extensionality* and *success*, and proceeding as before, it follows from $\delta \in A - \beta \wedge \delta$ that $\vdash \delta$. Thus, by *relative closure*, $\delta \in A - \alpha \wedge \delta$.
Case 2) $\nvdash \alpha$. Then $\nvdash \alpha \wedge \beta$. From $\beta \in A - \alpha \wedge \beta$, it follows, by *success*, that $\alpha \notin A - \alpha \wedge \beta$. Assume, by *reduction ad absurdum*, that $\delta \notin A - \alpha \wedge \delta$. It follows, by *ST*, that $\alpha \in A - \alpha \wedge \beta$. Contradiction. Hence $\delta \in A - \alpha \wedge \delta$.
■

PROOF OF THEOREM 3.2. Let $-$ be an operator to $A$ that satisfies *success, inclusion, vacuity, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*. We will

start by proving that $\preceq$, defined by ($\mathbf{C_{EB}} \preceq$), is an ensconcement relation.
**($\preceq$ is total)** Let $\alpha, \beta \in A$ be such that $\alpha \not\preceq \beta$. By ($C_{EB} \preceq$), it follows that $\alpha \in A - \alpha \wedge \beta$ and $\nvdash \alpha \wedge \beta$. It follows, by *success*, that $\beta \notin A - \alpha \wedge \beta$. Therefore $\beta \preceq \alpha$, by ($\mathbf{C_{EB}} \preceq$).
**($\preceq$ is transitive)** Let $\alpha, \beta, \delta \in A$ be such that $\alpha \preceq \beta$ and $\beta \preceq \delta$. We wish to prove that $\alpha \preceq \delta$. It follows trivially by ($\mathbf{C_{EB}} \preceq$) if $\vdash \alpha \wedge \delta$. Assume now that $\nvdash \alpha \wedge \delta$. From $\alpha \preceq \beta$ and $\beta \preceq \delta$ it follows, by ($\mathbf{C_{EB}} \preceq$), that ($\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$) and ($\beta \notin A - \beta \wedge \delta$ or $\vdash \beta \wedge \delta$). By *relative closure* (Observation 2.13 (a)) there are only two possible cases to consider:

Case 1) $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Hence, by *transitivity*, $\alpha \notin A - \alpha \wedge \delta$. Therefore, by ($C_{EB} \preceq$), $\alpha \preceq \delta$.
Case 2) $\alpha \notin A - \alpha \wedge \beta$ and $\vdash \beta \wedge \delta$. It follows, by *failure* (Observation 2.13 (a)), that $\nvdash \alpha$ and, by *extensionality*, that $A - \alpha \wedge \delta = A - \alpha$. Thus, by *success*, $\alpha \notin A - \alpha \wedge \delta$. Therefore $\alpha \preceq \delta$, by ($\mathbf{C_{EB}} \preceq$).
**($\preceq$1)** Let $\gamma \in A \setminus Cn(\emptyset)$ and let $H = \{\alpha \in A : \gamma \prec \alpha\}$. We will show that $H \subseteq A - \gamma$. Let $\alpha \in H$. If $\vdash \alpha$, then by *relative closure* $\alpha \in A - \gamma$. Assume now that $\nvdash \alpha$. From $\alpha \in H$ it follows that $\gamma \prec \alpha$. By ($\mathbf{C_{EB}} \preceq$) this means that:
($\gamma \notin A - \alpha \wedge \gamma$ or $\vdash \alpha \wedge \gamma$) and ($\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$). This condition holds if and only if ($\gamma \notin A - \alpha \wedge \gamma$, $\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$) or ($\vdash \alpha \wedge \gamma$, $\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$). In the latter case we have a contradiction. From the former, and since $\nvdash \alpha$, it follows, by *conjunctive factoring* and *success* that $A - \alpha \wedge \gamma = A - \gamma$. Hence $\alpha \in A - \gamma$. Therefore $H \subseteq A - \gamma$ and thus $H \nvdash \gamma$, by *success*.
**($\preceq$2)** Let $\alpha, \beta \in A$, $\nvdash \alpha$ and $\vdash \beta$. We wish to prove that $\alpha \prec \beta$. From $\vdash \beta$, it follows by *extensionality* that $A - \alpha \wedge \beta = A - \alpha$. Hence, by *success*, $\alpha \notin A - \alpha \wedge \beta$. Therefore $\alpha \preceq \beta$, by ($\mathbf{C_{EB}} \preceq$). It remains to prove that $\beta \not\preceq \alpha$. Assume by *reduction ad absurdum* that $\beta \preceq \alpha$. By ($\mathbf{C_{EB}} \preceq$), this means that $\beta \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$. The latter contradicts $\nvdash \alpha$ and the former contradicts *relative closure*. Hence $\beta \not\preceq \alpha$, from which it follows that $\alpha \prec \beta$.
**($\preceq$3)** Follows trivially by ($\mathbf{C_{EB}} \preceq$).
**(EBC)** It remains to show that $-$ satisfies (**EBC**), i.e., that $A - \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}$.
We will prove by cases:
1. $\vdash \alpha$. Follows trivially by *failure*.
2. $\nvdash \alpha$
2.1 $A \nvdash \alpha$. By *vacuity* and *inclusion* it follows that $A - \alpha = A$. On the other hand, by Lemma 1 (c), $cut_{\prec}(\alpha) = A$, from which it follows that $\{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\} = A$.
2.2 $A \vdash \alpha$.
We will prove (**EBC**), i.e. $A - \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}$, by double inclusion.
($\subseteq$) Let $\beta \in A - \alpha$. It follows, from *inclusion* that $\beta \in A$. We intend to prove that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. It is trivial if $\vdash \alpha \vee \beta$. Consider now that $\nvdash \alpha \vee \beta$ and assume by *reduction ad absurdum* that $cut_{\prec}(\alpha) \nvdash \alpha \vee \beta$. It follows that $\{\delta \in A : \{\gamma \in A : \delta \preceq \gamma\} \nvdash \alpha\} \nvdash \alpha \vee \beta$. From ($\mathbf{C_{EB}} \preceq$) it holds that $Z \nvdash \alpha \vee \beta$ where $Z = \{\delta \in A : \{\gamma \in A : \delta \notin A - \delta \wedge \gamma$ or $\vdash \delta \wedge \gamma\} \nvdash \alpha\}$. According to *EB2*, $Y \vdash \alpha \vee \beta$, where $Y = \{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\}$. Let $\theta \in Y$. Hence $\theta \in A - \theta \wedge \alpha$. We will prove that $\theta \in Z$ i.e., that $W \nvdash \alpha$ where $W = \{\gamma \in A : \theta \notin A - \theta \wedge \gamma$ or $\vdash \theta \wedge \gamma\}$. Let $\lambda \in W$. Hence $\lambda \in A$ and $\theta \notin A - \theta \wedge \lambda$ or $\vdash \theta \wedge \lambda$.
Now we will show that, $\lambda \in A - \alpha$. If $\vdash \lambda$ then, by *relative closure*, $\lambda \in A - \alpha$. Assume now that $\nvdash \lambda$. Hence, since $\vdash \theta \wedge \lambda$ can not hold, it follows that $\theta \notin A - \theta \wedge \lambda$, and thus $\nvdash \theta$ by *failure*. To prove that $\lambda \in A - \alpha$ it is enough to show, by *EB1*, that $X = \{\gamma \in A : \lambda \notin A - \lambda \wedge \gamma\} \nvdash \alpha$. We will show that $X \subseteq A - \alpha$. If $\delta \in X$ is such that $\vdash \delta$, then, by *relative closure*, $\delta \in A - \alpha$.
For all $\delta \in X \setminus Cn(\emptyset)$ it follows that $\lambda \notin A - \lambda \wedge \delta$. By *transitivity* and *ST* (Observation 3.1) it follows, from $\theta \notin A - \lambda \wedge \theta$, $\lambda \notin A - \lambda \wedge \delta$ and $\theta \in A - \theta \wedge \alpha$, that $\delta \in A - \alpha \wedge \delta$. It follows, by *success* and *conjunctive factoring*, that $\delta \in A - \alpha$. Hence $X \subseteq A - \alpha$. Thus, by *success*, $X \nvdash \alpha$.
Therefore, $\lambda \in A - \alpha$. Hence $W \subseteq A - \alpha$. By *success* it follows that $W \nvdash \alpha$. Hence $Y \subseteq Z$. Contradiction,

since $Y \vdash \alpha \vee \beta$ and $Z \nvdash \alpha \vee \beta$.
($\supseteq$) We will start by proving that $cut_{\prec}(\alpha) \subseteq A - \alpha$. Let $\delta \in cut_{\prec}(\alpha)$. If $\vdash \delta$, then $\delta \in A - \alpha$ by *relative closure*. Consider now that $\nvdash \delta$ and assume by *reduction ad absurdum* that $\delta \notin A - \alpha$. Hence, by *EB1*, it follows that $\{\gamma \in A : \delta \notin A - \delta \wedge \gamma\} \vdash \alpha$. Let $\psi \in A$ be such that $\delta \notin A - \delta \wedge \psi$. Thus, by ($\mathbf{C_{EB}} \preceq$), $\delta \preceq \psi$. Therefore $\{\gamma \in A : \delta \notin A - \delta \wedge \gamma\} \subseteq \{\gamma \in A : \delta \preceq \gamma\}$. It follows that $\{\gamma \in A : \delta \preceq \gamma\} \vdash \alpha$. Hence $\delta \notin cut_{\prec}(\alpha)$. Contradiction.
Hence $cut_{\prec}(\alpha) \subseteq A - \alpha$. Therefore, if $\beta \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$, then $A - \alpha \vdash \alpha \vee \beta$ from which, by *disjunctive elimination*, it follows that $\beta \in A - \alpha$. ■

PROOF OF THEOREM 3.3.
**Success:** Let $\nvdash \alpha$ and assume by *reduction ad absurdum* that $A - \alpha \vdash \alpha$. Then it follows by compactness that there exists a finite subset of $A - \alpha$, $A' = \{\beta_1, \ldots, \beta_k\}$, such that $A' \vdash \alpha$. Then it follows from the definition of $-$ that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta_i$, for $i = 1, \ldots, k$. Then $cut_{\prec}(\alpha) \vdash \alpha \vee (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)$. Hence $cut_{\prec}(\alpha) \vdash \alpha$. Contradiction by Lemma 1 (b).
**Inclusion:** Trivial:
**Vacuity:** Let $A \nvdash \alpha$ and let $\beta \in A$. By Lemma 1 (c) it follows that $cut_{\prec}(\alpha) = A$, from which it follows that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$, hence, according to the definition of $-$, $\beta \in A - \alpha$. Therefore $A \subseteq A - \alpha$.
**Extensionality:** Let $\vdash \alpha \leftrightarrow \beta$. Then $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$ by Lemma 1 (e), and the rest follows trivially.
**Disjunctive Elimination:** Let $\beta \in A$ and $\beta \notin A - \alpha$. Then it follows from the definition of $-$ that $cut_{\prec}(\alpha) \nvdash \alpha \vee \beta$. Assume by *reduction ad absurdum* that $A - \alpha \vdash \alpha \vee \beta$. Then compactness yields that there exists a finite subset of $A - \alpha$, $A' = \{\beta_1, \ldots, \beta_k\}$, such that $A' \vdash \alpha \vee \beta$. It follows from the definition of $-$ that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta_i$, for $i = 1, \ldots, k$. Then $cut_{\prec}(\alpha) \vdash \alpha \vee (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)$. Hence $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Contradiction.
**Conjunctive Factoring:** First we will consider the case when $\vdash \alpha \wedge \beta$. Hence $\vdash \alpha$ and $\vdash \beta$. It follows, by definition of $-$, that $A - \alpha \wedge \beta = A - \alpha = A - \beta = A$.
Let $\nvdash \alpha \wedge \beta$. We will prove by cases:

Case 1) $cut_{\prec}(\alpha) \vdash \beta$. Then by Lemma 1 (j) $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. We will prove by double inclusion that $A - \alpha \wedge \beta = A - \alpha$. Let $\gamma \in A - \alpha \wedge \beta$. It follows from the definition of $-$ that $\gamma \in A$ and $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$, then $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$. Hence $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$, from which we can conclude that $\gamma \in A - \alpha$.
For the other inclusion, let $\gamma \in A - \alpha$. Then it follows from the definition of $-$ that $\gamma \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$. Then, by Lemma 1 (d), $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$. On the other hand $cut_{\prec}(\alpha) \vdash \beta$ implies that $cut_{\prec}(\alpha) \vdash \beta \vee \gamma$. Then $cut_{\prec}(\alpha \wedge \beta) \vdash \beta \vee \gamma$. Hence $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$. Therefore $\gamma \in A - \alpha \wedge \beta$.
Case 2) $cut_{\prec}(\beta) \vdash \alpha$. Due to the symmetry of the case, it follows that $A - \alpha \wedge \beta = A - \beta$.
Case 3) $cut_{\prec}(\alpha) \nvdash \beta$ and $cut_{\prec}(\beta) \nvdash \alpha$. It follows by Lemma 1 (k) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha) = cut_{\prec}(\beta)$. Let $\gamma \in A - \alpha \wedge \beta$. According to the definition of $-$, $\gamma \in A$ and $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$ iff $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$ and $cut_{\prec}(\alpha \wedge \beta) \vdash \beta \vee \gamma$ iff $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$ and $cut_{\prec}(\beta) \vdash \beta \vee \gamma$ iff $\gamma \in A - \alpha$ and $\gamma \in A - \beta$. Hence $A - \alpha \wedge \beta = A - \alpha \cap A - \beta$.

**Transitivity:** Let $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. If $\alpha \notin A$, then by *inclusion* it follows that $\alpha \notin A - \alpha \wedge \delta$. Assume now that $\alpha \in A$. From $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$ it follows, by definition of $-$, that $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$ and $cut_{\prec}(\beta \wedge \delta) \nvdash \beta$ and thus $\nvdash \alpha$ and $\nvdash \beta$. We will prove by cases:

Case 1) $\vdash \delta$. It follows trivially by *extensionality* and *success* that $\alpha \notin A - \alpha \wedge \delta$.
Case 2) $\nvdash \delta$. From $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$ and $cut_{\prec}(\beta \wedge \delta) \nvdash \beta$ it follows by Lemma 1 (k) and (e) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$ and $cut_{\prec}(\beta \wedge \delta) = cut_{\prec}(\beta)$. Hence, from Lemma 1 (d), $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$ and $cut_{\prec}(\delta) \subseteq cut_{\prec}(\beta)$ from which it follows that $cut_{\prec}(\delta) \subseteq cut_{\prec}(\alpha)$. Assume by *reduction ad absurdum* that $\alpha \in A - \alpha \wedge \delta$. Hence, by *success*, it follows that $A - \alpha \wedge \delta \nvdash \delta$. Therefore, by Lemma 3 (b) and (c),

it follows that $cut_{\prec}(\alpha\wedge\delta)\vdash\alpha$ and $cut_{\prec}(\alpha\wedge\delta)\nvdash\delta$. From which it follows, by Lemma 1 (k) and (e), that $cut_{\prec}(\alpha\wedge\delta)=cut_{\prec}(\delta)$. Hence, $cut_{\prec}(\delta)\vdash\alpha$. Contradiction by Lemma 1 (b), since $cut_{\prec}(\delta)\subseteq cut_{\prec}(\alpha)$.
**EB1:** Let $\beta\in A$. If $\vdash\beta$, then $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$. From which it follows, by definition of $-$ that $\beta\in A-\alpha$. Assume now that $\nvdash\beta$. Let $X=\{\gamma\in A:\beta\notin A-\beta\wedge\gamma\}\nvdash\alpha$. We wish to prove that $\beta\in A-\alpha$, i.e., by definition of $-$, that $\beta\in A$ and $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$. To do so, it is enough to prove that $\beta\in cut_{\prec}(\alpha)$, i.e., that $\{\gamma\in A:\beta\preceq\gamma\}\nvdash\alpha$. Let $\theta\in A$ be such that $\beta\preceq\theta$. If $\vdash\theta$ then $\theta\in X$ by *extensionality* and *success*. Consider now that $\nvdash\theta$ and assume by *reduction ad absurdum*, that $\theta\notin X$. Hence $\beta\in A-\beta\wedge\theta$. Thus, by *success*, $\theta\notin A-\beta\wedge\theta$. By definition of $-$, it follows that $cut_{\prec}(\beta\wedge\theta)\nvdash\theta\vee(\beta\wedge\theta)$. Hence $cut_{\prec}(\beta\wedge\theta)\nvdash\theta$. By Lemma 1 (d) it follows that $cut_{\prec}(\beta)\nvdash\theta$. Therefore $\theta\notin cut_{\prec}(\beta)$ from which it follows, by Observation 2.9, that $\beta\not\preceq\theta$. Since $\preceq$ is a total relation it follows that $\theta\preceq\beta$. Hence $\theta=_{\preceq}\beta$. From Lemma 1 (i), it follows that $cut_{\prec}(\theta)=cut_{\prec}(\beta)=cut_{\prec}(\beta\wedge\theta)$. On the other hand, from $\beta\in A-\beta\wedge\theta$ and Definition 2.10 it follows that $cut_{\prec}(\beta\wedge\theta)\vdash\beta$. And so, $cut_{\prec}(\beta)\vdash\beta$ which contradicts Lemma 1 (b). Hence $\theta\in X$. Therefore $\{\gamma\in A:\beta\preceq\gamma\}\subseteq X$, from which it follows that $\{\gamma\in A:\beta\preceq\gamma\}\nvdash\alpha$ and consequently that $\beta\in A-\alpha$.
**EB2:** Let $\beta\in A-\alpha$ we will prove that $\{\gamma\in A:\gamma\in A-\gamma\wedge\alpha\}\vdash\alpha\vee\beta$.
It is trivial if $\vdash\alpha\vee\beta$. Assume now that $\nvdash\alpha\vee\beta$. From $\beta\in A-\alpha$ it follows that $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$. Let $\delta\in cut_{\prec}(\alpha)$, then by Lemma 3 (d), $\delta\in X=\{\gamma\in A:\gamma\in A-\alpha\wedge\gamma\}$. It follows that $cut_{\prec}(\alpha)\subseteq X$. Therefore $X\vdash\alpha\vee\beta$.
(**$C_{EB}\preceq$**) ($\Rightarrow$) Let $\alpha,\beta\in A$ be such that $\alpha\preceq\beta$. If $\vdash\alpha$, then by ($\preceq$2), $\vdash\beta$. Hence $\vdash\alpha\wedge\beta$. Consider now that $\nvdash\alpha$. Assume by *reduction ad absurdum* that $\alpha\in A-\alpha\wedge\beta$. Hence, $cut_{\prec}(\alpha\wedge\beta)\vdash\alpha\vee(\alpha\wedge\beta)$. Therefore, $cut_{\prec}(\alpha\wedge\beta)\vdash\alpha$. Thus, by Lemma 1 (b), $cut_{\prec}(\alpha\wedge\beta)\nvdash\beta$. Hence, by Lemma 1 (k) and (e), it follows that $cut_{\prec}(\alpha\wedge\beta)=cut_{\prec}(\beta)$. Hence $cut_{\prec}(\beta)\vdash\alpha$. Therefore, by Observation 2.9, it follows that $\{\gamma\in A:\beta\prec\gamma\}\vdash\alpha$. Thus, since $\alpha\preceq\beta$ and $\preceq$ is a transitive relation on $A$, it follows that $\{\gamma\in A:\alpha\prec\gamma\}\vdash\alpha$, which contradicts ($\preceq$1).
($\Leftarrow$) We will consider two cases:

Case 1) $\vdash\alpha\wedge\beta$. Then, by ($\preceq$3), $\alpha\preceq\beta$.
Case 2) $\alpha\notin A-\alpha\wedge\beta$. Assume by *reduction ad absurdum* that $\alpha\not\preceq\beta$. Hence $\beta\prec\alpha$, since $\preceq$ is a total relation. From which it follows, by Observation 2.9, that $\alpha\in cut_{\prec}(\beta)$. Thus, by Lemma 1 (d), $\alpha\in cut_{\prec}(\alpha\wedge\beta)$. Therefore, by Lemma 3 (a), $\alpha\in A-\alpha\wedge\beta$. Contradiction. ■

PROOF OF OBSERVATION 4.2.

**(a)** *Success, inclusion* and *vacuity* follow trivially by Observation 2.12. From Observation 2.12 we also know that $-$ satisfies: *strong inclusion, failure, relative closure* and *uniform behaviour*. *Extensionality* and *conjunctive factoring* follow trivially by Observations 2.13 (d) and 2.13 (f) respectively.
**Transitivity:** Let $\beta\in A$, $\alpha\notin A-\alpha\wedge\beta$ and $\beta\notin A-\beta\wedge\delta$. Hence, by *relative closure* $\nvdash\beta$. We intend to prove that $\alpha\notin A-\alpha\wedge\delta$. It follows trivially by *inclusion* if $\alpha\notin A$. Assume now that $\alpha\in A$. In this case, by *relative closure* $\nvdash\alpha$. We will prove by cases:

Case 1) $\vdash\delta$. It follows by *extensionality* and *success*.
Case 2) $\nvdash\delta$. From $\alpha\notin A-\alpha\wedge\beta$ and $\beta\notin A-\beta\wedge\delta$ it follows by *relative closure*, Lemma 2 and Observation 2.13 (e) that $A-\beta\subseteq A-\alpha\wedge\beta\subseteq A-\alpha$ and $A-\delta\subseteq A-\beta\wedge\delta\subseteq A-\beta$. Hence $A-\delta\subseteq A-\alpha$. Thus, by *decomposition* (Observation 2.13 (f)), it follows that $A-\alpha\wedge\delta\subseteq A-\alpha$. If $\alpha\in A-\alpha\wedge\delta$, then $\alpha\in A-\alpha$ which contradicts *success*. Hence $\alpha\notin A-\alpha\wedge\delta$.
**EB1:** Let $\beta\in A$ and $\{\gamma\in A:\beta\notin A-\beta\wedge\gamma\}\nvdash\alpha$. We will show that $\beta\in A-\alpha$.
We will prove by cases:

Case 1) $A\nvdash\alpha$. It follows by *vacuity*.

Case 2) $\vdash \beta$. It follows by *relative closure*.
Case 3) $A \vdash \alpha$ and $\nvdash \beta$. Let $X = \{\gamma \in A : \beta \notin A - \beta \wedge \gamma\}$. Assume by *reduction ad absurdum* that $\beta \notin A - \alpha$. By *relative closure* and *strong inclusion*, it follows that $A - \alpha \subseteq A - \beta$. We will consider two cases:
Case 3.1) $A - \alpha \subset A - \beta$. It follows that $A - \beta \not\subseteq A - \alpha$. From which it follows, by *strong inclusion*, that $A - \beta \vdash \alpha$. On the other hand, by Lemma 4 $A - \beta \subseteq X$. Therefore $X \vdash \alpha$. Contradiction.
Case 3.2) $A - \alpha = A - \beta$. By *uniform behaviour* it follows that $\alpha \in Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$. Let $Y = A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\}$. We will now show that $Y \subseteq X$. Let $\psi \in Y$. If $\vdash \psi$, then $\psi \in X$, by *extensionality* and *success*. Assume now that $\nvdash \psi$. We will consider two cases:
Case 3.2.1) $A - \beta = A - \psi$. Hence, by *decomposition* (Observation 2.13 (f)), $A - \beta \wedge \psi \subseteq A - \beta$.
Case 3.2.2) $\psi \in A - \beta$. By *linearity* (Observation 2.13 (d)) and *success*, $A - \psi \subseteq A - \beta$. Hence, by *decomposition*, $A - \beta \wedge \psi \subseteq A - \beta$.

Hence in both cases $\psi \in X$, since from $A - \beta \wedge \psi \subseteq A - \beta$ it follows, by *success*, that $\beta \notin A - \beta \wedge \psi$. Therefore $Y \subseteq X$. But this leads to a contradiction since $Y \vdash \alpha$ and $X \nvdash \alpha$.

**EB2:** Let $\beta \in A - \alpha$. We intend to prove that $X = \{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\} \vdash \alpha \vee \beta$. It is trivial if $\vdash \alpha \vee \beta$. Consider now that $\nvdash \alpha \vee \beta$. From $\beta \in A - \alpha$ it follows by *inclusion* that $\beta \in A$ and by Lemma 2 that $\beta \in A - \alpha \wedge \beta$. Hence $\beta \in X$. Therefore $X \vdash \alpha \vee \beta$.

**(b)** We will show that in general $-$ does not satisfy *disjunctive elimination*. Consider the following counter-example: Let $\alpha$ and $\beta$ be logically independent sentences. Let $A = \{\alpha, \beta, \alpha \vee \beta\}$ and $\preceq$ be the two-level ensconcement relation on $A$ defined by: $\alpha =_{\preceq} \beta \prec \alpha \vee \beta$. By Definition 2.11 it follows that $A - \alpha = \{\alpha \vee \beta\}$. Hence $\beta \in A$, $\beta \notin A - \alpha$ and $A - \alpha \vdash \alpha \vee \beta$. ■

Proof of Observation 4.3.

**(a)** *Success, inclusion* and *vacuity* follow trivially by Theorem 3.3. On the other hand, by Theorem 3.3, $-$ satisfies *disjunctive elimination*. Hence, by Observation 2.13 (a), it also satisfies *failure* and *relative closure*.

**(b)** For *strong inclusion* it is enough to consider the following counter-example: Let $\alpha, \beta$ and $\delta$ be logically independent sentences. Let $A = \{\alpha, \beta, \delta, \beta \vee \delta\}$ and $\preceq$ be an ensconcement relation on $A$ defined by: $\delta \prec \alpha \prec \beta \prec \beta \vee \delta$. By Definition 2.10 it follows that $A - \alpha = \{\beta, \beta \vee \delta\}$ and $A - \beta = \{\delta, \beta \vee \delta\}$. Hence $A - \beta \nvdash \alpha$ but $A - \beta \not\subseteq A - \alpha$.
For *uniform behaviour* consider the following counter-example: Let $A = \{\beta, \delta, \delta \rightarrow \alpha\}$ and let $\preceq$ be an ensconcement relation on $A$ such that all the formulae of A are at the same level. By Definition 2.10 it follows that $A - \beta = A - \alpha = \emptyset$, $A - \delta = \{\delta \rightarrow \alpha\}$ and $A - (\delta \rightarrow \alpha) = \{\delta\}$ (since $\vdash \delta \vee (\delta \rightarrow \alpha)$). Hence $\beta \in A$, $A \vdash \alpha$, $A - \alpha = A - \beta$, but $\alpha \notin Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$. ■

Proof of Observation 4.4.

**(a)** Let $(A, \preceq)$ be an ensconcement, $\alpha, \beta \in A$ and $-$ be the $\preceq$-based brutal contraction. We intend to prove that $\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$ holds if and only if $\alpha \notin A - \beta$ or $\vdash \beta$ holds. Assume first that $\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$ holds. We will prove by cases:
Case 1) $\vdash \alpha \wedge \beta$. Hence $\vdash \beta$.

Case 2) $\nvdash \alpha \wedge \beta$. Hence $\alpha \notin A - \alpha \wedge \beta$. We can consider two cases $\vdash \beta$ or $\nvdash \beta$. In the latter, from $\alpha \notin A - \alpha \wedge \beta$ it follows that $\alpha \notin cut_{\prec}(\alpha \wedge \beta)$. Thus, by Lemma 1 (d), it follows that

$\alpha \notin cut_{\prec}(\beta)$. Hence, $\alpha \notin A-\beta$.

Assume now that $\alpha \notin A-\beta$ or $\vdash \beta$. We will show that $\alpha \notin A-\alpha\wedge\beta$ or $\vdash \alpha\wedge\beta$. We will prove by cases:

Case 1) $\vdash \beta$. If $\vdash \alpha$, then $\vdash \alpha\wedge\beta$. Consider now that $\nvdash \alpha$. Hence, by Lemma 1 (e), $cut_{\prec}(\alpha\wedge\beta) = cut_{\prec}(\alpha)$. Therefore, by Lemma 1 (b), $\alpha \notin cut_{\prec}(\alpha\wedge\beta)$. Hence $\alpha \notin A-\alpha\wedge\beta$.

Case 2) $\nvdash \beta$. Then $\alpha \notin A-\beta$. Hence $\alpha \notin cut_{\prec}(\beta)$. It follows, by Observation 2.9, that $\beta \not\prec \alpha$. Thus $\nvdash \alpha$, by ($\preceq$2). Furthermore, since $\preceq$ is total, $\alpha \preceq \beta$. By Lemma 1 (h) and (i) it follows that $cut_{\prec}(\alpha\wedge\beta) = cut_{\prec}(\alpha)$. Hence, by Lemma 1 (b), $\alpha \notin cut_{\prec}(\alpha\wedge\beta)$. Therefore $\alpha \notin A-\alpha\wedge\beta$.

**(b)** Let $(A,\preceq)$ be an ensconcement, $\alpha,\beta \in A$ and $-$ be the $\preceq$-based brutal contraction.

From left to right: Since $\alpha \notin A-\beta$, then $\nvdash \beta$ otherwise $A-\beta = A$ which contradicts $\alpha \in A$. Hence $A-\beta = cut_{\prec}(\beta)$. Therefore $\alpha \notin cut_{\prec}(\beta)$, from which it follows, by Observation 2.9, that $\beta \not\prec \alpha$. Thus, by ($\preceq$2), $\nvdash \alpha$. Let $\delta \in A-\beta = cut_{\prec}(\beta)$. Hence, by Observation 2.9, it follows that $\beta \prec \delta$. Thus, since $\preceq$ is a transitive and total relation, it follows that $\delta \in cut_{\prec}(\alpha) = A-\alpha$. Therefore $A-\beta \subseteq A-\alpha$.

From right to left: Consider that $A-\beta \subseteq A-\alpha$ and $\nvdash \alpha$. Assume by *reduction ad absurdum* that $\alpha \in A-\beta$. Hence $\alpha \in A-\alpha = cut_{\prec}(\alpha)$, which contradicts Lemma 1 (b).

**(c)** Follows trivially from (a) and (b). ■

Proof of Observation 4.5. In the proof of the *postulates to construction* part of [11, Theorem 11] (whose statement we recalled in Observation 2.12 above) it was proven that if $-$ satisfies *success, inclusion, vacuity, failure, relative closure, strong inclusion* and *uniform behaviour*, then the relation $\preceq$ on $A$ defined by ($\mathbf{C'_{BR}} \preceq$) is an ensconcement and that, for all sentences $\alpha$, $A-\alpha = A-_{\preceq}\alpha$ where $-_{\preceq}$ is the operation defined by condition (**BC**). Hence, since according to Observation 4.4 (c), ($\mathbf{C_{BR}} \preceq$) is equivalent to ($\mathbf{C'_{BR}} \preceq$), we can conclude that the binary relation $\preceq$ on $A$ defined by ($\mathbf{C_{BR}} \preceq$) is an ensconcement relation and $-$ satisfies (**BC**). ■

Proof of Observation 4.6.

**(a)** It follows from Observation 4.4 (c) and Observations 2.12 and 4.5.

**(b)** Let $\alpha$ and $\beta$ be logically independent sentences. Let $A = \{\alpha,\beta,\alpha\vee\beta\}$ and $\preceq$ be an ensconcement relation on $A$ defined by: $\alpha \prec \beta \prec \alpha\vee\beta$. Let $-$ be the $\preceq$-based contraction on $A$. Hence $A-\alpha = \{\beta,\alpha\vee\beta\}$ and $A-\beta = \{\alpha,\alpha\vee\beta\}$. The relation $\preceq'$ defined by condition ($\mathbf{C_{BR}} \preceq$) based on the operation $-$ is not an ensconcement. Indeed, since $\beta \in A-\alpha$ and $\alpha \in A-\beta$, it follows from ($\mathbf{C_{BR}} \preceq$), that $\alpha \not\preceq' \beta$ and $\beta \not\preceq' \alpha$. Therefore $\preceq'$ is not total and thus, $\preceq'$ is not an ensconcement relation. ■

Proof of Observation 5.1.

**(a)–(d)** Straightforward.

**(e)** Let $\mathbf{K}$ be a belief set and $\div$ be an operator on $\mathbf{K}$ that satisfies ($\div$1), ($\div$2), ($\div$3) and ($\div$5). It follows from Lemma 5 (b) that $\div$ satisfies relevance. Therefore, since logical relevance follows trivially from relevance, it holds that $\div$ satisfies logical relevance. Finally, it follows from Observation 2.13 (b) that $\div$ satisfies disjunctive elimination.

**(f)** Let $\mathbf{K}$ be a belief set, $\beta \in \mathbf{K}$, $\alpha \notin \mathbf{K}\div\alpha\wedge\beta$, $\beta \notin \mathbf{K}\div\beta\wedge\delta$ and assume by *reduction ad absurdum* that $\alpha \in \mathbf{K}\div\alpha\wedge\delta$. From the latter it follows by *conjunctive trisection* (which is satisfied due to Lemma 6 and Observation 2.4) that $\alpha \in \mathbf{K}\div\alpha\wedge\delta\wedge\beta$. On the other hand, by ($\div V$) and ($\div$6), it follows that $\mathbf{K}\div\alpha\wedge\beta\wedge\delta$ is identical to: (i) $\mathbf{K}\div\alpha\wedge\beta$, (ii) $\mathbf{K}\div\alpha\wedge\beta \cap \mathbf{K}\div\beta\wedge\delta$ or (iii) $\mathbf{K}\div\beta\wedge\delta$. The first two cases can not hold, since $\alpha \in \mathbf{K}\div\alpha\wedge\beta\wedge\delta$ and $\alpha \notin \mathbf{K}\div\alpha\wedge\beta$.

Thus $\mathbf{K} \div \alpha \wedge \beta \wedge \delta = \mathbf{K} \div \beta \wedge \delta$. Hence $\beta \notin \mathbf{K} \div \alpha \wedge \beta \wedge \delta$, from which it follows, by $(\div 1)$, that $\alpha \wedge \beta \notin \mathbf{K} \div \alpha \wedge \beta \wedge \delta$. Hence, by $(\div 8)$ (which is satisfied due to Observation 2.4), $\mathbf{K} \div \alpha \wedge \beta \wedge \delta \subseteq \mathbf{K} \div \alpha \wedge \beta$. Contradiction, since $\alpha \in \mathbf{K} \div \alpha \wedge \beta \wedge \delta$ and $\alpha \notin \mathbf{K} \div \alpha \wedge \beta$.

**(g)** Let $\mathbf{K}$ be a belief set. Let $\beta \in \mathbf{K}$ and $X = \{\gamma \in \mathbf{K} : \beta \notin \mathbf{K} \div \beta \wedge \gamma\} \nvdash \alpha$. We wish to prove that $\beta \in \mathbf{K} \div \alpha$. This follows trivially by $(\div 1)$ if $\vdash \beta$ and by $(\div 3)$ if $\alpha \notin \mathbf{K}$. Assume now that $\nvdash \beta$ and $\alpha \in \mathbf{K}$. From $X \nvdash \alpha$ it follows that $\alpha \notin X$. Therefore, $\beta \in \mathbf{K} \div \beta \wedge \alpha$, from which it follows by $(\div 4)$ and $(\div V)$ that $\beta \in \mathbf{K} \div \alpha$.

**(h)** Let $\mathbf{K}$ be a belief set and $\beta \in \mathbf{K} \div \alpha$. We intend to show that $X = \{\gamma \in \mathbf{K} : \gamma \in \mathbf{K} \div \gamma \wedge \alpha\} \vdash \alpha \vee \beta$. It is trivial if $\vdash \alpha \vee \beta$. Assume now that $\nvdash \alpha \vee \beta$. From $\vdash \alpha \leftrightarrow (\alpha \vee \beta) \wedge \alpha$ it follows, by $(\div 6)$, that $\mathbf{K} \div \alpha = \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Hence $\beta \in \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Therefore, by $(\div 1)$, $\alpha \vee \beta \in \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Thus, by $(\div 2)$, $\alpha \vee \beta \in \mathbf{K}$. Hence $\alpha \vee \beta \in X$.

**(i)** Let $\mathbf{K}$ be a belief set and assume that $\mathbf{K} \vdash \alpha$ and $\mathbf{K} - \alpha = \mathbf{K} - \beta$. Then $\alpha \in \mathbf{K}$ and, furthermore, $\alpha \in \{\gamma \in \mathbf{K} : \mathbf{K} - \beta = \mathbf{K} - \gamma\}$. Therefore $\alpha \in Cn(\mathbf{K} - \beta \cup \{\gamma \in \mathbf{K} : \mathbf{K} - \beta = \mathbf{K} - \gamma\})$. ■

PROOF OF OBSERVATION 5.2.

**(a)–(c)** Straightforward.

**(d)** Follows trivially from (c) and Observation 2.13 (a).

**(e)** Let $A$ be a belief set and $-$ be an operator on $A$ that satisfies inclusion, vacuity and disjunctive elimination. Then, it follows from Lemma 7 that $-$ satisfies relevance. Therefore, by Lemma 5 (a) we can conclude that $-$ satisfies $(\div 5)$.

**(f)** Let $A$ be a belief set and $-$ be an operator on $A$ that satisfies inclusion, relative closure and strong inclusion. Then, it follows from (c) that $-$ satisfies $(\div 1)$. Therefore, we can conclude that $-$ satisfies $(\div 9)$, since this property follows trivially from strong inclusion and $(\div 1)$. ■

PROOF OF OBSERVATION 5.3. We will prove that statements 1. – 4. are equivalent by showing that 1. ⇔ 3., 3. ⇔ 4., 2. ⇒ 4. and 3. ⇒ 2..

(1. ⇔ 3.) Follows trivially from Observations 2.3 and 2.4.

(3. ⇔ 4.) Follows from Observations 5.1 and 5.2 and the fact that *inclusion* and $(\div 2)$ are two alternative designations of the same property, and this is also the case regarding *extensionality* and $(\div 6)$ as well as *conjunctive factoring* and $(\div V)$.

(2. ⇒ 4.) Follows trivially from Theorem 3.3.

(3. ⇒ 2.) Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$ that satisfies $(\div 1)$–$(\div 6)$ and $(\div V)$. Then it follows from Observation 5.1 (and from the fact that *inclusion* and $(\div 2)$, *extensionality* and $(\div 6)$, and *conjunctive factoring* and $(\div V)$ are pairs of alternative designations for one same property) that $\div$ satisfies the postulates of *inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*. Therefore, according to Theorem 3.2, $\div$ is an ensconcement-based contraction on $\mathbf{K}$. ■

PROOF OF OBSERVATION 5.4.

**(a)** Follows trivially from Theorem 3.3, Observation 5.2 and the fact that *inclusion* and $(\div 2)$, *extensionality* and $(\div 6)$, and *conjunctive factoring* and $(\div V)$ are pairs of alternative designations for one same property.

**(b)** We will show that in general $-$ does not satisfy $(\div 1)$ nor $(\div 3)$ nor $(\div 5)$. Consider the following counter-example: Let $\alpha$ and $\beta$ be logically independent sentences. Let $(A, \preceq)$ be an ensconcement where $A = \{\beta, \beta \to \alpha\}$ and $\preceq$ is the two-level ensconcement relation on $A$ defined by: $\beta \prec \beta \to \alpha$. Hence $cut_{\prec}(\alpha) = \{\beta \to \alpha\}$. Let $-$ be the $\preceq$-based contraction for

$A$. According to Definition 2.10, it holds that $A-\alpha=\{\beta\rightarrow\alpha\}$. Therefore $A-\alpha\neq Cn(A-\alpha)$ (hence $-$ does not satisfy ($\div$1)) and $A\not\subseteq Cn(A-\alpha\cup\{\alpha\})$ (hence $-$ does not satisfy ($\div$5)). Furthermore $\alpha\notin A$ and $A-\alpha\neq A$ (hence $-$ does not satisfy ($\div$3)). ■

Proof of Observation 5.5. Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Let $\div$ be the $\leq$-based contraction on $\mathbf{K}$ and $-$ be the ensconcement-based contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$. We start by recalling that (according to conditions ($\mathbf{C}_{\div_{\leq}}$) and (**EBC**)) it holds that:

$$\mathbf{K}\div\alpha=\begin{cases}\{\beta\in\mathbf{K}:\alpha<\alpha\vee\beta\} & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

and

$$\mathbf{K}-\alpha=\{\beta\in\mathbf{K}:cut_{<|_{\mathbf{K}}}(\alpha)\vdash\alpha\vee\beta\}.$$

We recall also that, since $\leq$ is an epistemic entrenchment relation with respect to $\mathbf{K}$, it follows from Observation 2.7 that $(\mathbf{K},\leq|_{\mathbf{K}})$ is an ensconcement.
We will prove by cases that, for all $\alpha\in\mathcal{L}$, $\mathbf{K}-\alpha=\mathbf{K}\div\alpha$.

Case 1) $\alpha\notin\mathbf{K}$. Hence $\mathbf{K}\nvdash\alpha$ and $\mathbf{K}\nvdash\perp$. Thus by Lemma 1 (c) $cut_{<|_{\mathbf{K}}}(\alpha)=\mathbf{K}$. Therefore $\mathbf{K}-\alpha=\mathbf{K}$. On the other hand, by definition of the operator $\div$, it holds that $\mathbf{K}\div\alpha\subseteq\mathbf{K}$. Let $\beta\in\mathbf{K}$. Since $\mathbf{K}$ is a logically closed set it holds that $\alpha\vee\beta\in\mathbf{K}$. Hence by Lemmas 8 and 9 it follows that $\alpha<\alpha\vee\beta$. Thus $\mathbf{K}\subseteq\mathbf{K}\div\alpha$. Hence $\mathbf{K}-\alpha=\mathbf{K}=\mathbf{K}\div\alpha$.
Case 2) $\vdash\alpha$. Hence $\mathbf{K}-\alpha=\mathbf{K}\div\alpha=\mathbf{K}$.
Case 3) $\alpha\in\mathbf{K}$ and $\nvdash\alpha$.

Let $\beta\in\mathbf{K}-\alpha$. Then, since $\mathbf{K}$ is a logically closed set, $\alpha\vee\beta\in\mathbf{K}$. If $\vdash\alpha\vee\beta$, then $\alpha<|_{\mathbf{K}}\alpha\vee\beta$ (by ($\preceq$2)). Then $\alpha<\alpha\vee\beta$. Hence $\beta\in\mathbf{K}\div\alpha$. Therefore $\mathbf{K}-\alpha\subseteq\mathbf{K}\div\alpha$. Assume now that $\nvdash\alpha\vee\beta$. It follows, by definition of $-$ that $cut_{<|_{\mathbf{K}}}(\alpha)\vdash\alpha\vee\beta$. Hence, by Observation 2.9, $\{\gamma\in\mathbf{K}:\alpha<|_{\mathbf{K}}\gamma\}\vdash\alpha\vee\beta$. Since $\leq|_{\mathbf{K}}$ is a total relation on $\mathbf{K}$ it follows that $\alpha<|_{\mathbf{K}}\alpha\vee\beta$ or $\alpha\vee\beta\leq|_{\mathbf{K}}\alpha$. In the latter case it would follow that $\{\gamma\in\mathbf{K}:\alpha\vee\beta<|_{\mathbf{K}}\gamma\}\vdash\alpha\vee\beta$, which contradicts ($\preceq$1). Therefore $\alpha<|_{\mathbf{K}}\alpha\vee\beta$. Then $\alpha<\alpha\vee\beta$, from which it follows that $\beta\in\mathbf{K}\div\alpha$. Hence $\mathbf{K}-\alpha\subseteq\mathbf{K}\div\alpha$.
Let $\beta\in\mathbf{K}\div\alpha$. Hence $\beta\in\mathbf{K}$ and $\alpha<\alpha\vee\beta$. Since $\mathbf{K}$ is a logically closed set it follows that $\alpha\vee\beta\in\mathbf{K}$. Thus $\alpha<|_{\mathbf{K}}\alpha\vee\beta$. Therefore, by Observation 2.9, $\alpha\vee\beta\in cut_{<|_{\mathbf{K}}}(\alpha)$. Hence $\beta\in\mathbf{K}-\alpha$.
Therefore $\mathbf{K}-\alpha=\mathbf{K}\div\alpha$. ■

Proof of Observation 5.6. Let $\mathbf{K}$ be a belief set and $(\mathbf{K},\preceq)$ be an ensconcement. Let $-$ be the $\preceq$-based contraction on $\mathbf{K}$ and $\div$ be the epistemic entrenchment-based contraction on $\mathbf{K}$ defined from the epistemic entrenchment relation $\leq_{\preceq}$. We start by recalling that (according to conditions ($\mathbf{C}_{\div_{\leq}}$) and (**EBC**)) it holds that:

$$\mathbf{K}\div\alpha=\begin{cases}\{\beta\in\mathbf{K}:\alpha<_{\preceq}\alpha\vee\beta\} & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

and

$$\mathbf{K}-\alpha=\{\beta\in\mathbf{K}:cut_{\prec}(\alpha)\vdash\alpha\vee\beta\}.$$

We will prove that, for all $\alpha\in\mathcal{L}$, $\mathbf{K}\div\alpha=\mathbf{K}-\alpha$.
If $\vdash\alpha$, then $\mathbf{K}\div\alpha=\mathbf{K}$ and $\vdash\alpha\vee\beta$. From the latter it follows that $\mathbf{K}-\alpha=\mathbf{K}$. Thus $\mathbf{K}\div\alpha=\mathbf{K}-\alpha$.
Assume now that $\nvdash\alpha$. Let $\alpha\notin\mathbf{K}$ (it follows that $\mathbf{K}\nvdash\alpha$ since $\mathbf{K}$ is a logically closed set). By definition of

$\div$, it follows that $\mathbf{K}\div\alpha\subseteq\mathbf{K}$. Let $\beta\in\mathbf{K}$. Thus $\alpha\vee\beta\in\mathbf{K}$ (since $\mathbf{K}$ is a logically closed set). Therefore, by Lemmas 8 and 9, $\alpha<_{\preceq}\alpha\vee\beta$. Hence $\mathbf{K}\subseteq\mathbf{K}\div\alpha$. Thus $\mathbf{K}\div\alpha=\mathbf{K}$. On the other hand, by Lemma 1 (c), $cut_{\prec}(\alpha)=\mathbf{K}$. Thus $\mathbf{K}-\alpha=\mathbf{K}$.

Assume now that $\alpha\in\mathbf{K}$ and $\nvdash\alpha$.

Let $\beta\in\mathbf{K}\div\alpha$. Hence $\beta\in\mathbf{K}$ and $\alpha<_{\preceq}\alpha\vee\beta$. We will prove by cases that $\beta\in\mathbf{K}-\alpha$.

Case 1) $\vdash\alpha\vee\beta$. Then $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$. Hence $\beta\in\mathbf{K}-\alpha$.

Case 2) $\nvdash\alpha\vee\beta$. Since $\mathbf{K}$ is a logically closed set and $\beta\in\mathbf{K}$ it follows that $\alpha\vee\beta\in\mathbf{K}$. From $\alpha<_{\preceq}\alpha\vee\beta$ it follows, by definition of $\leq_{\preceq}$, that $cut_{\preceq}(\alpha\vee\beta)\subset cut_{\preceq}(\alpha)$. Therefore there exists $\delta\in cut_{\preceq}(\alpha)$, such that $\delta\notin cut_{\preceq}(\alpha\vee\beta)$. Thus $\{\gamma\in\mathbf{K}:\delta\prec\gamma\}\vdash\alpha\vee\beta$. By ($\preceq 1$) and, since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\delta\prec\alpha\vee\beta$. From $\delta\in cut_{\preceq}(\alpha)$ it follows that $\{\gamma\in\mathbf{K}:\delta\prec\gamma\}\nvdash\alpha$. Hence $\{\gamma\in\mathbf{K}:\alpha\vee\beta\preceq\gamma\}\nvdash\alpha$. Therefore $\alpha\vee\beta\in cut_{\prec}(\alpha)$. Thus $\beta\in\mathbf{K}-\alpha$.

Let $\beta\in\mathbf{K}-\alpha$. We will prove that $\beta\in\mathbf{K}\div\alpha$. From $\beta\in\mathbf{K}-\alpha$ it follows that $\beta\in\mathbf{K}$ and $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$. Furthermore, since $\mathbf{K}$ is a logically closed set it follows that $\alpha\vee\beta\in\mathbf{K}$. If $\vdash\alpha\vee\beta$ it follows, by Lemma 10, that $\alpha<_{\preceq}\alpha\vee\beta$. Thus $\beta\in\mathbf{K}\div\alpha$. Assume now that $\nvdash\alpha\vee\beta$.

From $cut_{\prec}(\alpha)\vdash\alpha\vee\beta$, and since $\alpha\in\mathbf{K}$, it follows by Observation 2.9, that $\{\gamma\in\mathbf{K}:\alpha\prec\gamma\}\vdash\alpha\vee\beta$. Hence, by ($\preceq 1$) and since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\alpha\prec\alpha\vee\beta$.

Assume by *reduction ad absurdum* that $\alpha\vee\beta\leq_{\preceq}\alpha$. Thus, by the definition of $\leq_{\preceq}$, it follows that $cut_{\preceq}(\alpha)\subseteq cut_{\preceq}(\alpha\vee\beta)$. Since $\alpha\in cut_{\preceq}(\alpha)$, it follows that $\alpha\in cut_{\preceq}(\alpha\vee\beta)$. Hence, by Lemma 11, $\alpha\vee\beta\preceq\alpha$. Contradiction. Hence $\alpha\vee\beta\not\leq_{\preceq}\alpha$. Thus, by Lemma 8, $\alpha<_{\preceq}\alpha\vee\beta$, from which it follows that $\beta\in\mathbf{K}\div\alpha$. ■

PROOF OF THEOREM 5.7. We start by noticing that Observation 2.6 states exactly that $1.\Leftrightarrow 3.$. Hence, in order to prove that statements $1.-4.$ are equivalent, we will show that $3.\Leftrightarrow 4.$, $2.\Rightarrow 4.$ and $3.\Rightarrow 2.$.

($3.\Leftrightarrow 4.$) Follows from Observations 5.1 and 5.2 and the fact that *inclusion* and ($\div 2$) are two alternative designations of the same property, and this is also the case regarding *failure* and ($\div 3'$).

($2.\Rightarrow 4.$) Follows trivially from Observation 2.12.

($3.\Rightarrow 2.$) Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$ that satisfies ($\div 1$), ($\div 2$), ($\div 3$), ($\div 3'$), ($\div 4$) and ($\div 9$). Then it follows from Observation 5.1 (and from the fact that *inclusion* and ($\div 2$), and *failure* and ($\div 3'$) are pairs of alternative designations for one same property) that $\div$ satisfies the postulates of *relative closure, inclusion, vacuity, failure, success, strong inclusion* and *uniform behaviour*. Therefore, according to Observation 2.12, $\div$ is a brutal contraction on $\mathbf{K}$. ■

PROOF OF OBSERVATION 5.8.

- **(a)** Follows trivially from Observation 2.12, Observation 5.2 and the fact that *inclusion* and ($\div 2$), and *failure* and ($\div 3'$) are pairs of alternative designations for one same property.
- **(b)** We will show that in general $-$ does not satisfy ($\div 1$) nor ($\div 3$) nor ($\div 9$). Consider the following counter-example: Let $\alpha$ and $\beta$ be logically independent sentences. Let $(A,\preceq)$ be an ensconcement where $A=\{\beta,\alpha\vee\beta,\beta\rightarrow\alpha\}$ and $\preceq$ is the three-level ensconcement relation on $A$ defined by: $\beta\prec\alpha\vee\beta\prec\beta\rightarrow\alpha$. Let $-$ be the $\preceq$-based brutal contraction on $A$. According to Definition 2.11, it holds that $A-\alpha=cut_{\prec}(\alpha)=\{\beta\rightarrow\alpha\}$ and $A-\beta=cut_{\prec}(\beta)=\{\alpha\vee\beta,\beta\rightarrow\alpha\}$. Therefore $A-\beta\neq Cn(A-\beta)$ (hence $-$ does not satisfy ($\div 1$)) and, however $\alpha\notin A-\beta$, it does not hold that $A-\beta\subseteq A-\alpha$ (hence $-$ does not satisfy ($\div 9$)). Furthermore $\alpha\notin A$ and $A-\alpha\neq A$ (hence $-$ does not satisfy ($\div 3$)). ■

PROOF OF OBSERVATION 5.9. Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Let $\div$ be the $\leq$-based severe withdrawal on $\mathbf{K}$ and $-$ be the brutal contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$ (notice that, according to Observation 2.7, $(\mathbf{K},\leq|_{\mathbf{K}})$ is an ensconcement). We start by recalling that (according to conditions ($\mathbf{R}_{\div_{\leq}}$) and (**BC**)) it holds that:

$$\mathbf{K}\div\alpha=\begin{cases}\{\beta\in\mathbf{K}:\alpha<\beta\} & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

and

$$\mathbf{K}-\alpha=\begin{cases}cut_{<|_{\mathbf{K}}}(\alpha) & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

We will prove by cases that, for all $\alpha\in\mathcal{L}$, $\mathbf{K}-\alpha=\mathbf{K}\div\alpha$.

Case 1) $\alpha\notin\mathbf{K}$. Hence $\mathbf{K}\nvdash\alpha$ and $\mathbf{K}\nvdash\bot$. Thus by Lemma 1 (c) $cut_{<|_{\mathbf{K}}}(\alpha)=\mathbf{K}$. Therefore $\mathbf{K}-\alpha=\mathbf{K}$. On the other hand, by definition of the operator $\div$, it holds that $\mathbf{K}\div\alpha\subseteq\mathbf{K}$. Let $\beta\in\mathbf{K}$. Hence by Lemmas 8 and 9 it follows that $\alpha<\beta$. Thus $\mathbf{K}\subseteq\mathbf{K}\div\alpha$. Hence $\mathbf{K}-\alpha=\mathbf{K}=\mathbf{K}\div\alpha$.
Case 2) $\vdash\alpha$. Hence $\mathbf{K}-\alpha=\mathbf{K}\div\alpha=\mathbf{K}$.
Case 3) $\alpha\in\mathbf{K}$ and $\nvdash\alpha$.
Let $\beta\in\mathbf{K}-\alpha$. Hence $\beta\in cut_{<|_{\mathbf{K}}}(\alpha)$. Therefore, by Observation 2.9, $\alpha<|_{\mathbf{K}}\beta$. Hence $\alpha<\beta$. Thus $\beta\in\mathbf{K}\div\alpha$.
Let $\beta\in\mathbf{K}\div\alpha$. Then $\alpha<\beta$. Thus, $\alpha<|_{\mathbf{K}}\beta$. By Observation 2.9, it follows that $\beta\in cut_{<|_{\mathbf{K}}}(\alpha)$. Therefore $\beta\in\mathbf{K}-\alpha$. Hence $\mathbf{K}-\alpha=\mathbf{K}\div\alpha$. ■

Proof of observation 5.10. Let $\mathbf{K}$ be a belief set and $(\mathbf{K},\preceq)$ be an ensconcement. Let $-$ be the $\preceq$-based brutal contraction on $\mathbf{K}$ and $\div$ be the severe withdrawal on $\mathbf{K}$ defined from the epistemic entrenchment relation $\leq_{\preceq}$. We start by recalling that (according to conditions ($\mathbf{R}_{\div_{\leq}}$) and (**BC**)) it holds that:

$$\mathbf{K}\div\alpha=\begin{cases}\{\beta\in\mathbf{K}:\alpha<_{\preceq}\beta\} & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

and

$$\mathbf{K}-\alpha=\begin{cases}cut_{\prec}(\alpha) & \text{if } \nvdash\alpha\\ \mathbf{K} & \text{otherwise}\end{cases}$$

We will prove that, for all $\alpha\in\mathcal{L}$, $\mathbf{K}\div\alpha=\mathbf{K}-\alpha$.
If $\vdash\alpha$, then $\mathbf{K}-\alpha=\mathbf{K}\div\alpha=\mathbf{K}$.
Assume now that $\nvdash\alpha$. Let $\beta\in\mathbf{K}-\alpha$. Hence $\beta\in\mathbf{K}$. We will prove by cases that $\beta\in\mathbf{K}\div\alpha$.

Case 1) $\vdash\beta$. Then, by Lemma 10, $\alpha<_{\preceq}\beta$. Thus $\beta\in\mathbf{K}\div\alpha$.
Case 2) $\mathbf{K}\nvdash\alpha$. Then $\alpha\notin\mathbf{K}$ and, by Lemmas 8 and 9, $\alpha<_{\preceq}\beta$. Thus $\beta\in\mathbf{K}\div\alpha$.
Case 3) $\nvdash\beta$ and $\mathbf{K}\vdash\alpha$. Hence $\alpha\in\mathbf{K}$. From $\beta\in\mathbf{K}-\alpha$ it follows that $\beta\in cut_{\prec}(\alpha)$. Therefore, by Observation 2.9, $\alpha\prec\beta$. Assume by *reduction ad absurdum* that $\beta\leq_{\preceq}\alpha$. Thus, by the definition of $\leq_{\preceq}$, $cut_{\preceq}(\alpha)\subseteq cut_{\preceq}(\beta)$. Since $\alpha\in cut_{\preceq}(\alpha)$ it follows that $\alpha\in cut_{\preceq}(\beta)$. Hence, by Lemma 11, $\beta\preceq\alpha$. Contradiction. Hence by Lemma 8, $\alpha<_{\preceq}\beta$. Therefore $\beta\in\mathbf{K}\div\alpha$.

Let $\beta\in\mathbf{K}\div\alpha$. Then $\beta\in\mathbf{K}$ and $\alpha<_{\preceq}\beta$. We will prove by cases that $\beta\in\mathbf{K}-\alpha$.

Case 1) $\mathbf{K}\nvdash\alpha$. Then $cut_{\prec}(\alpha)=\mathbf{K}$ (by Lemma 1 (c)). Then $\beta\in\mathbf{K}-\alpha$.
Case 2) $\vdash\beta$. Hence, by ($\preceq$2) and ($\preceq$3), $H=\{\gamma\in\mathbf{K}:\beta\preceq\gamma\}\subseteq Cn(\emptyset)$. Thus $H\nvdash\alpha$. Hence $\beta\in cut_{\prec}(\alpha)$. Therefore $\beta\in\mathbf{K}-\alpha$.

Case 3) $\mathbf{K} \vdash \alpha$ and $\nvdash \beta$. From $\alpha <_{\preceq} \beta$ it follows, by the definition of $\leq_{\preceq}$, that $cut_{\preceq}(\beta) \subset cut_{\preceq}(\alpha)$. Therefore, there exists $\delta \in cut_{\preceq}(\alpha)$ such that $\delta \notin cut_{\preceq}(\beta)$. By Lemma 11 it follows that $\beta \npreceq \delta$. Thus, since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\delta \prec \beta$. Hence $\beta \in cut_{\prec}(\alpha)$. Therefore $\beta \in \mathbf{K} - \alpha$. ■