

Local times for grey Brownian motion

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In this paper we study the grey Brownian motion, namely its representation and local time. First it is shown that grey Brownian motion may be represented in terms of a standard Brownian motion and then using a criterium of S. Berman, *Trans. Amer. Math. Soc.*, **137**, 277–299 (1969), we show that grey Brownian motion admits a λ -square integrable local time almost surely (λ denotes the Lebesgue measure). As a consequence we obtain the occupation formula and state possible generalizations of these results.

Keywords: Brownian motion; grey Brownian motion; local time.

1. Introduction

Grey Brownian motion (gBm) was introduced by W. Schneider^{1,2} as a model for slow anomalous diffusions, i.e., the marginal density function of the gBm is the fundamental solution of the time-fractional diffusion equation, see also Ref. 3. This is a class $\{B_\beta(t), t \geq 0, 0 < \beta \leq 1\}$ of processes which are self-similar with stationary increments. More recently, this class was extended to the, so called “generalized” grey Brownian motion (ggBm) to include slow and fast anomalous diffusions which contain either Gaussian or non-Gaussian processes e.g., fBm, gBm and others. The time evolution of the marginal density function of this class is described by partial integro-differential equations of fractional type, see Refs. 4,5. In this paper we investigate the class of gBm, namely their representation in terms of standard Brownian motion (Bm) and show the existence of local time.

In Section 2 we recall the construction and certain properties of gBm, in particular the representation of gBm as a product of Bm and an independent positive random variable, see (6) below. Here we would like to emphasize the fact that the representation (6) for gBm allow us to study certain properties of gBm in terms

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of those from Bm, e.g., path properties and simulations. Finally, in Section 3 we use the criterium due to S. Berman⁶ in order to show that gBm admits a λ -square integrable local time, almost surely, cf. Theorem 3.1 below. As a corollary we obtain the occupation formula.

2. Grey Brownian motion

In this section we recall the construction of gBm due to W. Schneider.² The grey noise space is the probability space $(S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu_\beta)$, where $S'(\mathbb{R})$ is the space of tempered distributions defined on \mathbb{R} , $\mathcal{B}(S'(\mathbb{R}))$ is the σ -algebra generated by the cylinder sets and μ_β is the grey noise measure given by its characteristic functional

$$\int_{S'(\mathbb{R})} e^{i\langle w, \varphi \rangle} d\mu_\beta(w) = E_\beta \left(-\frac{1}{2} \|\varphi\|^2 \right), \quad \varphi \in S(\mathbb{R}), \quad 0 < \beta \leq 1. \quad (1)$$

Here $\langle \cdot, \cdot \rangle$ is the canonical bilinear pairing between $S(\mathbb{R})$ and $S'(\mathbb{R})$, $\|\cdot\|$ the norm in $L^2(\mathbb{R})$ and E_β is the Mittag-Leffler function of order β defined by

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)}, \quad x \in \mathbb{R}.$$

The range $0 < \beta \leq 1$ is to ensure the complete monotonicity of $E_\beta(-x)$, see Ref. 7, i.e., $(-1)^n E_\beta^{(n)}(-x) \geq 0$ for all $x \geq 0$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. In other words, there exists a probability measure ν_β on \mathbb{R}_+ which is absolutely continuous with respect to the Lebesgue measure with density M_β such that

$$E_\beta(-x) = \int_0^\infty e^{-\tau x} d\nu_\beta(\tau) = \int_0^\infty e^{-\tau x} M_\beta(\tau) d\tau. \quad (2)$$

The density M_β , the so-called M -Wright probability density function, is related to the fundamental solution of the time-fractional diffusion equation, emerges as a natural generalization of the Gaussian distribution. It is also a special case of the Wright function, namely, $M_\beta(x) = W_{-\beta, 1-\beta}(-x)$, see eq. (3.5) in Ref. 8.

The absolute moments of M_β in \mathbb{R}_+ are given by (see eq. (4.7) in Ref. 8)

$$\int_0^\infty \tau^\delta M_\beta(\tau) d\tau = \frac{\Gamma(\delta + 1)}{\Gamma(\beta\delta + 1)}, \quad \delta > -1. \quad (3)$$

Remark 2.1. Let μ_τ , $\tau > 0$, denote the Gaussian measure on $\mathcal{B}(S'(\mathbb{R}))$ with intensity τ , i.e.,

$$\int_{S'(\mathbb{R})} e^{i\langle w, \varphi \rangle} d\mu_\tau(w) = e^{-\frac{\tau}{2} \|\varphi\|^2}, \quad \varphi \in S(\mathbb{R}).$$

Then (2) gives the decomposition

$$\mu_\beta = \int_0^\infty \mu_\tau M_\beta(\tau) d\tau,$$

which says that the grey noise measure μ_β is the mixture of the the Gaussian measures μ_τ , $\tau \geq 0$ with $\mu_0 = \delta_0$ the Dirac measure at zero.

It is easy to show that the random variable $X_\beta(\mathbb{1}_{[0,t]})(\cdot) = \langle \cdot, \mathbb{1}_{[0,1]} \rangle$, $t \geq 0$ is a well defined element in $L^2(S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu_\beta) =: L^2(\mu_\beta)$ and

$$\|X_\beta(\mathbb{1}_{[0,t]})\|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta + 1)} \|\mathbb{1}_{[0,t]}\|^2 = \frac{1}{\Gamma(\beta + 1)} t.$$

The gBm B_β is then defined as the stochastic process

$$B_\beta = \{B_\beta(t) := X_\beta(\mathbb{1}_{[0,t]}), \quad t \geq 0\}.$$

The following properties can be easily derived from (1) and the fact that $\|\mathbb{1}_{[0,t]}\|^2 = t$.

- (1) $B_\beta(0) = 0$ almost surely. In addition, for each $t \geq 0$, the moments of any order are given by

$$\begin{cases} \mathbb{E}(B_\beta^{2n+1}(t)) &= 0 \\ \mathbb{E}(B_\beta^{2n}(t)) &= \frac{(2n)!}{2^n \Gamma(\beta n + 1)} t^n. \end{cases}$$

Here \mathbb{E} denotes the expectation with respect to μ_β .

- (2) For each $t, s \geq 0$, the characteristic function of the increments is

$$\mathbb{E}(e^{i\theta(B_\beta(t) - B_\beta(s))}) = E_\beta \left(-\frac{\theta^2}{2} |t - s| \right), \quad \theta \in \mathbb{R}. \tag{4}$$

- (3) The covariance function has the form

$$\mathbb{E}(B_\beta(t)B_\beta(s)) = \frac{1}{2\Gamma(\beta + 1)} (t \wedge s), \quad t, s \geq 0. \tag{5}$$

All these properties may be summarized as follows. For any $0 < \beta \leq 1$, the gBm $B_\beta(t)$, $t \geq 0$, is $\frac{1}{2}$ -self-similar with stationary increments. It is clear that for $\beta = 1$ the gBm coincides with Bm.

It was shown in Ref. 4 that the gBm B_β admits the following representation

$$\{B_\beta(t), t \geq 0\} \stackrel{d}{=} \{\sqrt{Y_\beta} B(t), t \geq 0\}, \tag{6}$$

where $\stackrel{d}{=}$ denotes the equality of the finite dimensional distribution and B is Bm. Y_β is an independent non-negative random variable with probability density function $M_\beta(\tau)$, $\tau \geq 0$.

3. Local times for grey Brownian motion

In this section we prove the existence of local times for gBm using the criterium due to Berman,⁶ Lemma 3.1. For the readers convenience we recall the notion of occupation measure as well as occupation density.

Let $f : [0, T] \rightarrow \mathbb{R}$ be a (nonrandom) measurable function and define, for any set $F \in \mathcal{B}(\mathbb{R})$, the occupation measure μ_f on $[0, T]$ of f up to “time” T by

$$\mu_f(F) := \int_I \mathbb{1}_F(f(s)) ds = \lambda(\{t \in [0, T] : f(t) \in F\}),$$

where λ is the Lebesgue measure on $[0, T]$. Hence, $\mu_f(F)$ describes the amount of time spent by f in F during the time period $[0, T]$. In particular, if $X = (X_t)_{t \in [0, T]}$ is a stochastic process, then the occupation measure of the sample path

$$[0, T] \ni t \mapsto X_t(w) \in \mathbb{R}$$

is defined in the same way but now the measure $\mu_{X_t(w)}$ is a random measure, it depends on the sample point w of the probability space. We say that f has an occupation density over $[0, T]$ if μ_f is absolutely continuous with respect to the Lebesgue measure λ and denote this density by $L^f(\cdot, [0, T])$. In explicit form, for any $x \in \mathbb{R}$,

$$L^f(x, [0, T]) = \frac{d\mu_f}{d\lambda}(x).$$

Thus, we have

$$\mu_f(F) = \int_0^T \mathbb{1}_F(f(s)) ds = \int_F L^f(x, [0, T]) dx.$$

A continuous stochastic process X has an occupation density on $[0, T]$ if, for almost all $w \in \Omega$, $X(w)$ has an occupation density $L^X(\cdot, [0, T])$, also called local time of X , see Berman.⁶

The criteria for the existence of local times for stochastic processes are due to Berman,⁶ Section 3. More precisely, a stochastic process X admits a local time if and only if

$$\int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \mathbb{E}(e^{i\theta(X(t)-X(s))}) ds dt \right| d\theta < \infty. \tag{7}$$

In the following we show that (7) is fulfilled if the stochastic process X is the gBm B_β . In fact, from (4) the characteristic function of the increments of gBm B_β is given by

$$\mathbb{E}(e^{i\theta(B_\beta(t)-B_\beta(s))}) = E_\beta\left(-\frac{\theta^2}{2}|t-s|\right).$$

Using Fubini, and the change of variables $r = (2)^{-1/2}\theta|t-s|^{1/2}$, we have to compute at first

$$\int_{\mathbb{R}} E_\beta\left(-\frac{\theta^2}{2}|t-s|\right) d\theta = \sqrt{\frac{2}{|t-s|}} \int_{\mathbb{R}} E_\beta(-r^2) dr.$$

The integral in the rhs may be computed using the representation of the Mittag-Leffler function (2), Fubini theorem again and the Gaussian integral, namely

$$\int_{\mathbb{R}} E_\beta(-r^2) dr = \int_0^\infty M_\beta(\tau) \int_{\mathbb{R}} e^{-r^2\tau} dr d\tau$$

$$\begin{aligned}
&= \sqrt{\pi} \int_0^\infty \tau^{-1/2} M_\beta(\tau) d\tau \\
&= \sqrt{\pi} \frac{\Gamma(-\frac{1}{2} + 1)}{\Gamma(-\beta\frac{1}{2} + 1)}.
\end{aligned}$$

In the last equality we used the absolute moments of the M -Wright function M_β given in (3).

Finally, the t, s -integration is performed as follows.

$$\int_0^1 \int_0^1 \frac{1}{|t-s|^{1/2}} ds dt = 2 \int_0^1 \int_0^t \frac{1}{|t-s|^{1/2}} ds dt = 4 \int_0^1 t^{1/2} dt = \frac{8}{3}.$$

Therefore, putting all together, we obtain

$$\int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \mathbb{E}(e^{i\theta(X(t)-X(s))}) ds dt \right| d\theta = \frac{8\sqrt{2\pi}}{3} \frac{\Gamma(-\frac{1}{2} + 1)}{\Gamma(-\beta\frac{1}{2} + 1)} < \infty.$$

Thus, we have shown the main result of this subsection which we state in the following theorem.

Theorem 3.1. *The gBm process B_β admits a λ -square integrable local time $L^{B_\beta}(\cdot, [0, T])$ almost surely.*

Corollary 3.1. *As a consequence of the existence of the local time $L^{B_\beta}(\cdot, [0, T])$, we obtain the occupation formula*

$$\int_0^T f(B_\beta(t)) dt = \int_{\mathbb{R}} f(x) L^{B_\beta}(x, [0, T]) dx, \text{ a.s.} \quad (8)$$

Remark 3.1. The above results may be generalized/realized in various directions.

- (1) Theorem (3.1) may be generalized for the so-called “generalized” grey Brownian motion $B_{\beta, \alpha}$ introduced by A. Mura and F. Mainardi⁵ such that for $\alpha = 1$ we recover the gBm, i.e., $B_{\beta, 1} = B_\beta$. Moreover, the local times $L^{B_{\beta, \alpha}}(\cdot, [0, T])$ of $B_{\beta, \alpha}$ may be weak approximated by the number of crossings of a regularization by convolution of $B_{\beta, \alpha}$. For the details see Ref. 9.
- (2) On the other hand, we may develop the Appell system which is a biorthogonal system of polynomials associated to the grey noise measure μ_β in order to construct, describe and characterize test and generalized functions spaces. Then the local time of gBm may be understood as a generalized function in this framework. For the details see Ref. 10.

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