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Concept of an elementary work as introduction to the line integral in engineering studies

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Abstract

The paper is devoted to the line integral topic belonging to the section of vector calculus in Mathematical Analysis applied to the undergraduate Mechanical Engineering program. An efficient way of teaching line integrals is proposed and developed based on the elementary work/force principles. By this way, the mathematical concept of the line integral is supposed to be learned in harmony with the elementary mechanics to appreciate its diversity and to set up the right idea about the scientific area covered by Mechanical Engineering and related academic and technical fields. In terms of practical training, the research also offers two new techniques of analytical calculus for line integrals containing singularities and provides a new coherent engineering approach to deal with vector fields as integrands in line integrals.

Keywords

Line integral, vector fields with singularities, elementary work, analytical calculus

Introduction

Line integrals are mandatory for all undergraduate engineering studies and constitute the subject of vector calculus.[1–3] Because of their common usage in such a vast number of engineering fields (mechanical, civil, marine, industrial, aerospace, chemical, electrical,

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etc.), the methodology of teaching and specific area of applications should be carefully observed and developed.

In terms of mandatory courses required for the first and second years of engineering undergraduate programs, Mechanical Engineering degree is the most challenging one[4] where the course on General and Theoretical Mechanics is a two-semester long, and is based on all three fundamental parts of Theoretical mechanics: statics, kinematics and dynamics.[5,6] The first part which is taught at the very beginning of the Bachelor program consists of General Mechanics (like notions of forces, work, energy, heat and basic laws of nature) and Statics (geometry of forces and all their actions and interactions in the equilibrium context). This discipline is going on simultaneously with the other two: Mathematical Analysis (where the concept of definite integral is introduced not only as a powerful tool of calculus but also as a mathematical model for a work/force conjuncture) and General Physics (where the students learn more about the extension of an elementary work/force applications to the gravitational, electrical and magnetic effects on the surroundings in the steady environment). Clarification of the connection between Mathematical Analysis (namely, Mathematical Analysis II) where the line integral topic is taught and General and Theoretical Mechanics along with General Physics from where the idea of this concept comes from, and, consequently, may be easily understood is shown in Figure 1 for Mechanical Engineering degree.

Single-semester course on General Mechanics with fundamental elements from Theoretical Mechanics: Statics and Kinematics is also required to achieve Bachelor degree in Industrial Engineering, Civil Engineering, Aerospace and Electronics majors. Figure 2 shows again how the courses on Mathematical Analysis are taught alongside with General Mechanics and Statics on basis of which the concept of line integral may be learned more efficiently. Here, the textbook by Bird[7] could be recommended as a supplementary material due to its simplicity of understanding and practical training motivating the students to proceed with more challenging explanations.

In spite of the fact that the same Calculus books[1–3] are used for other academic degrees like Physical Engineering, Electrical Engineering or Chemical one, the intrinsic nature of these programs is different (as it is shown in Figure 3) with the focus on the study of

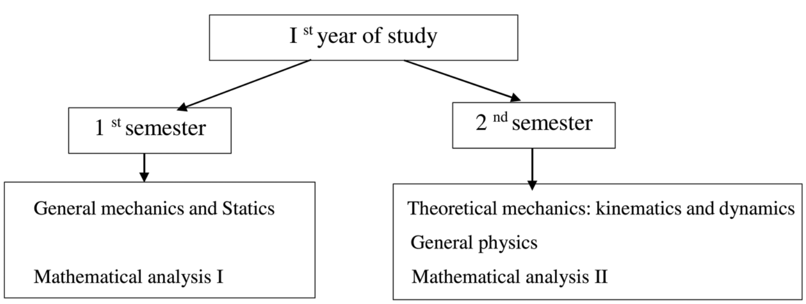


Figure 1. Demonstrative diagram showing Curriculum Units ongoing in parallel in the first year of study towards Mechanical Engineering degree (in general).

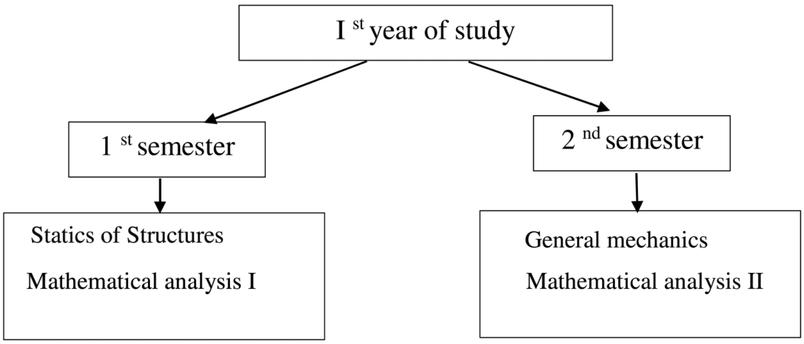


Figure 2. Demonstrative diagram showing Curriculum Units ongoing in parallel in the first year of study towards Civil Engineering degree (in general).

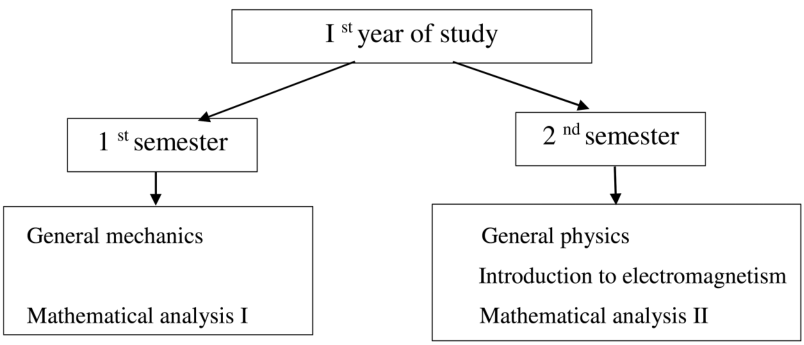


Figure 3. Demonstrative diagram showing Curriculum Units ongoing in parallel in the first year of study towards Engineering Physics degree (in general).

physics fundamentals, electromagnetism in non-steady environment and so on. To this end, the mechanical (physical) concepts of work, force, energy, heat and mathematical concept of integral should be taught coherently but still differently taking into account the principle major and, what is even more important, some of those differences should be well explained to the students to avoid further confusion while pursuing their professional careers.

So, the purpose and the novelty of the present research is to create an efficient way of teaching line integral topic based on the elementary work/force principles which are common and easily understood by various engineering groups of students and, particularly, beneficial for Mechanical Engineering major. By this approach, the mathematical concept of line integral will be learned in harmony with the elementary mechanics to appreciate its diversity and to set up the right idea about the scientific area covered by Mechanical Engineering and related branches. It is also important to note that the courses on Mathematical Analysis are taught in parallel (with respect to time-semester-schedules) with elementary or intermediate mechanics/physics courses as it follows from Figures 1–3.

In terms of practical training, the research also offers two new techniques of analytical calculus for line integrals containing singularities and provides a new coherent engineering approach to deal with vector fields as integrands in line integrals.

Differences and challenges while defining line integrals

Generally, Calculus books, for example,[1–3] have in fact two definitions of the line integral, namely, via an arc length, ds, as a differential element:

$$\int f(x, y)ds \tag{1}$$

and via a vector segment, d**r**, as a differential element:

$$\int \mathbf{F} \bullet \mathrm{d}\mathbf{r} \tag{2}$$

In equation (1), $f(x, y)$ is a scalar function of two variables x and y. Here and in what follows, only two-dimensional version of Euler space will be considered to avoid clutter (three-dimensional space is conceptually the same with an addition of one more variable/coordinate). In equation (2), $\mathbf{F} = \mathbf{F}(x, y)$ is a vector-valued function of the same two variables x and y, and it makes a dot (vector) product with the differential element d**r**, contrary to a simple scalar multiplication in equation (1). Of course, equations (1) and (2) are related, some books even consider them as the same equations with different notations only.[2] In spite of the fact that this statement is correct mathematically, these equations are still different in engineering practice, at least, for teaching purposes followed by a specific field of applications.[8] Commonly, equation (1) is suitable for calculations of surfaces' areas (could be together with the Green's theorem), masses and moments of inertia, for example, in electrical engineering. Meanwhile equation (2) occupies almost all branches of mechanical engineering starting from vector fields in statics and ending up in electromagnetism in dynamic interactions.

Interestingly, a popular undergraduate calculus textbook by Thomas[2] introduces the line integral based on the Riemann sum as an extension of the real one-dimensional definite integral. So, the line integral ends up as an integration of a scalar function. After that, the term *vector field* is pronounced on the purely mathematical basis like being a vector-valued function such that the connection between equations (1) and (2) becomes possible:

$$\int \mathbf{F} \bullet \mathbf{T}\,\mathrm{d}s = \int \mathbf{F} \bullet \mathrm{d}\mathbf{r} \tag{3}$$

where **T** stands for a unit vector in a specific direction. The drawback of this approach for mechanical engineering students is a creation of a wrong idea that equation (1) constitutes the primary definition of the line integral, and, as a result, everything about scalar functions of two variables and arc length should be studied first. However, it looks like unnecessary to acquire such a solid knowledge about those issues depriving the students from the acquiring in a more coherent way the topic of line integrals in engineering.

Such a way has been offered by Salas.[1] However, it was based on another, even more involved mathematical concept, like differential geometry. In spite of the fact that the formula produced for line integral calculations is straightforward, that approach did not permit to derive this formula from the desirable mechanical basis.

One more classical Calculus textbook for general undergraduate engineering programs by Stewart[3] considers first some mechanical examples of vector fields (gravitational and electric ones) as a prerequisite to study line integrals. However, again the theoretical part of the line integral introduction starts with the Riemann sum, scalar function of two variables and arc length as in.[2]

So, in this aspect, the logical way of teaching mathematics with its proper background contradicts the specific necessities of using line integrals in mechanical engineering applications. Besides, some new methodology is required also to respect the time-table of studies coherent with other subjects learned simultaneously, namely, Mathematical Analysis, General Mechanics/Physics, during initial years of Mechanical Engineering program. Hence, the present research is devoted to offer a new methodology of teaching line integrals for Mechanical Engineering and other related majors like Civil Engineering or Electronics based on the elementary work concept. This methodology is easy to be implemented but still rigorous mathematically and can be taught relying on the standard textbooks[1–3] as primary source of knowledge.

Calculation of an elementary work in various contexts

The methodology of teaching line integrals presumes, first of all, to define the line integral (it is better to say – to choose between *arc-length-based* definition (1) and *vector-segment-based* definition (2)). Then, suitability of examples provided for the study should be carefully respected to guarantee learning efficiency.

For Mechanical Engineering major, it is desirable to introduce line integral via the vector-segment definition (2) to come up later on with *force-field* applications like those which consider objects moving in a plane or space but still not dependent (as a mathematical model) on the real time factor (like velocity or real time). Here and in what follows, the term *object* will be substituted for a *ball* to save the usage of the term *object* for mathematical models and to reserve the usage of the term *particle* for physical contexts.

Surely, it should be kept in mind the motivation behind the *arc-length-based* definition (1). It even can be called primary definition because of its similarity with the one-dimensional definite integral. Meanwhile being excellent methodology for mathematically oriented students, it does not attend the goal of mechanical engineering programs. Other than that, it involves as a pre-requisite some pure mathematical concepts such as the Riemann sum and the theory of functions of two variables (which may be consulted as extra-curriculum in[9]). Contrary, the dot product of two vectors, i.e., the *vector-segment-based* definition (2), requires only basic curve-parameterization skill and general concept of an elementary work.

Another important note to the methodology proposed is to point out at the very beginning what it means *force movement*. When a force is moving a ball, it does not mean necessarily that the time factor is explicitly involved in the mathematical model. This is a characteristic issue of line integrals in mechanical engineering (like considering gravitational or electric forces in steady environment[3] where the time notation t serves as a parameter[10]). It is exactly opposite to Physics where the time t is the actual time like in Electromagnetism (Maxwell equations) in non-steady environment.[2]

To this end, the work/energy principle while serving for mathematical models may or may not include time factor as a real time. In mechanical engineering applications, this law is usually studied extensively in Dynamics section and is related also (in a more subtle way) to the second law of Newton. The term *work* presented in this law, however, has another context which is not essential for the study of line integrals.

So, to gain a knowledge about the line integral for mechanical engineering applications, it is sufficient to consider just an elementary work studied even at high schools. There are several cases to be considered consecutively for the correct methodology proposed in the present research.

Case 1) Movement of a ball subjected to a force (as a vector) in one-dimensional context. Here four sub-cases should be reflected:

(a) The force is a constant and aligned with the motion along the x-axis.

Consider a ball moving along the horizontal x-axis from $x = a$ to $x = b$ under the action of a force $\mathbf{F}$. If the vector $\mathbf{F}$ is horizontal (along the x-axis) and has constant magnitude, $\|\mathbf{F}\| = F$, then the work done on the ball is

$$W = F\,s \tag{4}$$

where $s = b - a$ is the displacement along the straight line traveled by the ball as depicted in Figure 4(a).

The force is a function of x but, as a vector, still aligned with the x-axis.

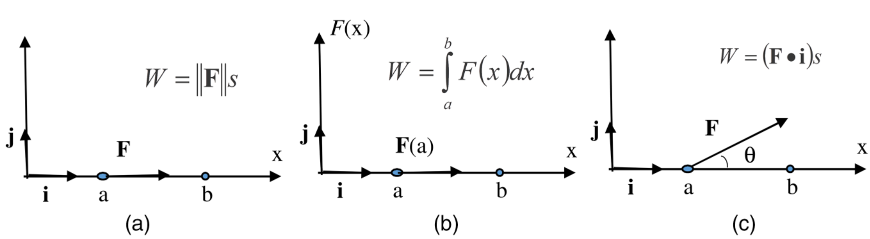


Figure 4. Movement of a force as a vector in one-dimensional context: (a) the force is a constant and aligned with the x-axis; (b) the force is a function of x and aligned with the x-axis; (c) the force is a constant and inclined at an angle θ to the x-axis.

If a force is a variable but a continuous function of magnitude $F = F(x)$, then the work done on the ball is

$$W = \int_a^b F(x)dx \tag{5}$$

Here, in fact, the displacement $s = b - a$ or, mathematically speaking, the distance traveled by the ball is divided into equally-spaced intervals, like $a = x_0,\ x_1,\ \ldots,\ x_i,\ x_{i+1},\ x_n = b$, and the total work is the sum (integral) over all sub-intervals $dx = x_{i+1} - x_i$. In that sense, the force on each sub-interval is actually evaluated as $F(x_i)$ and geometrically it is placed (as a vector) horizontally at the initial point of each sub-interval, let's say $x = a$, as illustrated in Figure 4(b).

The force is a constant but inclined at an angle θ to the direction of a straight-line motion along the x-axis.

If the vector $\mathbf{F}$ is inclined to the straight-line trajectory but still has a constant magnitude, $\|\mathbf{F}\| = F$, Figure 4(c), then the formula for the work is better off to be written via the dot product of two vectors rather than the scalar quantities such as

$$W = (\mathbf{F} \bullet \mathbf{i})\, s \tag{6}$$

where $(\mathbf{F} \bullet \mathbf{i})$ is exactly the scalar component of $\mathbf{F}$ onto the direction of motion indicated by the standard unit vector $\mathbf{i}$ along the x-axis.

The force is a function of x and inclined to the direction of motion along the x-axis.

This is the most general sub-case of a movement of a force as a vector in one-dimensional context. So, the force here is described by a vector quantity with scalar argument x, $\mathbf{F} = \mathbf{F}(x)$, and the work done on the ball should be calculated via integral

$$W = \int_a^b (\mathbf{F} \bullet \mathbf{i})\, dx \tag{7}$$

As one may expect, these formulas (4)-(7) serve as the basic preliminary model to deal with the line integral concept in engineering. So, the next step is to extend one-dimensional context to two-dimensional (or three-dimensional) one.

Case 2) Movement of a ball subjected to a force (as a vector) in two-dimensional context. Here several sub-cases may also be distinguished. But, first of all, some changes in terminology are required. If a force as a vector is placed into two-dimensional plane, i.e., $\mathbf{F} = \mathbf{F}(x,\ y)$, then the force technically is a *vector-field* (in mathematics) or *force-field* (in engineering). What is predominant is that it is now a vector-valued function of two scalar arguments $(x,\ y)$:

$$\mathbf{F}(x,\ y) = P(x,\ y)\,\mathbf{i} + Q(x,\ y)\,\mathbf{j} \tag{8}$$

where $P(x, y)$ and $Q(x, y)$ are some scalar functions of two variables and **i**, **j** are standard unit vectors, in x and y directions, respectively. (Three-dimensional version is completely analogous to two-dimensional one, so, it will not be targeted in this paper to avoid clutter).

Being back to the two-dimensional context, in the simplest sub-case a), the vector-field is a constant, i.e., $\mathbf{F}(x, y) = \mathbf{F}(x_0, y_0)$, such as depicted in Figure 5 with the motion of a ball along straight-line. Here, (x_0, y_0) is a fixed pair of coordinates of vector **F**. The qualitative picture of the vector-field is drawn via small-sized vectors, and only one of them is drawn bigger and placed at the starting point on the trajectory of the ball just for the sake of visualization convenience. Thus, the work of the vector-field done on the ball will be the scalar projection of $\mathbf{F}(x_0, y_0)$ onto the direction of motion indicated by the unit vector $\frac{\mathbf{s}}{\|\mathbf{s}\|}$. In fact, **s** here is the line-trajectory vector such as the distance traveled by the ball is the length of this vector, i.e., $\|\mathbf{s}\|$. So, the work done on the ball is

$$W = \left(\mathbf{F} \bullet \frac{\mathbf{s}}{\|\mathbf{s}\|}\right)(\|\mathbf{s}\|) = \mathbf{F} \bullet \mathbf{s} \tag{9}$$

which is similar to equation (6), i.e., no integral is required as soon as the vector-field is constant. As one may see, this conclusion is logic, easy to grasp, understand and remember by the students. So, let us proceed by the same methodology to other more involved cases.

Sub-case b). Movement of a ball through a two-dimensional variable vector-field. An example of a simple, nice-looking but varying vector-field $\mathbf{F}(x, y) = -\langle x, y\rangle$ is depicted in Figure 6. In terms of the vector-field components, equation (8), one has $P(x, y) = -x$ and $Q(x, y) = -y$. To start considering this sub-case, two possibilities of the ball's motion should be taken into account, namely, the straight-line trajectory and any other curved trajectory, like a circle, for example, which is also drawn in Figure 6. However, as soon as the vector-field is not a constant, the work calculation requires not only an integration procedure but also parameterization of the trajectory

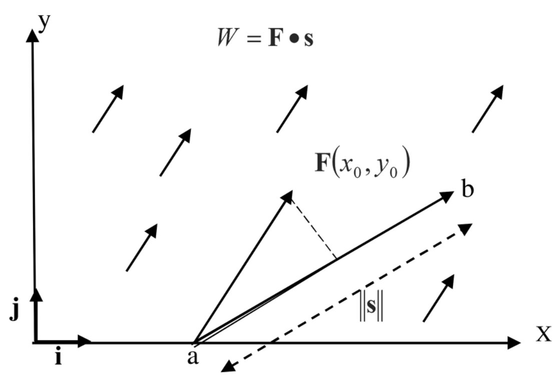


Figure 5. Movement of a force as a vector in two-dimensional context.

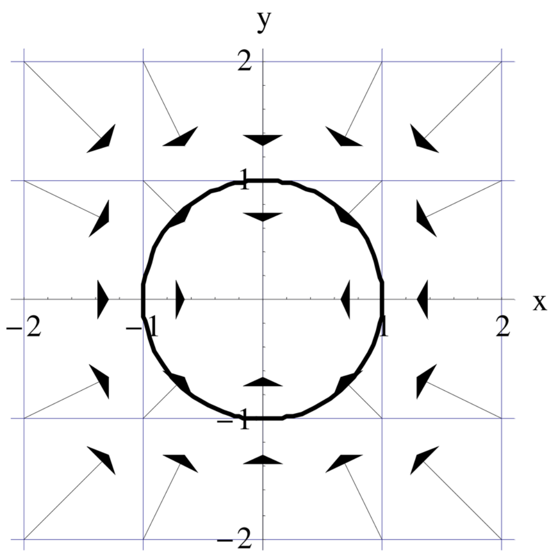


Figure 6. Two-dimensional variable vector-field $\mathbf{F}(x,\ y) = -\langle x,\ y\rangle$ with a circular ball's trajectory.

(independently, straight one or curved). So, parameterization is also the property of the variable vector-field and not the path taken by the ball. This is another important conclusion to appreciate.

Following the classic parametric-curve theory,[3] Figure 7 describes some rectilinear segment AB on a line L characterized by an initial point A given by a position vector $\mathbf{r}_0$ in the direction of another given position vector $\mathbf{v}$ which has the same sense as the displacement vector $\mathbf{s}$ from the point A to the point B. Point B is described by an arbitrary position vector $\mathbf{r}_1$. It means that in parametric form with t being a parameter interpreted, without loss of generality,[10] as a time factor, the best parameterization for a line segment is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,(\mathbf{r}_1 - \mathbf{r}_0), \quad 0 \le t \le 1 \tag{10}$$

where the difference $(\mathbf{r}_1 - \mathbf{r}_0)$ serves as a direction vector $\mathbf{v}$. For a circle of radius R centered at the coordinates $\begin{Bmatrix} c_x \\ c_y \end{Bmatrix}$ the parameterization is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c_x \\ c_y \end{Bmatrix} + \begin{Bmatrix} R\ \cos\ t \\ R\ \sin\ t \end{Bmatrix}, \quad 0 \le t \le 2\pi \tag{11}$$

In general, from the mathematical point of view, the interpretation of t-parameter is not such a trivial task, and may require some specific care in advanced mechanical courses[11] or while dealing with pure mathematics.[12] However, it is much simpler for any introductory mechanics course.

The linearization of any curve C (as for the straight-line segment, equation (10), or for a part of the circle, equation (11)) between two coordinates $x = a$ and $x = b$ corresponding to some step or time interval h is drawn in Figure 8(a). Hence, when the usage of a parametric curve theory is necessary for further engineering applications, the displacement vector **s** should be substituted for an infinitesimal differential (direction) vector d**r** due to Leibnitz

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} \tag{12}$$

as depicted in Figure 8(b). In fact, this vector is exactly the vector-segment in the definition of the line integral (2). To this end, the following mathematical concepts are relevant to be introduced for the students to appreciate the coherence between mathematical and

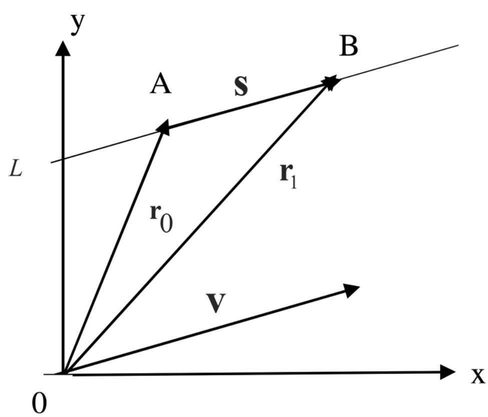


Figure 7. Parametric vector description of a straight-line L via the given point A and the direction vector **v**.

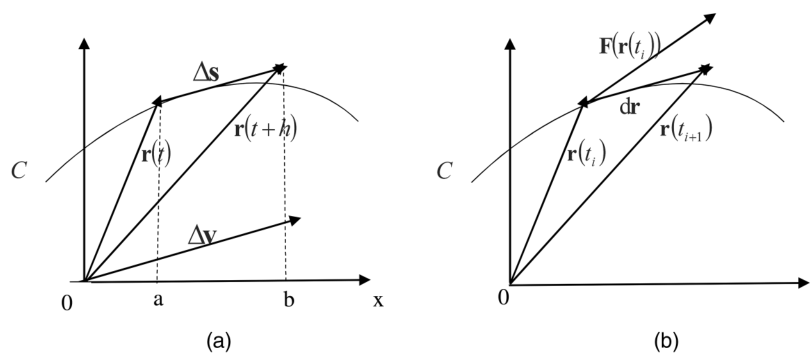


Figure 8. Geometric linearization of a curve C: (a) between two coordinates $x = a$ and $x = b$; (b) infinitesimal linearization corresponding to differential element d**r**.

engineering interpretations: $\Delta\mathbf{s} = \mathbf{r}(t+h) - \mathbf{r}(t)$ is the displacement (vector) for a time interval $[t,\ t+h]$. Then, $\Delta\mathbf{v} = \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ is an average velocity (vector) which has the same direction as $\Delta\mathbf{s}$. So, an instant velocity (vector) will be $\mathbf{r}'(t) = \lim_{h->0} \Delta\mathbf{v}$. Then, as before, the path is divided into short segments, in order to find the work done in moving the ball along each short segment. The total work is determined via integration procedure taking into account that, in terms of infinitesimal quantities, $d\mathbf{r} = \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i) \approx \mathbf{r}'(t_i)(t_{i+1} - t_i)$. It was assumed here that $\mathbf{r}'(t_{i+1}) \approx \mathbf{r}'(t_i)$ for curved segments. It is, of course, a point of attention that rigorous equality, $\mathbf{r}'(t_{i+1}) = \mathbf{r}'(t_i)$, maintains for straight lines. Finally, the practical engineering rule for the work calculation will be what is technically called *line integral*

$$W = \int_{\alpha}^{\beta} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \sum_{i=1}^{n} \mathbf{F}[\mathbf{r}(t_i)] \bullet \mathbf{r}'(t_i)(t_{i+1} - t_i) = \int_{\alpha}^{\beta} \mathbf{F}[\mathbf{r}(t)] \bullet \mathbf{r}'(t)dt, \quad \alpha \le t \le \beta \quad (13)$$

So, the *line integral* is in fact the analog of a regular integral but related to some vector-field rather than a simple one-dimensional force. This conclusion is of utmost importance for the students pursuing Mechanical Engineering career and all other relevant engineering degrees.

Taking into account equations (8) and (12), the work defined by equation (13) can also be viewed as

$$W = \int_{\alpha}^{\beta} \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \int_{C} [P\,dx + Q\,dy] \quad (14)$$

which again, in terms of the practical engineering rule, would be

$$W = \int_{a}^{b} [P(x,\ f(x))\,dx + Q(x,\ f(x))\,f'(x)\,dx] \quad (15)$$

for the cases when the curve C is represented by a graphic of the function $y = f(x)$. Clearly, here $dy = f'(x)\,dx$ as in one-dimensional calculus. The limits of integration are also changed with respect to x-coordinate: $a \le x \le b$.

Classification of vector-fields in terms of mathematical modeling

Correct mathematical modeling of vector-fields is of utmost importance in practical calculations of line integrals. For some models these calculations may be done even without direct evaluation of integrals.

For example, Figure 6 represents an axisymmetric vector-field $\mathbf{F}(x,\ y) = -\langle x,\ y\rangle = -x\mathbf{i} - y\mathbf{j}$. To analyze the features of this field, the polar coordinates $\{r,\ \theta\}$ should be preferred where no dependence on θ takes place, i.e., $\mathbf{F}(x,\ y) = \mathbf{F}(\mathbf{r})$ such as

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j},\ \|\mathbf{r}\| = \sqrt{x^2 + y^2},\ r = \|\mathbf{r}\| = \sqrt{x^2 + y^2},\ r^2 = \|\mathbf{r}\|^2 = x^2 + y^2 \quad (16)$$

In fact, the famous gravitational effect in Newton's mechanics can be classified as the axisymmetric vector-field or force-field

$$\mathbf{F}(x,\ y) = -\frac{C_0}{\left(\sqrt{x^2 + y^2}\right)^3}(x\mathbf{i} + y\mathbf{j}) = -\frac{C_0}{\|\mathbf{r}\|^3}\mathbf{r} = -\frac{C_0}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = \mathbf{F}(\mathbf{r}) \quad (17)$$

where C_0 is a physical constant. To this end, the most rigorous way of modeling vector-fields in general is via the product of the magnitude of the vector-field, like $\frac{C_0}{\|\mathbf{r}\|^2}$, and the unit vector in the positive direction (outward from the origin of the Cartesian coordinate system), like $\frac{\mathbf{r}}{\|\mathbf{r}\|}$. The *minus* sign in equation (17) comes from the mechanics of the gravitation effect.

Indeed, while teaching calculus for engineering majors, it is instructive to show that the constant C_0 in equation (17) is $C_0 = GMm$ where G is the Newton's constant responsible for the strength of the gravitational force (or, technically, force-field), M is the global (external) mass of the earth (i.e., what is called *gravity*, - it is exactly related to M), and m is the local (internal) mass of a ball moving in the presence of the gravity. So, from the engineering perspective, equation (17) reflects the force exerted by the gravitational field on the ball with mass m. This is the Newton's famous inverse-square law and the force (technically, the force-field as a vector) points towards the origin which explains *minus* sign in equation (17).

It is also worth to be mentioned that it is common in engineering practice to use both terms: the gravitational field and the force-field. The difference is that the term *gravitational field* is reserved exclusively for the property of the global mass M alone, i.e., it is defined by the same equation (17) while taking $m = 1$ such as

$$\mathbf{F}(x,\ y) = -\frac{GM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \quad (18)$$

From here, one may conclude that equation (18) stands for the gravitational field which is defined as a force-field per unit internal mass m such as the strength of the gravitational field is proportional to the force that acts on a given internal mass.

The same engineering concept is applied to the electric field and Coulomb's force-field. A charge Q placed at the origin of the Cartesian coordinate system produces an electric field

$$\mathbf{E}(x,\ y) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \quad (19)$$

where ε_0 is the *permittivity* constant in physics. So, the force exerted on a local charge q

corresponds exactly to the Coulomb's force-field:

$$\mathbf{F^c}(x,\ y) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{20}$$

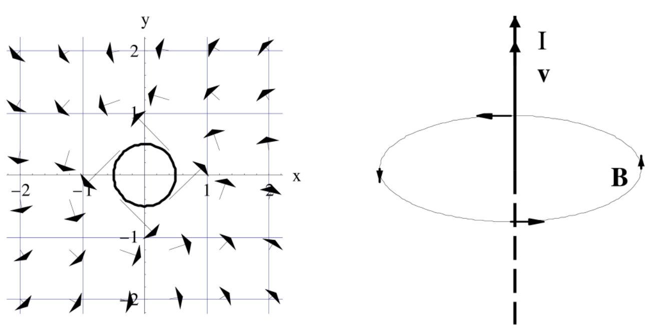
Obviously, equation (19) and equation (20) are proportional and classified as the axisymmetric vector-fields. No specific sign (*plus* or *minus*) is attributed here to the charge q since electric effect is two-sided (attractive or repulsive) contrary to the gravitational force which is directed towards the global mass.

It is instructive to mention that any axisymmetric field can be represented as a function of $\mathbf{r}$ only. So, the work done by the axisymmetric field on a ball moving along a circle, for example, will be always zero on the basis of a simple geometrical property of two orthogonal vectors: one which represents the field, due to equation (13), $\mathbf{F}[\mathbf{r}(t)]$, and another one which represents the trajectory of the ball's motion, $\mathbf{r}'(t)$. The dot product of these vectors in equation (13) gives zero-value to the respective integral without any calculus of the integral itself.

It is worth to be also mentioned that an axisymmetric field is a special case of the general three-dimensional model as well. To see the difference between an axisymmetric field and a particular case of another three-dimensional magnetic vector-field (with K being a relevant constant in physics)

$$\mathbf{B}(x,\ y) = \frac{K}{r^2}\langle -y,\ x\rangle = \frac{K}{r^2}(-y\,\mathbf{i} + x\,\mathbf{j}) \tag{21}$$

one should notice that the field (21) cannot be represented as a function of $\mathbf{r}$ only contrary to equation (17). The planar visualization of this filed is drawn in Figure 9(a) and reflects the original magnetic field surrounding a current in a straight long wire. To help viewing the magnetic vector-field as a three-dimensional one, Figure 9(b) interprets geometrically the Ampere's law, $\int_C \mathbf{B} \bullet d\mathbf{r} = \mu_0 \mathrm{I}$, which is widely used for magnetic fields around long



(a) (b)

Figure 9. Geometric representation of a magnetic vector-field: (a) planar visualization of magnetic field surrounding a current in a straight long wire; (b) geometric interpretation of the Ampere's law $\int_C \mathbf{B} \bullet d\mathbf{r} = \mu_0 \mathrm{I}$.

straight wires. Here the wire is positioned along the z-axis with $\boldsymbol{v}$ being the velocity-vector of moving charges, I stands for a steady current (that flows up to the z-axis) and μ_0 is the *permeability* physical constant. Hence, the correct writing of equation (21) is

$$\mathbf{B}(x,\ y,\ z) = \frac{K}{r^2}\{-y,\ x,\ 0\} = \frac{K}{r^2}(-y\,\mathbf{i} + x\,\mathbf{j} + 0\,\mathbf{k}) \tag{22}$$

On the other hand, if one considers the motion of a ball in a constant magnetic field $\mathbf{B}(x,\ y,\ z) = B_0$ (for steady charges), the same Figure 9(a) is applicable as well, however, in this case, the magnetic field lies in the z-direction (with unit vector $\mathbf{k}$) such as $\mathbf{B}(x,\ y,\ z) = (0,\ 0,\ B_0)$. By looking at Figure 9(a) and noticing how (in which direction) the circle is traveled by the ball through the constant magnetic field, it is easy to predict the sign of the work calculated by the integral (17) and interpret the result. If the computed work comes positive then the ball withdraws (and accumulates) the mechanical energy from the field, but, if the work comes negative, - the ball loses its energy. Anyway, for both constant and variable magnetic fields, the field classification is the three-dimensional one (neither the radial (axisymmetric), nor two-dimensional).

Vector-fields as mathematical object

Usually, physical effect (which is some *field* like gravitational or electrical) is directly related to the respective force (like Newton's force or Coulomb's force) which is also a vector-field. So, in both cases, the mathematical model is characterized by, what is called, *vector-field*. So, the term *vector-field* can be viewed as some mathematical object to which necessary criteria should be attributed for a specific engineering usage.

But, first of all, it should be recognized that the only correct form of representing any vector-field is equation (8). From here, the principle criteria are about the function of two-variables (like domain, continuity, differentiability), and the line integral theory (like requirements for the curve, region connectedness, integrability). These criteria are essential in order to obtain a meaningful solution to practical engineering problems arisen from physical effects.

Take $\mathbf{F}(x,\ y)$ to be a force-field, then the work done by the field over a curve C is given by equations (13)-(15). Smooth curve C is defined as continuous with a parameterization $\mathbf{r}(t)$ such as $\mathbf{r}'(t) \neq 0$. The modeling requirements are: take $\mathbf{F}(x,\ y)$ to be a continuous vector-field defined on a smooth (or piecewise-smooth) curve C given by a vector-valued function $\mathbf{r}(t),\ \ \alpha \leq t \leq \beta$. Then the line integral of $\mathbf{F}(x,\ y)$ along C is most easily evaluated by equation (13).

Let $\mathbf{F}(x,\ y) = P(x,\ y)\,\mathbf{i} + Q(x,\ y)\,\mathbf{j}$ (as it appears in equation (8) and equation (14)) be a vector-field on an open simply-connected region D. This condition concerns the domain of $\mathbf{F}(x,\ y)$ and is extremely important. The region D is called *open simply-connected* in two-dimensional plane if each and every simple closed curve C like a circle, ellipse or rectangle (without any self-intersections) encloses only points in proper region D, basically, it means that no holes are permitted inside D (Figure 10(a)).[1–3] If such holes exist

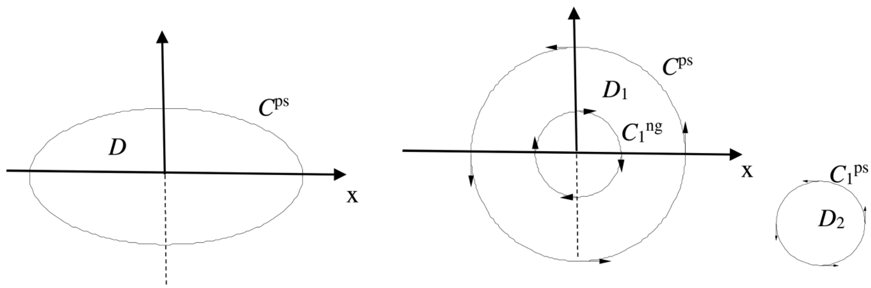


Figure 10. Connectedness of a region enclosed by a simple smooth curve C: (a) open simply-connected region *D* in two-dimensional plane; (b) open simply-connected region *D* split into two parts.

then they might be called *points of singularity* (for example, the origin of the Cartesian coordinate system in Figure 10(b)) and usually they are those where the function $\mathbf{F}(x, y)$ is not defined and consequently is not differentiable.

Suppose further that $P(x, y)$ and $Q(x, y)$, besides being well-defined, have continuous first-order partial derivatives in D then these functions are called continuously differentiable which permits to evaluate the fulfilment of the equality

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x} \tag{23}$$

It is of utmost importance to recognize physical effects which can be modeled as vector-fields to obtain meaningful results in calculations. For example, magnetic field (22) fulfills the principle requirement for a vector-field (8). However, the related magnetic force

$$\mathbf{F}^m = q\,\mathbf{v} \times \mathbf{B} \tag{24}$$

does not, i.e., equation (24) cannot be classified as a force-field.

However, equation (24) is the same for both constant and variable magnetic fields where the first one is directed along the z-axis, and the second one rotates around this axis. So, the direction of the magnetic force changes in various situations but obeys the same physical law (24).

Analysis of point singularities in vector-fields

Analysis of singularities is, in general, an important task of any analytical study. It is instructive to transmit to the students a knowledge that, looking for singularities in equations and developing analytical methods to handle them, is one of the primary goals of any successful mathematical modeling. As it was demonstrated above, the correct modeling of physical laws is not such a trivial problem and a careful procedure is always required while making computer calculations. There are so many types of singularities involved while studying mechanical engineering, for example, Boedo[13] not only analyzed various singularity functions but also incorporated them inside the definite integral as integrands for the effective way of constructing shear-moment diagrams in beams.

The present paper deals with the evaluation of geometric singularities in the expressions of vector-fields, equations (18)-(22), namely, the denominator with the point-singularity of the following form

$$r^n, \ r = \|\mathbf{r}\| = \sqrt{x^2 + y^2} \tag{25}$$

where n is an integer.

Particularly, in many branches of engineering, it is necessary to know if the work done by a force on a closed loop is zero. So, as soon as the vector-field has a singularity inside the loop – special procedures are always required. Such problems involve solutions of exact differentiable equations in mathematics, energy conservation criteria in mechanics, topology considerations in civil engineering, usage of electro-magnetic principles in physics, etc.

The modeling problem, in fact, considers such a geometry of the curve C that represents a unit circle or any circle of a tiny radius to model a hole around the origin of the Cartesian coordinate system (point singularity). Here, there are three different cases to be pointed out.

Case 1^0) One or both of the functions, $P(x, y)$ and $Q(x, y)$ in equation (8), have singularity (25) in the denominator but the work-integrals (13)-(15) are calculated over the curve C (let's say a ball is moving in positive contour-clockwise direction C^{ps}) which is the border of a simply-connected region D (Figure 10(a)) located out of the point singularity. Then the work W will be zero as soon as the requirement (23) is fulfilled which follows from the famous Green's theorem[1–3]:

$$W = \int_C [P\,dx + Q\,dy] = \iint_D \left[\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y}\right] dx\,dy \tag{26}$$

Case 2^0) One or both of the functions, $P(x, y)$ and $Q(x, y)$ in equation (8), have singularity (25) in the denominator but the field itself is axisymmetric, for example, described by equations (17)-(20). Then the work-integrals (13)-(15) will be always zero no matter what the curve C is, i.e., the region D may contain a hole, like region D_1 in Figure 10(b). The criterion of zero-work here is not mathematical but geometrical one based on the orthogonality of the vector-field $\mathbf{F}[\mathbf{r}(t)]$ and trajectory-vector $\mathbf{r}'(t)$. So, the proper definition of the line integral (13) gives zero-value to the work without even starting making any calculations.

Case 3^0) One or both of the functions, $P(x, y)$ and $Q(x, y)$ in equation (8), have singularity (25) in the denominator and the work-integrals (13)-(15) are calculated over the curve C^{ps} (positive contour-clockwise direction) which is the outer border of a region D containing a point singularity. This is the most involved case where the region D should be split first into two sub-regions: D_1 with the outer border C^{ps} traveled in the positive contour-clockwise direction and C_1^{ng} traveled in the opposite negative clockwise direction and represented by a unit circle as in Figure 10(b). To maintain the region D simply-connected, the removed part D_2 is depicted separately in Figure 10(b) while the outer counter C_1^{ps} of the unit circle is traveled in the positive direction. This way, it can be easily seen that the geometric sum of the regions D_1 and D_2 removes the hole from the region D. So, as soon as the condition (23) is fulfilled in the region D_1, the work-integrals (13)-(15) will be zero in this region similar to the Case 1^0. As concerning D_2, the extended version of the Green's theorem (26) yields

$$W = \int_{C^{ps}} [P\,dx + Q\,dy] = \int_{C_1^{ps}} [P\,dx + Q\,dy] \tag{27}$$

which gives the final formula for the work calculations. Notice that the result of these calculations could be any value, not necessary zero, in spite of the fact that the condition (23) is fulfilled in the region D_1.

New calculation techniques and examples of checking the zero-value work on a closed loop for vector fields with singularities

All the problems mentioned above are somehow related to the term $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2}$ and its powers in the evaluation of line integrals. Two new analytical techniques are proposed in the present research to speed up substantially the calculus involved in the direct integration of equations (13)-(15) and in verifying equation (23) for the usage of the Green's theorem (26).

Technique No 1. Direct integration of equation (13) over a circle centered at the origin.

Let's consider a general form of the work-integral

$$W = \int_{\alpha}^{\beta} \mathbf{F}[\mathbf{r}(t)] \bullet \mathbf{r}'(t)dt, \quad \alpha \leq t \leq \beta \tag{28}$$

where the vector-field $\mathbf{F}[\mathbf{r}(t)]$ contains the term r^n, $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2}$, n being any rational number, over a circle with radius R centered at the origin of the Cartesian coordinate system and parameterized by

$$x = R\cos\ t,\ y = R\sin\ t,\ 0 \leq t \leq 2\pi \tag{29}$$

Some general vector-field to consider is

$$\mathbf{F}(x,\ y) = r^n(-y\,\mathbf{i} + x\,\mathbf{j}) \tag{30}$$

So, for $n = 1$, one has $r = \sqrt{x^2 + y^2}$.

If $R = 1$, then $1 = r = \sqrt{(1\ \cos\ t)^2 + (1\ \sin\ t)^2} = 1$,
If $R = 2$, then $2 = r = \sqrt{(2\ \cos\ t)^2 + (2\ \sin\ t)^2} = 2$,
If $R = 3$, then $3 = r = \sqrt{(3\ \cos\ t)^2 + (3\ \sin\ t)^2} = 3$, etc.

For $n = 2$, one has $r^2 = x^2 + y^2$.

If $R = 1$, then $1^2 = r^2 = (1\ \cos\ t)^2 + (1\ \sin\ t)^2 = 1^2$,
If $R = 2$, then $2^2 = r^2 = (2\ \cos\ t)^2 + (2\ \sin\ t)^2 = 2^2$,
If $R = 3$, then $3^2 = r^2 = (3\ \cos\ t)^2 + (3\ \sin\ t)^2 = 3^2$, etc.

The procedure outlined above is verified for the negative integers as well, for example, for $n = -1$, one has $\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$. As concerning $n = 0$, such as $r^0 = 1 = \sqrt{x^2 + y^2}$, it covers only the case $R = 1$ which is already included as part of $n = 1$.

Based on this procedure, equation (28) for the field (30) with parameterization (29) can be transformed into much easier one such as

$$W = R^{n+1} \int_0^{2\pi} (-y\,\mathbf{i} + x\,\mathbf{j}) \bullet (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt \tag{31}$$

So, in equation (31), the integrand consists only of the dot product of the pure vector field, $\mathbf{F}^P(x,\ y)$ - equation (30) without r^n -term, and part of what is left after taking derivatives in equation (29) with respect to t-parameter. All other manipulations (concerning the r^n -term and radius of the circle R) are simplified and put before the integration symbol.

Thus, the final formula proposed in this research for vector-fields containing r^n -term, $\mathbf{F}(x,\ y) = r^n\mathbf{F}^P(x,\ y)$, n being any rational number, integrated over a circle with any radius R centered at the origin is

$$W = R^{n+1} \int_0^{2\pi} \mathbf{F}^P(x,\ y) \bullet (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})dt \tag{32}$$

Example No 1. To see how this new technique works, let's consider the qualitative behavior of a rotational (similar to magnetic) field

$$\mathbf{F}(x,\ y) = \frac{-y}{r^2}\mathbf{i} + \frac{x}{r^2}\mathbf{j} \tag{33}$$

over a circle of radius $R = 3$ centered at the origin of the Cartesian coordinate system. Here, $r = \sqrt{x^2 + y^2}$, $n = -2$, $\mathbf{F}^P(x,\ y) = \langle -y,\ x \rangle$, $x = 3\cos t$, $y = 3\sin t$. So, due to proposed equation (32),

$$\begin{aligned} W &= 3^{-2+1} \int_0^{2\pi} \langle -y,\ x \rangle \bullet \langle -\sin t,\ \ \cos t \rangle dt = \\ &= \frac{1}{3} \int_0^{2\pi} \langle -3\sin t,\ 3\cos t \rangle \bullet \langle -\sin t,\ \ \cos t \rangle dt = \frac{1}{3} \int_0^{2\pi} 3\ dt = 2\pi \end{aligned} \tag{34}$$

Since the result of the integration is non-zero and positive, the field (33) produces some *work* while moving a ball over the circle, and the ball (on which this field acts upon) withdraws the mechanical energy from the field (this is the mechanical significance of the positive sign in the result).

For circles centered at other positions, not $(0,\ 0)$, the integrals considered in this paper have no point singularity and consequently are covered by the Case 1^0 of the previous paragraph. It means that the parameterization (11) removes the singularity from the denominator.

Technique No 2. Simplification of the calculus of the term $\frac{\partial Q(x,\ y)}{\partial x} - \frac{\partial P(x,\ y)}{\partial y}$ in the Green's theorem (26) when both or one of the functions $P(x,\ y)$ and $Q(x,\ y)$ have the common term $(x^2 + y^2)^n$, n being any rational number.

Let's introduce here again *pure* functions $P^{\mathrm{Pr}}(x,\ y)$ and $Q^{\mathrm{Pr}}(x,\ y)$ from which the common term was removed, and designate this term as $R_q = x^2 + y^2$.

Preliminary calculus involves the following mathematical procedure: power-law partial derivatives are calculated first for the common term, such as, with respect to x-variable, one gets $\frac{\partial}{\partial x}(x^2 + y^2)^n = n(x^2 + y^2)^{n-1}\,2x$ and, with respect to y-variable, $\frac{\partial}{\partial y}(x^2 + y^2)^n = n(x^2 + y^2)^{n-1}\,2y$. Second, partial derivatives of two products $P^{\mathrm{Pr}}(x,\ y)\,(x^2 + y^2)^n$ and $Q^{\mathrm{Pr}}(x,\ y)\,(x^2 + y^2)^n$ are calculated to get, respectively, $\frac{\partial Q(x,\ y)}{\partial x}$ and $\frac{\partial P(x,\ y)}{\partial y}$:

$$\frac{\partial Q(x,\ y)}{\partial x} = \frac{\partial Q^{\mathrm{Pr}}(x,\ y)}{\partial x}\,(x^2 + y^2)^n + 2n\,Q^{\mathrm{Pr}}(x,\ y)\,(x^2 + y^2)^{n-1}\,x$$

$$\frac{\partial P(x,\ y)}{\partial y} = \frac{\partial P^{\mathrm{Pr}}(x,\ y)}{\partial y}\,(x^2 + y^2)^n + 2n\,P^{\mathrm{Pr}}(x,\ y)\,(x^2 + y^2)^{n-1}\,y$$

To fulfill the requirement (23), the right-hand sides of these two equalities should be equal which yields, after some algebraic simplifications, a new formula for checking zero-value of the work-integral:

$$\frac{\partial Q^{\mathrm{Pr}}(x,\ y)}{\partial x} - \frac{\partial P^{\mathrm{Pr}}(x,\ y)}{\partial y} = 2n\,(R_q)^{-1}(y\,P^{\mathrm{Pr}}(x,\ y) - x\,Q^{\mathrm{Pr}}(x,\ y)) \tag{35}$$

It means that condition (23) where it is necessary to evaluate the full derivatives of functions $P(x,\ y)$ and $Q(x,\ y)$ reduces to calculation of the derivatives of much shorter functions $P^{\mathrm{Pr}}(x,\ y)$ and $Q^{\mathrm{Pr}}(x,\ y)$ which significantly speeds up analytical calculations.

To see how this new technique works, let's be back to Example No 1. Due to equation (35), $P^{\mathrm{Pr}}(x, y) = -y$, $Q^{\mathrm{Pr}}(x, y) = x$, $R_q = x^2 + y^2$, $n = -1$, so, $1 - (-1) = 2(-1)(x^2 + y^2)^{-1}(-y^2 - x^2)$, which results in *True* statement $2 = 2$. It means that the condition (35) and, consequently, the condition (23) are both satisfied for the vector-field (33).

Two more examples are relevant to be considered for a comparison purpose with other papers available about this topic.[14]

Example No 2. The vector-field considered[14] is

$$\mathbf{F}(x,\ y) = \left(\frac{x}{x^2 + y^2} + x^2\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2}\right)\mathbf{j} \tag{36}$$

To apply equation (35), one notices that $P^{\mathrm{Pr}}(x,\ y) = x + x^2(x^2 + y^2)$, $Q^{\mathrm{Pr}}(x,\ y) = y$, $R_q = x^2 + y^2$, $n = -1$, so, $\frac{\partial Q^{\mathrm{Pr}}(x,\ y)}{\partial x} = 0$, $\frac{\partial P^{\mathrm{Pr}}(x,\ y)}{\partial y} = 2yx^2$, and the condition

(35) yields $0 - 2yx^2 = 2(-1)(R_q)^{-1}(yx + yx^2R_q - xy)$, which results in *True* statement: $-2yx^2 = -2yx^2$. It means that the conditions (35) and (23) are both satisfied for the vector-field (36).

Example No 3. The vector-field considered[14] is

$$\mathbf{F}(x, y) = \left(\frac{x - y}{x^2 + y^2} + x^2\right)\mathbf{i} + \left(\frac{x + y}{x^2 + y^2}\right)\mathbf{j} \tag{37}$$

To apply equation (35), one notices that $P^{\mathrm{Pr}}(x, y) = x - y + x^2(x^2 + y^2)$, $Q^{\mathrm{Pr}}(x, y) = x + y$, $R_q = x^2 + y^2$, $n = -1$, so, $\dfrac{\partial Q^{\mathrm{Pr}}(x, y)}{\partial x} = 1$, $\dfrac{\partial P^{\mathrm{Pr}}(x, y)}{\partial y} = -1 + 2yx^2$, and the condition (35) yields

$1 + 1 - 2yx^2 = 2(-1)(R_q)^{-1}(yx - y^2 + yx^2R_q - x^2 - xy)$, which results again in *True* statement:$2 - 2yx^2 = -2(R_q)^{-1}(-R_q + yx^2R_q) = 2 - 2yx^2$. It means that the conditions (35) and (23) are both satisfied for the vector-field (37).

Conclusions

Teaching Mathematical Analysis course for Mechanical Engineering students and other related majors should involve the overall tendency at universities of incorporating research, creativity and learning activities early in the carrier.

The paper provides a new methodology of teaching the topic of line integrals which makes part of Mathematical Analysis: Vector Calculus suitable for specific needs of engineers. We believe that the proposed methodology is beneficial not only for Mechanical Engineering students but also for other related engineering majors. Indeed, as it follows from Figures 1–3, the students learn first Mathematical Analysis I and some introductory courses on mechanics or physics. From Mathematical Analysis I, the students learn all necessary mathematics such as functions, limits, derivatives and integrals with respect to the function of one variable. On the other hand, from introductory courses on mechanics or physics, the students learn the concepts of work, force as a vector, equilibrium of a particle and steady motion of a rigid body. On the basis of this background, it will be intrinsically natural for the students to acquire the right fillings about the line integral topic since the concepts of mechanical work and forces are actually not problematic ones and have a lot of experimental and practical real-world manifestations. So, the approach of teaching the line integral topic using the concept of work/force principles will help the students to achieve also high performance in advanced mathematics as well.

Other than that, in the present research, two new analytical methods are also developed and implemented for targeting such an important issue in Mathematical Analysis as well as in general Mechanical Engineering courses as geometric singularities. Complementary analysis of vector fields and clarification of some terminology are also discussed to make the teaching methodology coherent and well understood.

As concerning future scientific fields to be explored, the present paper offers several teaching and scientific activities. Namely, the core idea of the paper (connection of the mathematical concept of line integral with the mechanical concept of work/force principles) may be used to explore: a) more advanced mechanical engineering concepts such as energy conservation criteria; b) some mathematical concepts related to engineering such as criteria for exact differential equations; c) some mixed (engineering/mathematics) concepts such as criteria for the usage (in terms of its necessity) of the Green's theorem and related importance of simply-connected domains.

Methodological and scientific contributions of the paper may be highlighted as follows:

- A new methodology of teaching line integrals is elaborated based on the elementary work/force principles;
- Two new techniques of analytical calculus for line integrals containing singularities are developed and implemented;
- A new coherent engineering approach to deal with vector fields as integrands in line integrals is proposed;
- Techniques developed are supplemented by examples, and results are compared with those of other authors.

The paper is believed to be of value for didactic and scientific reasons in courses related to mechanical engineering, both for research and learning purposes.

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