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Geometry and Global Stability of 2D Periodic Monotone Maps

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Abstract
We establish conditions to ensure global stability of a competitive periodic system from hypotheses on individual maps. We study planar competitive maps of Kolgomorov type. We show how conditions for global stability for individual maps will remain invariant under composition and hence establish a globally stable cycle. Our main theoretical contribution is to show that stability for monotone non-autonomous periodic maps can be reduced to a problem of global injectivity. We provide analytic conditions that can be checked and illustrate our results with important competition models such as the planar Leslie-Gower and Ricker maps.

Keywords Competition models · Global stability · Kolmogorov maps · Monotone maps · 2D Periodic maps

1 Introduction

A fundamental problem in the area of discrete dynamical systems is the global stability of fixed or periodic points of a map. In this article we consider the global dynamics a non-autonomous periodic planar maps of the type

$$\mathbf{x}_{n+1} = F_n(\mathbf{x}_n), \text{ for } n \in \mathbb{Z}^+, \tag{1}$$

where $\mathbf{x}_n \in \mathbb{R}^2_+ = [0, \infty)^2$ and for some positive integer $p > 1$, we have $F_{n+p} = F_n$, for all n.

We address the question of whether the global dynamics of the periodic system (1) can be determined from the knowledge of the dynamics of the individual maps F_n. We provide new insights to this problem by focusing on competitive Kolmogorov maps, that is, competitive maps of type $F_n(x, y) = (x f_n(x, y), y g_n(x, y))$. These maps arise naturally in applications

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to mathematical biology and mathematical economics. For instance, they represent important problems in population models with fluctuation habitat, also commonly called periodically forced systems (see [5–8] for further examples).

We will focus on a set on conditions that are sufficient to ensure global stability of individual maps F_n. Some of these conditions will remain invariant under composition and thus trivially carry over to the periodic case. While for other conditions we will establish criteria that can be analytically verified. Most importantly, in applications for periodic systems of period two, conditions for global stability of competitive maps F_n alone will be sufficient to that the periodic system (1) is globally stable

Let us now begin to describe our conditions. We say that map $F : \Omega \to \mathbb{R}^k$ is *monotone* if whenever $\mathbf{x} \leq_{\mathcal{O}} \mathbf{y}$, then $F(\mathbf{x}) \leq_{\mathcal{O}} F(\mathbf{y})$, where $\mathcal{O}$ is an orthant of $\mathbb{R}^k$ and $\mathbf{x} \leq_{\mathcal{O}} \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in \mathcal{O}$. Simply said, a map is monotone if it preserves order. When dealing with planar maps we have two orderings. One is the order induced by the first or third quadrant and maps preserving this ordering are called *cooperative*. The other is the order induced by the second or fourth quadrant (denoted by $\leq_K$ or K-order) and maps preserving this ordering are called *competitive*. We say a region $\Omega \subseteq \mathbb{R}^2_+$ is a *competitive region* or K*-convex* if it contains the line segment joining any two of its points that are ordered with respect to $\leq_K$.

As we examine planar competitive maps and its applications, we will focus on sufficient conditions that can be checked analytically. A planar map is called *strongly* competitive if the inverse of the Jacobian Matrix of the map is a positive matrix. When the planar map is orientation preserving this is easily observed by checking that the entries on the main diagonal are positive and the entries in the off-diagonal are negative. This was first introduced in [21] and later generalized in [3] for higher dimensional maps to establish global stability. Since we are concerned with analytical conditions, this will be our working definition, that is, when we say a map is competitive, we mean strongly competitive.

We also remark, as mentioned in [21], that positiveness is not the most general condition that could be stated but will suffice for our purposes. Recent work in [12] shows that it is possible to consider nonnegative instead of positive conditions, but these do not change the main ideas and conclusions in this article since we do not consider degenerate cases, or limiting cases for the parameters in our study. In fact, as we work with *repellor* and *saddle* points we consider the strict inequality of being greater (or less) than one instead of greater or equal (less or equal) than one. More precisely, we say a fixed point is a repellor when both eigenvalues of the Jacobian Matrix at the fixed point have absolute value greater than one and a fixed point a saddle when one of the eigenvalues of the Jacobian Matrix at the fixed point have absolute value greater than one and the other has absolute value less than one. This is also called a hyperbolic saddle.

Now we list our main assumptions, similar to the hypotheses made in the articles [3,11, 13,21,22] when studying competitive maps. We shall assume that $F : \Omega \subseteq \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is a class C^1 and Ω is a competitive region. In addition, F satisfies the following hypotheses.

(H1) The map F is *competitive*.
(H2) The *extinction* fixed point $(0, 0)$ is a repellor.
(H3) The map F has two *exclusion* fixed points $E_1 = (\widehat{x}, 0)$ and $E_2 = (0, \widehat{y})$ that are globally asymptotically stable when restricted to each axis, but a saddle in Ω.

As we are concerned with global stability, we are interested in a unique fixed point or a unique cycle. One of our main contributions is to frame this question as a problem on global injectivity. In particular, we use classical Theorem of Gale-Nikaido [9], which has been recently given a geometric generalization in [1], to set analytic conditions for the uniqueness of the interior fixed point. Indeed, to say that $F(x, y) = (xf(x, y), yg(x, y))$

has a unique interior coexistence fixed point is to say that $(f(x, y), g(x, y)) = (1, 1)$ has a unique solution. Let us denote by $\widetilde{F}$ the *reduced* map associated to F to be given by $\widetilde{F}(x, y) = (f(x, y), g(x, y))$. Our new hypothesis is simply the following:

(H4) The *reduced* map $\widetilde{F}$ is injective.

Our main result is as follows.

Theorem 1 *Consider a periodic system* (1) *with period p. Assume that $\Omega \subseteq \mathbb{R}^2_+$ is a competitive region and $F_n : \Omega \to \mathbb{R}^2_+$ is a Kolmogorov planar map of class C^1 that satisfies* (H1)–(H4). *If $\Phi_p = F_{p-1} \circ \cdots \circ F_1 \circ F_0$ satisfies* (H3)–(H4)*, then the periodic system has a globally asymptotically stable periodic cycle.*

We remark that in applications, with concrete expressions for the maps F_n, we will show that if individual maps satisfy (H1)–(H4), then the composition map Φ_p automatically satisfies (H1)–(H4) as well and the system has a globally asymptotically stable cycle.

We organize this paper as follows. We provide the necessary background on competitive maps and determine the analytical conditions to check our hypotheses. Later, we apply our results to global stability of periodic competitive systems such as the planar Leslie-Gower competition model [14] and the Ricker competition model [19].

2 Competitive Maps and Main Result

For more comprehensive details on the theory of monotone planar maps, we refer to [21]. In applications, the following result tells us how to analytically check if a planar map is competitive and when that is the case, the orbit of every point will converge to some fixed point.

Theorem 2 (Smith [21]) *Let $\Omega \subseteq \mathbb{R}^2_+$ be a competitive region and $F : \Omega \to \mathbb{R}^2_+$ be a planar C^1 map that satisfies the following:*

i. The map F is orientation preserving, i.e., $\det DF(\mathbf{x}) > 0$, *for* $\mathbf{x} \in \mathbb{R}^2_+$.
ii. The entries on the main diagonal of $DF(\mathbf{x})$ are positive and the entries in the off-diagonal are negative.
iii. The map F is injective.

Then $\{F^n(\mathbf{x})\}$ is eventually componentwise monotone. In addition, if the orbit has compact closure, then it converges to a fixed point of F.

It can be difficult to show that the map F is injective on Ω, but in [2], the authors use singularity theory to determine regions of invertibility of F so that the result above can apply for the Ricker competition map.

As mentioned in the introduction, we restrict ourselves to competitive maps (H1) that have a repellor extinction point (H2) and one *exclusion* fixed point in each axis that are saddle points (H3). Under these three hypotheses, we can already tell quite a bit on the global dynamics of the map.

Theorem 3 (Smith [21]) *Let $\Omega \subseteq \mathbb{R}^2_+$ be a competitive region and $F : \Omega \to \mathbb{R}^2_+$ be a planar C^1 such that* (H1)–(H3) *hold. Then there exist positive fixed points $E_* = (x_*, y_*)$ and $E^* = (x^*, y^*)$, not necessarily distinct, such that the exclusion fixed points (E_1 and E_2) and the positive fixed points (E_* and E^*) are K-ordered as $E_2 \leq_K E^* \leq_K E_* \leq_K E_1$. In* *addition, the orbit of every interior point converges to a fixed point* (u, y) *belonging to the rectangle with corners* E_* *and* E^* *and sides parallel to the axis, that is,* $x^* \leq u \leq x_*$ *and* $y_* \leq v \leq y^*$.

Therefore, if it can be shown that F has a *unique positive coexistence* fixed point, then the rectangle with corners E_* and E^*, in the result above, reduces to a single unique fixed point. Hence F is globally stable. This is a subtle but important issue, it means that we can utilize results on global injectivity to determine global stability results. Indeed, from this discussion we have the following straightforward result.

Proposition 1 *Let* $\Omega \subseteq \mathbb{R}^2_+$ *be a monotone region and* $F : \Omega \to \mathbb{R}^2_+$ *be a planar* C^1 *such that* (H1)–(H4) *hold. Then there exists a unique positive coexistence fixed point* E^* *that is globally asymptotically stable in the interior of* Ω.

We now consider a non-autonomous periodic planar map of period $p > 1$ as denoted by (1). For each i, let $\Phi^i_p = F_{i+p-1} \circ \cdots \circ F_{i+1} \circ F_i$ be the associated composition operators. When $i = 0$, we write Φ^0_p as Φ_p. (For more details in non-autonomous periodic difference equations we refer the reader to [16]). We seek to determine if the maps F_n satisfy hypotheses (H1)–(H4), then Φ^i_p will also satisfy hypotheses (H1)–(H4).

We begin with (H1), in order to check if an orientation preserving map is competitive and thus satisfies (H1), it suffices to check that the inverse of the Jacobian matrix is a positive matrix, that is, entries of $DF^{-1}(\mathbf{p})$ are positive, see [3, Lemma 4.1]. Hence competitiveness is preserved under composition of maps *in any dimensions* using the chain rule.

Next, a simple computation will also show that the origin will remain a repellor under composition, thus (H2) is also preserved under composition. Indeed, let $F_i(x, y) = (xf_i(x, y), yg_i(x, y))$, these maps have the origin as a common extinction fixed point, so the origin is also an extinction fixed point of Φ^i_p. Let DF_i be the Jacobian matrix of F_i. A quick computation shows that

$$DF_i(x, y) = \begin{pmatrix} f_i(x, y) + x\partial_x f_i(x, y) & x\partial_y f_i(x, y) \\ y\partial_x g_i(x, y) & g_i(x, y) + y\partial_y g_i(x, y) \end{pmatrix}. \tag{2}$$

Hence, at the extinction fixed point (0,0), the eigenvalues are $f_i(0, 0)$ and $g_i(0, 0)$. By assumption (H1) that the origin is a repellor of each individual map, we have that $|f_i(0, 0)| > 1$ and $|g_i(0, 0)| > 1$. Using the chain rule, the eigenvalues at the origin for Φ^i_p are given by $\prod_{j=i}^{p+i-1} f_j(0, 0)$ and $\prod_{j=i}^{p+i-1} g_j(0, 0)$. Thus, both are clearly greater than one in absolute value.

Let us now focus on checking (H3). Since we must restrict the analysis to each axis, we begin with the dynamics of periodic systems of one-dimensional maps.

Let $F_0,\ F_1 : D \subset \mathbb{R}^+_0 \to D$ be competitive Kolmogorov maps given by $F_0(x) = xf_0(x)$ and $F_1(x) = xf_1(x)$ where D is a closed interval. Motivated by the applications in population dynamics, we further assume that $f_0(x), f_1(x) > 0$ and $f'_0(x), f'_1(x) < 0$ for all $x \in D$. Since F_0 and F_1 are competitive, $F'_0(x) \geq 0$ and $F'_1(x) \geq 0$, or equivalently,

$$f_0(x) + xf'_0(x) \geq 0 \text{ and } f_1(x) + xf'_1(x) \geq 0 \text{ for all } x \in D.$$

It is a straightforward computation to verify that the composition maps $\Phi^i_2 = F_{i+1} \circ F_i$ are Kolgomorov maps and the associated reduced maps are injective. Indeed, let $i = 0$ (the case when $i = 1$ is similar) and $\Phi_2(x) = (F_1 \circ F_0)(x) = xf_0(x) f_1(xf_0(x))$.

The reduced map associated to Φ_2 is given by $\widetilde{\Phi}_2(x) = f_0(x) f_1(xf_0(x))$. We can show that $\widetilde{\Phi}_2(x)$ is injective by showing that its derivative is nowhere vanishing. A computation shows that

$$\widetilde{\Phi}_2'(x) = f_0'(x) f_1(x f_0(x)) + f_0(x) f_1'(x f_0(x)) \left(f_0(x) + x f_0'(x) \right),$$

or equivalently,

$$\widetilde{\Phi}_2'(x) = f_0^2(x) f_1'(x f_0(x)) + f_0'(x) \left(f_1(x f_0(x)) + x f_0(x) f_1'(x f_0(x)) \right). \quad (3)$$

The first term in (3) is negative as $f_1'(x) < 0$. For the second term, let us denote $y = x f_0(x)$. Then $f_0'(x) < 0$ and from the assumption that F_1 is monotone, we have that $f_1(y) + y f_1'(y) \geq 0$. Thus $\widetilde{\Phi}_2'(x) < 0$ for all $x \in D$. This means that $\widetilde{\Phi}_2$ is injective.

The upshot of the computation above, is that when D is compact, the orbit of any interior point under Φ_2 must converge to some fixed point. As we assume that the origin is a repellor, the orbit must converge to an interior fixed point. Recall that an interior fixed point of Φ_2 would satisfy $\widetilde{\Phi}_2(x) = 1$. If the interior fixed point were not unique, by Rolle's Theorem, there would be a point where $\widetilde{\Phi}_2'(x) = 0$ contradicting the computation in (3) where we show $\widetilde{\Phi}_2'(x) < 0$ for all $x \in D$. Hence, the fixed point must be unique and globally stable. We also note that since the domain and image of F_0 and F_1 is D, all possible fixed points of Φ_2 would be in D as well.

As we turn to planar maps, the discussion above establishes the first part in (H3). Namely, for each exclusion fixed point in the axis, the restriction to each axis will be globally stable. We now need to establish the conditions for it to be a saddle and we do so by computing the eigenvalue of the eigenvector not in the axis. Without loss of generality, let us focus on the x-axis and consider the composition operator Φ_p.

For each i, we assume that $F_i(x, y) = (x f_i(x, y), y g_i(x, y))$ satisfies (H3) and the exclusion point of F_i on the x-axis is $(\widehat{x_i}, 0)$. The Jacobian Matrix of F_i at the exclusion point is

$$DF_i(\widehat{x_i}, 0) = \begin{pmatrix} f_i(\widehat{x_i}, 0) + \widehat{x_i} \frac{\partial f_i}{\partial x}(\widehat{x_i}, 0) & x \frac{\partial f_i}{\partial y}(\widehat{x_i}, 0) \\ 0 & g_i(\widehat{x_i}, 0). \end{pmatrix}. \quad (4)$$

Thus, the eigenvalue corresponding to the eigenvector not in the axis is $g_i(\widehat{x_i}, 0)$ which we assume to be greater than one. Next, consider $\{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{p-1}\}$ to be the globally asymptotically stable p-cycle on the x-axis. This is indeed the globally stable p-cycle from the one-dimensional analysis above. By the chain rule, $D\Phi_p(\widehat{x}, 0)$ will be a triangular matrix and by (4), the eigenvalue associated to the eigenvector not in the x-axis is $\prod_{i=0}^{p-1} g_i(\overline{x}_i, 0)$. Similarly, for the exclusion point of F_i on the y-axis we have that the eigenvalue associated to the eigenvector not in the y-axis is $\prod_{i=0}^{p-1} f_i(0, \overline{y}_i)$.

Therefore, to say that Φ_p satisfies (H3) whenever F_i satisfies (H3) is equivalent to say that if $|g_i(\widehat{x_i}, 0)| > 1$ for each i, then

$$\prod_{i=0}^{p-1} |g_i(\overline{x}_i, 0)| > 1 \text{ and } \prod_{i=0}^{p-1} |f_i(0, \overline{y}_i)| > 1. \quad (5)$$

In view of the computation above, we have the following result as a generalization of Theorem 3:

Theorem 4 *Let $\Omega \subseteq \mathbb{R}_+^2$ be a monotone region and $F_n : \Omega \to \mathbb{R}_+^2$ be a planar C^1 with $F_n(x, y) = (x f_n(x, y), y g_n(x, y))$ such that* (H1)–(H3) *hold. If*

$$|f_n(0, y)| > 1, \forall (0, y) \in \Omega \text{ and } |g_n(x, 0)| > 1, \forall (x, 0) \in \Omega, \quad (6)$$

then the periodic system (1) has two cycles $\left\{\left(\overline{x}_0^*, \overline{y}_0^*\right), \left(\overline{x}_1^*, \overline{y}_1^*\right), \ldots, \left(\overline{x}_{p-1}^*, \overline{y}_{p-1}^*\right)\right\}$ *and* $\left\{\left(\overline{x}_{0,*}, \overline{y}_{0,*}\right), \left(\overline{x}_{1,*}, \overline{y}_{1,*}\right), \ldots, \left(\overline{x}_{p-1,*}, \overline{y}_{p-1,*}\right)\right\}$ *such that the orbit of every interior point converges to a cycle* $\left\{(u_0, v_0), (u_1, v_1), \ldots, \left(u_{p-1}, v_{p-1}\right)\right\}$ *with* $\overline{x}_n^* \leq u_n \leq \overline{x}_{n,*}$ *and* $\overline{y}_{n,*} \leq v_n \leq \overline{y}_n^*$.

Remark 1 The condition in hypotheses (6) is much stronger that condition (5) needed for the composition map to satisfy (H3). In applications to concrete maps, such as the Ricker Map, we are able to show directly that (5) holds from $|g_i(\widehat{x_i}, 0)| > 1$ and $|f_i(0, \widehat{y_i})| > 1$ since we can relate the individual fixed points $(\widehat{x_i}, 0)$ and $(0, \widehat{y_i})$ to the cycle $\{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{p-1}\}$ and $\{\overline{y}_0, \overline{y}_1, \ldots, \overline{y}_{p-1}\}$, respectively.

Lastly, we will now consider how one checks condition (H4), that is, we provide analytic conditions to determine when the reduced map is injective. There are many important results for detecting invertibility of a map, see [1,9,10,17,18]. We define the *leading principal minors* of an $n \times n$ matrix as the determinant of the submatrix obtained by deleting the last $n-k$ rows and columns of the matrix for $k = 1, 2 \ldots, n$. This means that the first leading principal minors is the first diagonal entry and subsequent leading principal minors are the square matrices along the diagonal. For our purposes, we shall use the following result that provides two simple analytical conditions.

Theorem 5 (Gale-Nikaido [9]) *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *planar map given by* $F(x, y) = (f(x, y), g(x, y))$. *If all the leading principal minors of the Jacobian Matrix do not change sign, that is,*

$$|f_x| \neq 0 \text{ and} \tag{GN1}$$

$$|f_x g_y - f_y g_x| \neq 0, \tag{GN2}$$

then F *is injective.*

We remark that Theorem 5 works in higher dimensions as well and the leading principal minors can be replaced by any nested sequence of square matrices along the diagonal of the Jacobian Matrix starting from diagonal entry.

We will illustrate for two-periodic maps how to check the conditions above to show that the reduced map of the composition operator is injective. Let us consider two Kolmogorov maps defined by $F_i : D \subset \mathbb{R}_+^2 \to D$ given by $F_i(x, y) = (x f_i(x, y), y g_i(x, y))$, for $i = 0, 1$, such that $f_i(x, y)$ and $g_i(x, y)$ are decreasing in both variables.

Denote $\Phi_2(x, y) = (F_1 \circ F_0)(x, y)$ as the map given by

$$\Phi_2(x, y) = (x f_0(x, y) f_1(x f_0(x, y), y g_0(x, y)), y g_0(x, y) g_1(x f_0(x, y), y g_0(x, y)))).$$

The reduced map is given by

$$\widetilde{\Phi}_2(x, y) = (f_0(x, y) f_1(x f_0(x, y), y g_0(x, y)), g_0(x, y) g_1(x f_0(x, y), y g_0(x, y)))).$$

The first leading principal minor of the Jacobian Matrix of $\widetilde{\Phi}_2$ is given by

$$\begin{aligned} J_{11}(x, y) = f_0^2(x, y) \frac{\partial f_1}{\partial u}(u, v) + \frac{\partial f_0}{\partial x}(x, y) \frac{\partial}{\partial u}(u f_1(u, v)) \\ + y f_0(x, y) \frac{\partial f_1}{\partial v}(u, v) \frac{\partial g_0}{\partial x}(x, y), \end{aligned}$$

where we write $u(x, y) = xf_0(x, y)$ and $v(x, y) = yg_0(x, y)$ for simplicity.

Similarly, the determinant of the Jacobian Matrix of $\widetilde{\Phi}_2$ is given by

$$\det \widetilde{\Phi}_2(x, y) = J_{11}(x, y)J_{22}(x, y) - J_{12}(x, y)J_{21}(x, y),$$

where J_{ij} are the entries of $D\widetilde{\Phi}_2$, namely, $J_{11}(x, y)$ as above and,

$$J_{12}(x, y) = \frac{\partial f_0}{\partial y}(x, y)\frac{\partial}{\partial u}(uf_1(u, v)) + f_0\frac{\partial f_1}{\partial v}(u, v)\frac{\partial}{\partial y}(yg_0(x, y)),$$
$$J_{21}(x, y) = \frac{\partial g_0}{\partial x}(x, y)\frac{\partial}{\partial v}(vg_1(u, v)) + g_0\frac{\partial g_1}{\partial u}(u, v)\frac{\partial}{\partial x}(xf_0(x, y)),$$

and

$$J_{22}(x, y) = g_0^2(x, y)\frac{\partial g_1}{\partial v}(u, v) + \frac{\partial g_0}{\partial y}(x, y)\frac{\partial}{\partial v}(vg_1(u, v))$$
$$+xg_0(x, y)\frac{\partial g_1}{\partial u}(u, v)\frac{\partial f_0}{\partial y}(x, y).$$

Thus conditions (GN1) and (GN2) that will be checked in specific maps are

$$J_{11}(x, y) \neq 0, \text{ for all } (x, y) \in D, \tag{7}$$

and

$$\det \widetilde{\Phi}_2(x, y) \neq 0 \text{ for all } (x, y) \in D. \tag{8}$$

In the case of the planar Leslie-Gower model and the Ricker competition model, we can check (7) and (8), under some conditions on their parameters. We will see this fact in details in the next section.

3 Applications

Let us now consider particular maps from Mathematical Biology in order to illustrate our results. We restrict most of our work to 2-periodic maps due to the complexity of computations since the amount of parameters involved is significant for analytical computations.

3.1 Periodic Leslie-Gower Model

Consider the 2-periodic Leslie-Gower model $\mathbf{x}_{n+1} = F_n(\mathbf{x}_n)$ where $F_n : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ and $F_{n+2} = F_n$ for all n such that the sequence of maps is given by

$$F_0(x, y) = \left(\frac{r_0 x}{1 + c_{11}x + c_{12}y}, \frac{s_0 y}{1 + c_{21}x + c_{22}y}\right)$$

and

$$F_1(x, y) = \left(\frac{r_1 x}{1 + d_{11}x + d_{12}y}, \frac{s_1 y}{1 + d_{21}x + d_{22}y}\right).$$

We remark that the analysis that follows can be extended to a more general period of the equation.

Note that that F_n is a diffeomorphism. We shall consider the case when $r_i > 1$ and $s_i > 1$ for $i = 0, 1$, which corresponds to the situation where the system does not go to extinction. In addition, all the competition parameters c_{ij} and d_{ij} are assumed to be positive constants.

The conditions under which the autonomous Leslie-Gower model satisfies conditions (H1)–(H4) are analyzed in detail in [3]. We state below the conditions that will be used in this article for completeness. First, the competition parameters must satisfy

$$c_{21} < c_{11}, c_{12} < c_{22} \text{ and } d_{21} < d_{11}, d_{12} < d_{22}. \tag{9}$$

In addition, the condition that each individual map F_i must satisfy (H3) is given by:

$$g_0(\widehat{x}_0, 0) = g_0\left(\frac{r_0 - 1}{c_{11}}, 0\right) = \frac{c_{11}s_0}{c_{21}(r_0 - 1) + c_{11}} > 1 \tag{10}$$

and

$$g_1(\widehat{x}_1, 0) = g_1\left(\frac{r_1 - 1}{d_{11}}, 0\right) = \frac{d_{11}s_1}{d_{21}(r_1 - 1) + d_{11}} > 1. \tag{11}$$

As we begin checking our hypotheses, we know that (H1) and (H2) are trivially satisfied. For (H3) and (H4), we will provide analytical conditions that can be checked. Let us consider the eigenvalues associated to the 2-cycle in the x-axis and determine how to verify condition (5) to establish (H3).

We denote $\{(\overline{x}_0, 0), (\overline{x}_1, 0)\}$ to be the 2-cycle in the $x-$axis. A straightforward computation shows that

$$\overline{x}_0 = \frac{r_0 r_1 - 1}{c_{11} + d_{11}r_0} \text{ and } \overline{x}_1 = \frac{r_0 r_1 - 1}{d_{11} + c_{11}r_1}$$

In fact, there are similar expressions for general p-periodic case. In a paper by Clark and Gross [4], the authors showed that the p-periodic equation $x_{n+1} = \frac{r_n x_n}{1+c_n x_n}$, $r_{n+p} = r_n$ and $c_{n+p} = c_n$ for all n, has a globally asymptotically stable p-cycle of the form $\{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{p-1}\}$ where

$$\overline{x}_0 = \frac{\prod_{i=0}^{p-1} r_i - 1}{c_0 + \sum_{i=1}^{p-1}\left(\prod_{j=0}^{i-1} r_j\right) c_i}, \quad \overline{x}_{i+1} = f_i(\overline{x}_i), \quad f_i(x) = \frac{r_i x}{1 + c_i x}.$$

Now, to check condition (5) for the 2-periodic case we need to show that

$$\prod_{i=0}^{1} |g_i(\overline{x}_i, 0)| = \frac{s_0}{c_{21}\overline{x}_0 + 1}\frac{s_1}{d_{21}\overline{x}_1 + 1} > 1.$$

We note that the condition for $\prod_{i=0}^{1} |f_i(0, \overline{y}_i)|$ is analogous. Using the expressions for $\overline{x}_i$ this is equivalent to check if

$$\frac{s_0 s_1 - 1}{r_0 r_1 - 1} > \frac{c_{21}d_{21}(r_0 r_1 - 1) + c_{21}(d_{11} + c_{11}r_1) + d_{21}(c_{11} + d_{11}r_0)}{(c_{11} + d_{11}r_0)(d_{11} + c_{11}r_1)}. \tag{12}$$

Now, to see whether the periodic Leslie-Gower model satisfies (H4) by checking if the Gale-Nikaido conditions (7) and (8) hold for the composition map $\Phi_2(x, y) = (F_1 \circ F_0)(x, y)$.

The reduced map associated to $\Phi_2(x, y)$ is given by

$$\widetilde{\Phi}_2(x, y) = \left(\frac{r_0 r_1}{(c_{11}x + c_{12}y + 1)\left(\frac{d_{11}r_0 x}{c_{11}x+c_{12}y+1} + \frac{d_{12}s_0 y}{c_{21}x+c_{22}y+1} + 1\right)}, \frac{s_0 s_1}{(c_{21}x + c_{22}y + 1)\left(\frac{d_{21}r_0 x}{c_{11}x+c_{12}y+1} + \frac{d_{22}s_0 y}{c_{21}x+c_{22}y+1} + 1\right)} \right).$$

The first leading principal minor $J_{11}(x, y)$ of the Jacobian Matrix of $\widetilde{\Phi}_2$ is given by

$$-\frac{r_0 r_1((c_{11} + d_{11}r_0)(c_{21}x + c_{22}y + 1)^2 + d_{12}s_0 y\,(y(c_{22}c_{11} - c_{12}c_{21}) + (c_{11} - c_{21})))}{((c_{21}x + c_{22}y + d_{12}s_0 y + 1)(1 + c_{11}x + c_{12}y) + d_{11}r_0 x\,(c_{21}x + c_{22}y + 1))^2}$$

Clearly, from relations in (9) it follows that $J_{11}(x, y) < 0$ and the first Gale-Nikaido condition (7) is satisfied.

In order to check the second Gale-Nikaido condition (8) we need to show that $\det D\widetilde{\Phi}_2 \neq 0$. In the general case, the computations are exceedingly long and unreasonable to manipulate due the number of parameters. Therefore we will focus on the *symmetric* case where we can concretely check the hypotheses of our results. We highlight that in this special symmetric case, we can show that condition (8) is satisfied and hence (H4) holds and we can also show that conditions (10) and (11) implies (12) which in turn verifies (H3).

In the Leslie-Gower competition model, the symmetric case means the case where there is no intraspecific competition (competition between members of the same species), i.e., $c_{ii} = d_{ii} = 1$ and the interspecific competition is equal (competition between members of different species), i.e., $c_{ij} = d_{ij} = a$, for $i \neq j$. Under this assumption on the parameters we have that

$$\det D\widetilde{\Phi}_2(x, y) = (1 - a) r_0 r_1 s_0 s_1 \left(\frac{A}{B}\right),$$

where

$$\begin{aligned} A =\ & r_0 \left(x\left(a^2 y + a + y + 1\right) + ax^2 + (y + 1)(ay + 1)\right) \\ & \times \left(2a^3x^2 + a^2x(2x + 3y + 3) + s_0\left(3a^3xy + a^2\left(x^2 + x(3y + 2) + y(y + 2)\right)\right.\right. \\ & + a\left(x^2 + x(y + 2) + (y + 1)^2\right) + (x + 1)(y + 1)\Big) + a\left(x^2 + x(2y + 3) + (y + 1)^2\right) \\ & + (x + 1)(y + 1)\Big) + (ay + x + 1)^2\left(s_0\left(2a^3xy + a^2\left(x^2 + 2xy + x + 2y(y + 1)\right)\right.\right. \\ & + a(y + 1)(2x + 2y + 1) + (y + 1)^2\Big) + as_0^2 y(ay + y + 1) + (a + 1)(ax + y + 1)^2\Big) \\ & + ar_0^2 x(ax + x + 1)(ax + y + 1)^2 \end{aligned}$$

and

$$\begin{aligned} B =\ & (ar_0 x(ax + y + 1) + (ay + x + 1)(ax + s_0 y + y + 1))^2 \\ & \times (r_0 x(ax + y + 1) + (ay + x + 1)(as_0 y + ax + y + 1))^2 . \end{aligned}$$

Since $A/B > 0$, as both of these expressions contain only positive terms, and from relation (9) we have that $a < 1$, it follows that $\det D\widetilde{\Phi}_2(x, y) > 0$. Thus, from Theorem 5 the map $\widetilde{\Phi}_2$ is injective and (H4) holds.

For (H3), we can combine (10) and (11) and we have in the symmetric case that

$$(s_0 s_1 - 1) > a\left[a(r_0 - 1)(r_1 - 1) + (r_1 - 1) + (r_0 - 1)\right], \tag{13}$$

and also (12) reduces to

$$\frac{s_0 s_1 - 1}{r_0 r_1 - 1} > a \left[\frac{a(r_0 r_1 - 1) + (1 + r_1) + (1 + r_0)}{(1 + r_0)(1 + r_1)} \right]. \tag{14}$$

We claim that

$$a\,(r_0 - 1)\,(r_1 - 1) + r_0 + r_1 - 2 > \frac{(r_0 r_1 - 1)\,[a(r_0 r_1 - 1) + r_0 + r_1 + 2]}{(1 + r_0)(1 + r_1)}. \tag{15}$$

Indeed, a long but straightforward manipulation shows that inequality (15) is equivalent to the inequality $(1 - a)\,(r_0 - r_1)^2 > 0$ which is clearly valid as $a < 1$. Next, we combine (15) and (13) to establish that (14) is true and hence (H3) holds for Φ_2.

Finally, from Theorem 1 the 2-periodic Leslie-Gower competition model has a 2-periodic globally asymptotically stable cycle with respect to the interior of $\mathbb{R}^2_+$.

3.2 Periodic Ricker Map

Consider the 2-periodic Ricker model $\mathbf{x}_{n+1} = F_n(\mathbf{x}_n)$ where $F_n : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ and $F_{n+2} = F_n$ for all n, such that the sequence of maps is given by

$$F_0(x, y) = \left(x e^{r_0 - x - a_0 y}, \, y e^{s_0 - y - b_0 x} \right),$$

and

$$F_1(x, y) = \left(x e^{r_1 - x - a_1 y}, \, y e^{s_1 - y - b_1 x} \right).$$

We are interested in the analyses of a system where the individual maps satisfy (H1)–(H4). This is the case where the system does not go to extinction and has a unique positive fixed point. Therefore we assume that the parameters for each individual map F_i satisfies $a_i s_i < r_i$, $b_i r_i < s_i$, and $a_i b_i < 1$, $i = 0, 1$. In addition, because we are restricting our analysis for the monotone case, we further assume that $r_i, s_i < 1$, $i = 0, 1$. A detailed analysis of the autonomous Ricker competition model can be found in [2,3,15] and for completeness, we will mention the results that will be needed for our analysis.

As we assume that the individual maps are monotone, this requires that the maps are locally invertible on their domains. Following the work in [2] where the domain of invertibility for the Ricker map was studied, we need to find a region in the domain bounded by the critical curve. Let $F(x, y) = (x e^{r - x - ay}, y e^{s - y - bx})$ and the critical curve of a the 2-dimensional continuous map F is given by the set of points for which the Jacobian determinant of F vanishes, or for which the map F is not differentiable. In other words,

$$LC_{-1} = \{p \in U : \det DF(p) = 0, \text{ or } F \text{ is not differentiable in } p\}.$$

For the Ricker map F, we have that

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}^2_+ : y = \frac{1 - x}{1 - (1 - a_i b_i)x}, \; x \neq \frac{1}{1 - a_i b_i} \right\}, \tag{16}$$

and it is formed by two branches:

(i) LC^1_{-1}, a curve connecting the points $(0, 1)$ and $(1, 0)$, for $x < \frac{1}{1 - a_i b_i}$.
(ii) LC^2_{-1}, an unbounded curve for $x > \frac{1}{1 - a_i b_i}$.

It has been shown in [2] that the dynamics of the Ricker maps is determined by points under LC^1_{-1} whenever $r_i, s_i < 1$. Namely,

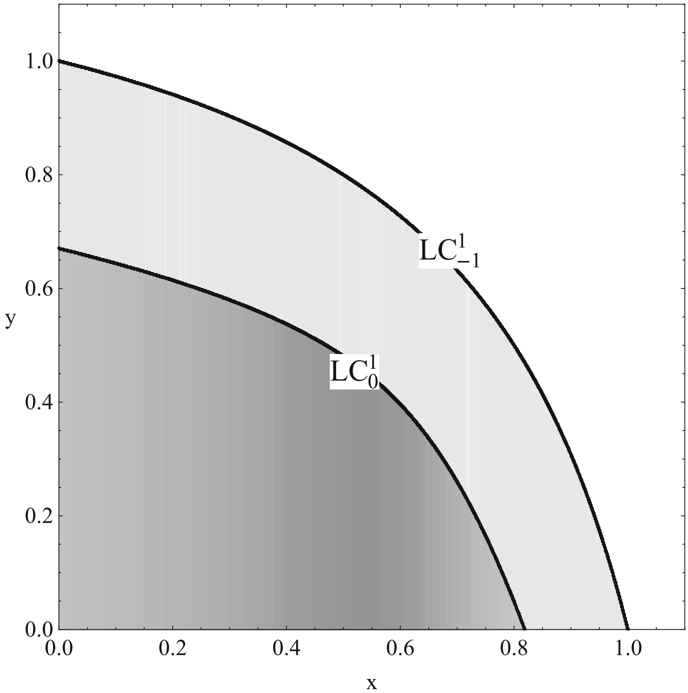

Fig. 1 The relative position between the critical curves LC^1_{-1} and LC^1_0 of the Ricker competition model. The region under the curve LC^1_0 is the image of the region under the curve LC^1_{-1} and it denoted by $\mathcal{D}$ in Theorem 6

Theorem 6 [2, Theorem 4.7] *Consider the Ricker map given by* $F(x, y) = (xe^{r-x-ay}, ye^{s-y-bx})$ *as above. Let* $\mathcal{D}$ *be the region enclosed by the curve* $LC^1_0 = F_i\left(LC^1_{-1}\right)$ *and the axes. Then* $Im(F) = \mathcal{D}$.

Observe that from (16) and condition (i) above, the curve LC^1_{-1} is contained in the unit square. Also, in the monotone case when $r_i, s_i < 1$, the curve LC^1_0 lies under the curve LC^1_{-1}. Thus, from Theorem 6, we may restrict the domain of the Ricker map to the region Ω enclosed by the curve LC^1_{-1} as the image of any point will lie inside it after one iteration as depicted in Fig. 1, see [2] for more details. In particular, in the analysis that follow we then assume that $0 \le x \le 1$ and $0 \le y \le 1$.

Another important consequence of Theorem 6 for the periodic case is that the domain of F_i where it is invertible is the region Ω which depends on a_i and b_i. Since this region must be common to all maps, we require that $a_i = a$ and $b_i = b$.

As we proceed to check (H3) for the periodic map, we remark that Sacker in [20] proved that the p-periodic Ricker equation $x_{n+1} = x_n e^{r_n - x_n}$, $r_{n+p} = r_n$, for all n, has a globally asymptotically stable p-cycle whenever $r_i \in (0, 2)$, not just for the monotone case.

Since we assume that each F_i satisfies (H3), then we have

$$|g_i(\widehat{x}_i, 0)| = e^{r_i - b\widehat{x}_i} > 1. \tag{17}$$

Now we verify that (5) is satisfied by the following computation,

$$\prod_{i=0}^{p-1} |g_i(\overline{x}_i, 0)| = \prod_{i=0}^{p-1} e^{r_i - b\overline{x}_i} = e^{\sum (r_i - b\overline{x}_i)} = e^{\sum r_i - b\sum \overline{x}_i}$$
$$= e^{\sum r_i - b\sum \widehat{x}_i} = e^{\sum (r_i - b\widehat{x}_i)} = \prod_{i=0}^{p-1} e^{r_i - b\widehat{x}_i} > 1.$$

We note that for the Ricker Map, it is straightforward to see that $\sum \overline{x}_i = \sum \widehat{x}_i$. Thus Φ_i satisfies (H3) whenever F_i satisfies (H3).

As we turn our attention to checking (H4), we again restrict the analysis for the 2-periodic system. We will show that for each $i = 0, 1$, the composition map $\Phi_2^i : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\Phi_2^i = F_{i+1} \circ F_i$ is injective by checking the Gale-Nikaido conditions.

Without loss of generality, let us consider the case when $i = 0$, i.e, $\Phi_2 = F_1 \circ F_0$. Thus, the composition map Φ_2 is given by

$$\Phi_2(x, y) = \left(x e^{\Theta_1(x,y)}, y e^{\Theta_2(x,y)} \right),$$

where

$$\Theta_1(x, y) = r_0 + r_1 - x\left(1 + e^{r_0 - x - ay}\right) - y\left(a + a e^{s_0 - bx - y}\right)$$

and

$$\Theta_2(x, y) = s_0 + s_1 - x\left(b + b e^{r_0 - x - ay}\right) - y\left(1 + e^{s_0 - bx - y}\right).$$

The reduced map $\widetilde{\Phi}_2(x, y)$ is given by $\widetilde{\Phi}_2(x, y) = \left(e^{\Theta_1(x,y)}, e^{\Theta_2(x,y)}\right)$.

The first leading principal minor of the Jacobian matrix of $\widetilde{\Phi}_2$ is given by

$$J_{11}(x, y) = \left((x-1)e^{r_0 + bx + y} + ab y e^{s_0 + ay + x} - e^{(b+1)x + (a+1)y} \right) e^{\Theta_1(x,y) - (1+b)x - (1+a)y}.$$

Clearly $e^{\Theta_1(x,y) - (1+b)x - (1+a)y} > 0$ and $(x-1)e^{r_0 + bx + y} < 0$. Now let us show that

$$ab y e^{s_0 + ay + x} - e^{(b+1)x + (a+1)y} < 0, \tag{18}$$

or equivalently

$$y - \ln(aby) > s_0 - bx. \tag{19}$$

First, $y - \ln(aby) \geq 1 - \ln(ab) > 1$ for all y. Indeed, $\ln(ab) < 0$ as $ab < 1$. Since $0 < s_0 < 1$ and $0 < bx$ (i.e., $-bx < 0$), it follows that $s_0 - bx < 1$. Consequently, $y - \ln(aby) > 1 > s_0 - bx$ is always satisfied. Thus $J_{11}(x, y)$ is bounded away from zero and the first Gale-Nikaido condition is satisfied.

Next, for the second Gale-Nikaido condition, let us compute the determinant of the Jacobian matrix of $\widetilde{\Phi}_2$.

$$\begin{aligned} \det D\widetilde{\Phi}_2(x, y) = \Big[&(1-x)e^{r_0 + bx + y} + (1-y)e^{s_0 + x + ay} + (1-x)(1-y)(1-ab)e^{r_0 + s_0} \\ &+ (1-ab)e^{(1+b)x + (1+a)y} - ab(1-ab)xy e^{r_0 + s_0} \\ &+ ab(x-1)e^{r_0 + y + bx} + ab(y-1)e^{s_0 + x + ay} \Big] e^{\Theta_1 + \Theta_2 - (1+b)x - (1+a)y}. \end{aligned}$$

Simplifying the expression above, we have det $D\widetilde{\Phi}_2(x, y) > 0$ if $e^{(1+b)x+(1+a)y} > abxye^{r_0+s_0}$. This inequality is equivalent to

$$(1 + a)y - \ln(ay) > r_0 + s_0 + \ln(bx) - (1 + b)x. \quad (20)$$

Forward computations show that $(1 + a)y - \ln(ay) \geq 1 - \ln\left(\frac{a}{1+a}\right) > 1$ for all $a > 0$ and $y > 0$. Analogously, one can show that $r_0+s_0+\ln(bx)-(1+b)x \leq r_0+s_0+\ln(\frac{b}{1+b})-1 < 1$ for all $b > 0$ and $x > 0$. This shows that Inequality (20) is always verified.

Therefore, det $D\widetilde{\Phi}_2(x, y)$ is bounded away from zero and from Theorem 5 the map $\widetilde{\Phi}_2$ is injective.

Consequently, from our main result in Theorem 1 the 2-periodic competitive Ricker competition model has a 2-periodic globally asymptotically stable cycle with respect to the interior of $\mathbb{R}^2_+$.

4 Conclusion

In this paper, we have derived conditions to determine global stability of a competitive periodic planar systems from hypotheses on individual maps. Our main contribution was to show that global asymptotic stability for monotone non-autonomous periodic maps can be reduced to a problem of global injectivity. Our main result is illustrated with two important competition models in fluctuation habitats: the periodic Leslie-Gower model and the periodic Ricker competition model. We showed how the theoretical results are verified in applications.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Balreira, E.C.: Foliations and global inversion. Comment. Math. Helv. **85**(1), 73–93 (2010)
2. Balreira, E.C., Elaydi, S., Luís, R.: Local stability implies global stability for the planar Ricker competition model. Discrete Contin. Dyn. Syst. Ser. B **19**(2), 323–351 (2014)
3. Balreira, E.C., Elaydi, S., Luís, R.: Global stability of higher dimensional monotone maps. J. Differ. Equ. Appl. **23**(12), 2037–2071 (2017)
4. Clark, M.E., Gross, L.J.: Periodic solutions to nonautonomous difference equation. Math. Biosci. **102**, 105–119 (1990)
5. Cushing, J.M., Henson, S.M.: Global dynamics of some periodically forced, monotone difference equations. J. Differ. Equ. Appl. **7**(6), 859–872 (2001)
6. Elaydi, S., Sacker, R.: Global stability of periodic orbits of nonautonomous difference equations and populations biology. J. Differ. Equ. **208**, 258–273 (2005)
7. Elaydi, S., Sacker, R.: Skew-product dynamical systems: applications to difference equations. In: Proceedings of the Second Annual Celebration of Mathematics, United Arab Emirates (2005)
8. Elaydi, S., Sacker, R.: Periodic difference equations, population biology and the cushing-henson conjectures. Math. Biosci. **201**, 195–207 (2006)
9. Gale, D.: Nikaidô, Hukukane: the jacobian matrix and global univalence of mappings. Math. Ann. **159**, 81–93 (1965)
10. Gutierrez, C.: A Solution to the bidimensional global asymptotic stability conjecture. Ann. Inst. H. Poincaré Anal. Non. Linéaire **12**, 627–671 (1995)

11. Hirsch, M.W.: On existence and uniqueness of the carrying simplex for competitive dynamical systems. J. Biol. Dyn. **2**(2), 169–179 (2008)
12. Hou, Z.: On existence and uniqueness of a modified carrying simplex for discrete kolmogorov systems. J. Differ. Equ. Appl. **27**(2), 284–315 (2021)
13. Hsu, S.B., Smith, H.L., Waltman, P.: Competitive exclusion and coexistence for competitive systems on ordered banach spaces. Trans. Am. Math. Soc. **348**(10), 4083–4094 (1996)
14. Leslie, P., Gower, J.: The properties of a stochastic model for two competing species. Biometrica **45**, 316–330 (1958)
15. Luís, R., Elaydi, S., Oliveira, H.: Stability of a ricker-type competition model and the competitive exclusion principle. J. Biol. Dyn. **5**(6), 636–660 (2011)
16. Luís, R.: Nonautonomous Periodic Difference Equations: Applications to Populations Dynamics and Economics. Lambert Academic Publishing, Germany (2017)
17. Nollet, S., Xavier, F.: Global inversion via the palais-smale condition. Discrete Contin. Dyn. Syst. **8**, 17–28 (2002)
18. Parthasarathy, T.: On Global Univalence Theorems, vol. 77. Springer-Verlag, Berlin-Heidelberg-New York (1983)
19. Ricker, W.E.: Stock and recruitment. J. Fish. Res. Board Can. **11**(5), 559–623 (1954)
20. Sacker, R.J.: A note on periodic ricker maps. J. Differ. Equ. Appl. **13**(1), 89–92 (2007)
21. Smith, H.: Planar competitive and cooperative difference equations. J. Differ. Equ. Appl. **3**(5–6), 335–357 (1998)
22. Smith, H.L.: Periodic competitive differential equations and the discrete dynamics of competitive maps. J. Differ. Equ. **64**(2), 165–194 (1986)