AFFINE AND CURVATURE COLLINEATIONS IN SPACE-TIME

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August 1989

DECLARATION

I hereby declare that this thesis has been composed by myself and is based on work done by myself and that this thesis has not been accepted in any previous application for a degree, diploma or similar award. Also I declare that all quotations have been distinguished by quotation marks and the sources of information specifically acknowledged.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. Graham Hall for all his help, encouragement, advice and patience as well as for his hospitality during the past three years. I would also like to thank Dr. J. Pulham for providing advice and help in using both the computing facilities of the University of Aberdeen and his own programs Metric and Tetrad.

I would also like to thank Dr.E. Vaz and Prof. M.R. Valença of the University of Minho, and Prof.J.F. Carvalho of the University of Coimbra for their help encouragement and support.

Finally, I gratefully acknowledge the financial support of the Portuguese Institute for Scientific Research (INIC) and of the University of Minho.

To the memory of my father

SUMMARY

The purpose of this thesis is the study of the Lie algebras of affine vector fields and curvature collineations of space-time, the aim being, in the first case, to obtain upper bounds on the dimension of the Lie algebra of affine vector fields (under the assumption that the space-time is non-flat) as well as to obtain a characterization of such vector fields in terms of other types of symmetries. In the case of curvature collineations the aim was that of characterizing space-times which may admit an infinite-dimensional Lie algebra of curvature collineations as well as to find local characterizations of such vector fields.

Chapters 1 and 2 consist of introductory material, in Differential Geometry (Ch.1) and General Relativity (Ch.2).

In Chapter 3 we study homothetic vector fields which admit fixed points. The general results of Alekseevsky [a] and Hall [b] are presented, some being deduced by different methods. Some further details and results are also given.

Chapter 4 is concerned with space-times that can admit proper affine vector fields. Using the holonomy classification obtained by Hall [c] it is shown that there are essentially two classes to consider. These classes are analysed in detail and upper bounds on the dimension of the Lie algebra of affine vector fields of such space-times are obtained. In both cases local characterizations of affine vector fields are obtained.

Chapter 5 is concerned with space-times which may admit proper curvature collineations. Using the results of Halford and McIntosh [d] , Hall and McIntosh [e] and Hall [f] we were able to divide our study into several classes The last two of these classes are formed by those space-times which admit a (1 or 2-dimensional) non-null distribution spanned by vector fields which contract the Riemann tensor to zero. A complete analysis of each class is made and some general results concerning the infinite-dimensionality problem are proved. The chapter ends with some comments in the cases when the distribution mentioned above is null.

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[a] Ann. Glob. Anal. Geom., 3, (1985), 5.
[b] GRG, 20(7), (1988), 671.
[c] GRG, 20(4), (1988), 399.
[d] J. Phys. A.: Math. Gen., 14, (1981), 2331.
[e] Int. J. Theor. Phys., 22(5), (1983), 469.
[f] GRG 15(6), (1983), 581.

TABLE

INTRODUCTION

One of the most common ways of solving Einstein's equations consists of assuming that the metric one is looking for admits a certain group of symmetries. Such an assumption consists in general of a statement that the space-time admits a (local) group of isometries of a given dimension whose orbits are submanifolds of a certain type or, equivalently, a Lie algebra of Killing vector fields of a given dimension and spanning a distribution of a given type. If such is the case, one can then (at least in theory) reduce Einstein's equations to an easier set of equations. Because of such an important fact certain types of symmetries have been the object of extensive study in the past. Such studies concerned mostly Killing, homothetic and conformal vector fields (see for instance [57]).

There are however other types of symmetries one can consider in the same setting, and these are affine vector fields and curvature collineations. The first are the natural type of symmetry a linear connection can admit and it is well known that the set of such vector fields always forms a finite-dimensional Lie algebra which contains the Lie algebra of homothetic vector fields [55]. The second type (i.e. curvature collineations) is a natural generalization of the concept of affine vector field in the sense that they are vector fields which leave the Riemann tensor invariant, a property satisfied by affine vector fields, and has been introduced relatively recently in General Relativity by Katzin et al [52]. These types of vector fields seemed, however, less susceptible to study than conformal or homothetic vector fields.

In recent years, however, due mainly to the results obtained by McIntosh and Halford [61], Hall and McIntosh [45], Hall and Kay [44] in the study of the problem of "determination of the metric from the curvature" a more detailed study of such types of symmetries has become possible.

These results allowed Hall [35] to show that whenever the Riemann tensor of a given space-time - considered at each point as an endomorphism of the space of bivectors at that point - has rank ≥ 4, every curvature collineation (and, a posteriori, every affine vector field) is a homothetic vector field.

There are, nevertheless, cases when the rank of the Riemann tensor is smaller than 4 and, in such case, it is possible for the space-time in question to admit curvature collineations which are not homothetic vector fields and the study of such cases seems interesting in its own right. Furthermore, the results mentioned above provide us with a great deal of information about the tensor field $\mathcal{L}_X g$, where X is a curvature collineation and g is the metric of the space-time in question.

In the case of affine vector fields, the tensor $\mathcal{L}_X g$ defined above is covariantly constant (see [43]). Let us define then an affine vector field as being proper affine if it is not a homothetic vector field. In such case, the tensor field $\mathcal{L}_X g$ is not a constant multiple of the metric. As remarked by Hall [40], it follows that the space-time (M, g) if non-flat is reducible, that is its holonomy group is non-trivial. The classification af space-time according to its holonomy group has been achieved recently by Hall [40], and it provides us with a starting point to the study of proper affine vector fields.

The aim of this thesis is the study of the case when a given space-time (M, g) may admit proper affine vector fields or proper curvature collineations (that is, which are not affine vector fields).

Chapters 1 and 2 consist of introductory material in Differential Geometry (Ch.1) and General Relativity (Ch.2). In Chapter 1 we set up the basic results in Differential Geometry required in the sequel, from the concepts of manifold and tensor field, to those of linear connection, holonomy group and symmetry. In Chapter 2, after defining the basic concept of space-time we consider the several classification schemes we shall use, e.g. the Segre classification of the Ricci tensor, the Petrov classification of the Weyl tensor and the holonomy classification. The relationship between this classification schemes is also presented.

In Chapter 3 we study homothetic vector fields which admit fixed points. In a certain sense one could perhaps say that this is the only case when homothetic vector fields are of interest, since, in the neighbourhood of a point where a homothetic vector field does not vanish, one can always conformally rescale the metric so as to make that homothetic vector field a Killing vector field. In this Chapter the general results of Alekseevsky [3] and Hall [38] are presented, some being deduced by different methods. Further details and results are also given.

Chapter 4 is concerned with space-times that admit proper affine vector fields. Using the holonomy classification obtained by Hall [40] it is shown that there are essentially two classes to consider. These classes are analysed in detail and upper bounds on the dimension of the Lie algebra of affine vector fields of such space-times are obtained. In both cases local characterizations of affine vector fields are obtained. These show that apart from some natural proper affine vector fields which are easily found, the problem of finding others reduces to the problem of finding homothetic vector fields either in the same manifold or in some lower dimensional manifold. Under some global assumptions (geodesic completeness and simply-connectedness) some of these results are in fact global. The chapter ends with a generalization to affine vector fields of a theorem of Nomizu on the extension of Killing vector fields (a different proof of that same result has recently been published by Hall [32]).

Chapter 5 is concerned with space-times which may admit proper curvature collineations. Here, one of the main objectives of our work was that of characterizing all space-times which admit an infinite-dimensional Lie algebra of such vector fields, as well as that of finding local characterizations of such vector fields.Using the results of Halford and McIntosh [61], Hall and McIntosh [45] and Hall [35] mentioned above, we were able to divide our study into several classes which were then studied separately. For the first of theses classes (2+2-decomposable space-times) we were able to prove that in fact the Lie algebra of curvature collineations is always finite dimensional and that the problem of finding curvature collineations in such space-times reduces to that of finding curvature collineations in 2-dimensional Lorentzian or Riemannian manifolds. The second class analysed consisted of space-times which admit a (unique up to multiples) non-null vector field u which contracts the Riemann tensor to zero. In the case when there exists a curvature collineation parallel to such a vector field we were able to prove that the local Lie algebra of curvature collineations (that is the Lie algebra of curvature collineations of a suitably small open subset with the induced metric) is always infinite dimensional. A local characterization of such space-times and of their curvature collineations was obtained. Under some global assumptions, the infinite-dimensionality result is in fact global. In the general case we were forced to make the further assumption that the distribution spanned at each point by the evaluation at that point of all curvature collineations has constant dimension. Under this assumption we were able to prove that the local Lie algebra of curvature collineations is finite dimensional unless a (local) curvature collineation parallel to u exists. The last class analysed consisted of all those space-times wich admit a 2-dimensional non-null distribution spanned by vector fields which contract the Riemann tensor to zero. For this class results similar to those of the preceding class were obtained. The chapter ends with some comments in the cases when the vector field u or the 2-dimensional distribution mentioned above are null. These classes have already been the object of some investigations (see. [27], [77], [78]) which lead to the conclusion that clear cut results as those obtained in the other classes are perhaps not to be expected.

1. PSEUDO-RIEMANNIAN MANIFOLDS.

1.1.Conventions and some basic definitions.

In this section are gathered some of the basic definitions with which we shall be concerned in the sequel. Throughout this work we shall use **Einstein's summation convention**, that is, the convention that if in a mathematical expression (e.g. $x^i y_i$) an index appears repeated, once in lower and once in upper position, then it is implied that the index in question is to be summed over its range (e.g., if the range of i in the above expression is $1, ..., n$ then the expression means $x^1 y_1 + ... + x^n y_n$).

Let E be a real vector space of finite dimension n, (e_i) a basis of E; we denote by E^* the **dual** of E and by (e^i) the basis of E^* dual to (e_i), that is, the unique basis of E^* such that $e^i(e_j) = \delta^i{}_j$, were $\delta^i{}_j$ is the Kronecker delta. Given two finite dimensional real vector spaces E, F and a linear map $u : E \longrightarrow F$, we denote by ${}^t u$ its **transpose**, that is, the linear map from F^* to E^* given for $\alpha \in F^*$ by ${}^t u$ $(\alpha) = \alpha \circ u$.

For all integers $r, s \geq 0$ we define $T^r_s(E)$ as the vector space of all multilinear maps α:$(E^*)^r \times (E)^s \longrightarrow \mathbf{R}$. If (e_i) is a basis of E, the elements of $T^r_s(E)$,

$$e_{i_1} \otimes .. \otimes e_{i_r} \otimes e^{j_1} \otimes .. \otimes e^{j_s},$$

$1 \leq i_1, .., i_r, j_1, ..j_s \leq n$, which are characterized as the unique elements of $T^r_s(E)$ such that:

$$e_{i_1} \otimes .. \otimes e_{i_r} \otimes e^{j_1} \otimes .. \otimes e^{j_s}(e^{k_1}, .., e^{k_r}, e_{m_1}, .., e_{m_s}) = \delta^{k_1}_{i_1} ... \delta^{j_s}_{m_s}, \tag{1.1.1}$$

for all $1 \leq k_1, .., k_r, m_1, .., m_s \leq n$, form a basis of $T^r_s(E)$. Often we shall denote the elements of the above basis by $e_{i_1..i_r}{}^{j_1..j_s}$. If $\alpha \in T^r_s(E)$, defining

$$\alpha^{i_1..i_r}{}_{j_1..j_s} = \alpha(e^{i_1}, .., e^{i_r}, e_{j_1}, .., e_{j_s}),$$

we have

$$\alpha = \alpha^{i_1..i_r}{}_{j_1..j_s} e_{i_1..i_r}{}^{j_1..j_s}. \tag{1.1.2}$$

Let now h be a symmetric bilinear form on E. There exist then unique integers $p, q \geq 0$ with $p + q \leq n$ and a basis (e_i) of E such that $h(e_i, e_i) = -1$ for $1 \leq i \leq q$, $h(e_i, e_i) = 1$ for $q+1 \leq i \leq q+p$ all the remaining $h(e_i, e_j)$ being zero, $1 \leq i, j \leq n$.

We call (p,q) the signature of h. A basis of E such as the basis (e_i) above is called an **orthonormal basis** of E with respect to h. If h is nondegenerate we have $p+q=n$, in which case we denote its signature by $(p,n-p)$. In such case, if $p \geq 2(n-p)$, one can find a basis $(f_i)_i$ of E such that $h(f_i, f_{i+(n-p)}) = h(f_j, f_j) = 1$ for $1 \leq i \leq 2(n-p)$ and $2(n-p)+1 \leq j \leq n$, all the remaining $h(f_k, f_l)$ being 0 (take an orthonormal basis $(e_i)_i$ and define $f_i = e_i$ for $2(n-p)+1 \leq i \leq n$, $f_i = (e_i + e_{i+(n-p)})/\sqrt{2}$, $f_{i+(n-p)} = (e_i - e_{i+(n-p)})/\sqrt{2}$ for $1 \leq i \leq (n-p)$. Such a basis will be called a **null** basis of E (with respect to h) or, in the particular case when $n=4$ and $p=3$, a **null tetrad**.

Let g be a symmetric and nondegenerate bilinear form over a real vector space E of finite dimension n. Given a subspace V of E we denote by $V^{\perp}$ its **orthogonal**, that is, the subspace of those $y \in E$ such that $g(x,y)=0$ for all $x \in V$. $V^{\perp}$ is of dimension $n-k$ whenever V is of dimension k [15]. A subspace V of E will be called **null** if the subspace $V \cap V^{\perp}$ is not trivial; otherwise it will be said to be **non null**, in which case E is the direct sum of V and $V^{\perp}$.

Let g be a symmetric and nondegenerate bilinear form over a real vector space E of finite dimension n. For $x \in E$ the map $y \mapsto g(x,y)$ is a linear form over E; thus, we define a linear map $G : E \longrightarrow E^*$ by $x \mapsto (y \mapsto g(x,y))$; this map is an isomorphism since g is nondegenerate. Given a basis (e_i) of E, the matrix of G with respect to this basis is also the matrix of g, that is $G(e_i) = g_{ij}e^j$; thus, if $x = x^i e_i \in E$, we have $G(x) = g_{ij}x^j e^i$. We set $x_i = g_{ij}x^j$ and we say that G acts by **lowering indices**. As G is an isomorphism we can define G^{-1}, which is a map from E^* to E. If one denotes by $(g^{ij})_{i,j}$ its matrix with respect to the basis $(e^i), (e_j)$, then we have $g_{ij}g^{jk} = \delta_i{}^k$, and if $\alpha = \alpha_j e^j \in E^*$, then we have $G^{-1}(\alpha) = \alpha_j g^{jk} e_k$; we set $\alpha^j = g^{jk}\alpha_k$ and we say that G^{-1} acts by **raising indices**. This process of lowering and raising indices is extended to the spaces $T^r_s(E)$ in the obvious way.

An automorphism u of E is said to be **orthogonal** with respect to g if and only if for all $x, y \in E$ one has $g(u(x), u(y)) = g(x,y)$. The set of all such maps is denoted by $O(E,g)$; it is a subgroup of the linear group $GL(E)$ and it is called the **orthogonal group** of g. One can show that, defining, for $f \in GL(E)$, f^* by $f^* = G^{-1} o\, {}^t f o G$ (called the **adjoint** of f with respect to g) then $f \in O(E,g)$ if and only if $f^* = f^{-1}$ [15]. In particular, one sees that $\det(u) = \pm 1$ for all u in $O(E,g)$.

A **pseudo-euclidean** space is a finite dimensional real vector space equiped with a nondegenerate symmetric bilinear form. Given two pseudo-euclidean spaces $(E,g),(F,h)$ we say that they are **linearly equivalent** if there exists an isomorphism $u : E \longrightarrow F$ such that $h(u(x),u(y)) = g(x,y)$ for all $x,y \in E$; linear equivalence is an equivalence relation in the set of pseudo-euclidean spaces. Two pseudo-euclidean spaces lie in the same equivalence class for this relation if and only if their bilinear forms have the same signature. We call **Minkowski space** any representative of the class of signature $(3,1)$, that is, any 4-dimensional real vector space equiped with a nondegenerate and symmetric bilinear form of signature $(3,1)$. If $(E,g),(F,h)$ are linearly equivalent the groups $O(E,g)$ and $O(F,h)$ are isomorphic [15]; this justifies the notation $O(p,q)$ for both groups, (p,q) being the signature of g and h. The group $O(3,1)$, i.e. the orthogonal group of Minkowski space, is called the **Lorentz** group and sometimes denoted by $\mathcal{L}$.

Let G be a group, whose zero element shall be denoted e and whose composition law we denote multiplicatively. Let X be a set. A **group action on the right** of G on X is a map $\Phi : X \times G \longrightarrow X$ such that the following properties are satisfied for all $a,b \in G$ and all $x \in X$:

$$\Phi(x,e) = x; \tag{1.1.3}$$

$$\Phi(\Phi(x,a),b) = \Phi(x,ab). \tag{1.1.4}$$

Let X be a set, G be a group and Φ be an action of G on X on the right; for all $x \in X$ we denote by O_x the set of elements y of X such that $y = \Phi(x,a)$ for some a in G and call it the **orbit** (or G-**orbit**) of x; the set of all $a \in G$ such that $\Phi(x,a) = x$ is called the **isotropy** set of x and shall be denoted by I_x; I_x is a subgroup of G (in general not normal); I_x and I_y are conjugate subgroups of G whenever y lies in the orbit of x. The action Φ is said to be **transitive** if for any pair x,y of elements of X there exists a in G such that $\Phi(x,a) = y$. Φ is said to be **effective** (resp. **free**) if $\Phi(x,a) = x$ for all (resp. some) $x \in X$) implies $a = e$. The relation $\mathcal{R}$ defined on X by "$x\mathcal{R}y$ if and only if there exists a in G such that $\Phi(x,a) = y$" is obviously an equivalence relation on X whose equivalence classes are exactly the orbits of the elements of X under the action Φ.

Finally, if H is a subgroup of G then the restriction Φ' of Φ to $X \times H$ is an action of H on X on the right; if, for $x \in X$, we denote by O'_x its orbit and by I'_x its isotropy under this action, we have $O'_x \subset O_x$ and $I'_x \subset I_x$.

Given a set X and an action on the right Φ of a group G on X the **problem of classification** of the elements of X under the action Φ consists in finding a description of the orbits of Φ or, equivalently, in finding necessary and sufficient conditions for two elements of X to lie in the same orbit (for Φ). This is usually done by finding invariants of Φ, that is, maps $f : X \longrightarrow Y$ (where Y is some set) which are constant on each orbit of Φ.

A fundamental example, playing an important role in the sequel is the **Jordan equivalence problem** which we now describe [60]. Let E be a real vector space of finite dimension n and let $\mathcal{L}(E)$ be its algebra of endomorphisms. The group $GL(E)$ acts on the right in $\mathcal{L}(E)$ by $\Psi : \mathcal{L}(E) \times GL(E) \longrightarrow \mathcal{L}(E)$, $(f, u) \mapsto \Psi(f, u) = u^{-1} \circ f \circ u$. The Jordan equivalence problem is the classification problem underlying this group action. The minimal and characteristic polynomials, as well as the determinant and trace functions are invariants of this action [60].

Let $f \in \mathcal{L}(E)$, and assume that all its eigenvalues $\lambda_1, \lambda_2, .., \lambda_r$, with respective multiplicities $\zeta_1, .., \zeta_r$ are real. Define, for $k \geq 1$, $\lambda \in \mathbf{R}$, $J(k; \lambda_i)$ as being the $k \times k$ matrix:

$$\begin{pmatrix} \lambda_i & 1 & 0 & . & . & . & 0 \\ 0 & \lambda_i & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & \lambda_i & 1 \\ 0 & . & . & . & . & 0 & \lambda_i \end{pmatrix}$$

It can be proved [60] that there exist integers $s_{11}, s_{12}, .., s_{1\beta_1}, .., s_{r1}, .., s_{r\beta_r}$, and a basis of E such that, with respect to this basis, the matrix of f takes the form:

$$\begin{pmatrix} J(s_{11}, \lambda_1) & 0 & . & . & . & . & 0 \\ 0 & J(s_{12}; \lambda_1) & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & J(s_{r\beta_r}; \lambda_r) \end{pmatrix}$$

Thus, in this case, f is completely characterized when its eigenvalues, their multiplicities and the integers s_{ij} are given, and it can be shown that two elements

v, u of $\mathcal{L}(E)$ lie in the same orbit under Ψ if and only if their respective sets $\lambda_i, \beta_j, s_{lk}$ are equal (up to their respective orders)[60].

The symbol:

$$\{s_{11} ... s_{1\beta_1} .... s_{r1} ... s_{r\beta_r}\}, \tag{1.1.5}$$

contains most of the above information (except for the number of different eigenvalues and their actual values). The symbol:

$$\{(s_{11}, .., s_{1\beta_1}), ..., (s_{r1}, .., s_{r\beta_r})\}, \tag{1.1.6}$$

differs from the above one in the sense that it tells us the number of different eigenvalues (i.e. the number of pairs ()) and their multiplicities ($s_{i1} + ... + s_{i\beta_i} = \zeta_i$). The first symbol is called the **Segre type** [20] of the endomorphism in question; the second will be called a Segre type degeneracy.

As an example take the Segre type $\{2, 1, 1\}$ and its degeneracy $\{(2, 1), (1)\}$ for an endomorphism f of $\mathbf{R}^4$; this means that f has two distinct real eigenvalues. λ_1, λ_2, the first with multiplicity 3, and that there exists a basis of $\mathbf{R}^4$ with respect to which the matrix of f is:

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

NOTE.1.1.1. If, instead of real vector spaces, we take complex vector spaces then, due to the fact that $\mathbf{C}$ is algebraically closed, the above classification is valid for all endomorphisms. In the real case the above classification is not complete as f can have complex eigenvalues; however, it happens that for our needs it will only be necessary to consider the case when f has at most a conjugate pair of complex eigenvalues; we make the convention that we denote by

$$\{s_{11}, ..s_{1\beta_1}, ..., s_{r1}, .., s_{r\beta_r}, z, \overline{z}\}, \tag{1.1.7}$$

the Segre type of these linear maps. We keep a notation similar to (1.1.6) for the degeneracies of this Segre type.

Another important example is the following; let E be a finite dimensional real vector space, g be a symmetric bilinear form over E; let $\mathcal{S}_E$ be the set of subspaces of E. The group $O(E, g)$ acts on the right in $\mathcal{S}_E$ by $\Psi : \mathcal{S}_E \times O(E, g) \to \mathcal{S}_E$, $(F, u) \mapsto u^{-1}(F)$. The underlying classification problem has the following solution [15]:

THEOREM 1.1.1.(Witt). *Let F, H be elements of $\mathcal{S}_E$. Then F, H belong to the same orbit under Ψ if and only if they are of the same dimension and the restrictions of g to F and H have the same signature.*

In the case of a Minkowski space, (E, g), the above theorem leads to the well known classification of subspaces as **null**, **timelike** or **spacelike** according to whether the signature of the restriction of g is degenerate, nondegenerate and non-positive or positive definite (respectively). Given a vector $v \in E$ we say that v is **spacelike** if $g(v,v) > 0$, that it is **timelike** if $g(v,v) < 0$ and that it is **null** if it is nonzero but $g(v,v) = 0$. The following is then another characterization of the above classes [28]:

THEOREM 1.1.2. *Let (E, g) be a Minkowski space. If F is a proper subspace of E of dimension two or more, then:*

(a).F is timelike if and only if it contains more than one null subspace of dimension 1;

(b).F is null if and only if it contains a unique one-dimensional null subspace;

(c).F is spacelike if and only if it does not contain null subspaces;

(d).F is timelike if and only if it contains a timelike one-dimensional subspace;

(e).If F is timelike (resp. spacelike), (resp. null),$F^{\perp}$ is spacelike (resp. timelike), (resp. null).

In the case when F is null, it contains at most a null 1-dimensional subspace (cf. (b)); every element of this null subspace is orthogonal to all elements of F.

With this classification of subspaces of Minkowski space we can then proceed to a classification of the elements of the Lorentz group themselves. In fact, as follows from the solution to the Jordan equivalence problem, given any endomorphism u of (E, g) there exists at least one 2-dimensional subspace F_u of E stable by u, that is, such that $u(F_u) \subset F_u$. Now let $u \in O(3,1)$. Then, for $v \in {F_u}^{\perp}$ and $z \in F_u$ one has

$$g(u(v), z) = g(u(v), u(u^{-1}(z))) = g(v, u^{-1}(z)) = 0$$

and this shows that $F^{\perp}$ is also stable by u. One can therefore classify the elements of $O(3,1)$ according to the nature of F_u. Here, we shall concentrate our attention on only one type of orthogonal transformation.

Given a null vector $l \in E$ we say that $u \in O(3,1)$ is a **null rotation about** l if l is an eigenvector of u and, furthermore $\det(u) = 1$. If (l,k,x,y) is a null tetrad (l,k being the null vectors) it can be proved [28] that there exist real numbers $A, \alpha, \beta, \gamma, \delta$, such that $A > 0$, $\alpha^2 + \beta^2 = 1$ and

$$\begin{aligned} u(l) &= Al \\ u(k) &= -A(\gamma^2 + \delta^2)l + A^{-1}k + \sqrt{2}\gamma x - \sqrt{2}\delta y \\ u(x) &= -A\sqrt{2}(\alpha\gamma + \beta\delta)l + \alpha x - \beta y \\ u(y) &= A\sqrt{2}(\alpha\delta - \beta\gamma)l + \beta x + \alpha y. \end{aligned} \tag{1.1.8}$$

1.2.Manifolds; differentiable maps.

In all that follows the topological spaces considered are assumed to be Hausdorff with a countable basis of open sets.

Given a topological space M, a **chart** (or **coordinate chart**)of M is a pair (U, ϕ) where U is an open subset of M (called the **domain** of the chart), and ϕ is a homeomorphism of U onto an open subset of $\mathbf{R}^n$ (the integer n is called the **dimension** of the chart). Given two charts (U, ϕ) and (V, ψ) of M we say that they are C^k-**compatible** if and only if either $U \cap V$ is empty or the maps $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are C^k-differentiable in their domains of definition.

A C^k-**atlas** of M is a family $\mathcal{A}$ of charts of M whose domains cover M and such that any two charts of $\mathcal{A}$ are C^k-compatible. Two C^k-atlases of M are C^k-compatible if their union is also a C^k-atlas of M. A C^k-atlas $\mathcal{A}$ of M is said to be **complete** if and only if any atlas $\mathcal{A}'$ of M which is C^k-compatible with $\mathcal{A}$ is contained in $\mathcal{A}$. Every atlas $\mathcal{A}$ of M is contained in a unique complete atlas; we call this atlas the **completion** of $\mathcal{A}$ and we denote it by $\hat{\mathcal{A}}$.

A C^k-**manifold** is a topological space M (satisfying the topological conditions above) equiped with a complete C^k-atlas.

If M is connected (for manifolds this condition is equivalent to path connectedness [50]) one can prove that all the charts in a complete atlas of M have the same dimension; the integer in question is then called the **dimension** of M.

Given any integer n, consider $\mathbf{R}^n$ together with its usual topology; then $\mathcal{A} = \{(\mathbf{R}^n, \mathrm{id})\}$ is a C^k-atlas of $\mathbf{R}^n$ for all values of k, so $\mathbf{R}^n$ together with the completion atlas $\hat{\mathcal{A}}$ is a C^k-manifold for all values of k. In the sequel we shall always consider $\mathbf{R}^n$ as equiped with the C^∞- manifold structure obtained in this way.

Another example of a C^k-manifold is obtained in the following way; let M be a C^k-manifold and let $\mathcal{A}$ be its atlas, U be an open subset of M. For every chart of M, (V, ϕ) say, the pair $(V \cap U, \phi_{|V \cap U})$ defines a chart of U. The set of all charts thus obtained, using all elements of $\mathcal{A}$, defines an atlas of U. When equiped with the manifold structure defined by the completion of this atlas, we say that U is an **open submanifold** of M. An open subset of a manifold will always be considered as equiped with this manifold structure.

Given two C^k-manifolds M and N with respective (complete) atlasses $\mathcal{A}$ and $\mathcal{B}$ we define naturally over the topological space $M \times N$ (with the product topology) a structure of C^k-manifold by taking as charts of $M \times N$ the products of the charts of M and N (i.e. of the form $(U \times V, (\phi, \psi))$, where $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$) and taking then the completed atlas of the atlas thus obtained. With this manifold structure we say that $M \times N$ is the **product** of the (manifolds) M and N.

In all that follows we shall consider only manifolds of differentiability class C^∞; thus in what follows the word differentiable will mean C^∞- differentiable whilst the word manifold will always mean C^∞-manifold.

Given two manifolds M and N with respective (complete) atlasses $\mathcal{A}$ and $\mathcal{B}$ and a map $f : M \to N$, we say that f is **differentiable** at $p \in M$ if and only if for any chart (U, ϕ) of M such that $p \in U$ and every chart (V, ψ) of N such that $f(U) \subset V$, the map

$$\psi \circ f_{|U} \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is differentiable at $\phi(p)$. f is said to be differentiable if and only if it is differentiable at all points of M. A **diffeomorphism** of a manifold M onto a manifold N is a differentiable map f from M to N which is one-to-one and onto and such that its inverse is also differentiable . A diffeomorphism of M onto itself is called a **transformation** of M. The set of differentiable maps from a manifold M to a manifold N will be denoted by $C^\infty(M, N)$, except when $N = \mathbf{R}$, in which case we denote it by $\mathcal{F}(M)$. $\mathcal{F}(M)$ is a commutative unitary ring and a vector space over the reals. A differentiable map $\gamma : I \to M$, where I is an open interval of the real line and M is a manifold, is called a **curve** on M.

NOTE.1.2.1. It follows from the topological conditions imposed at the beginning of this section that every manifold admits a differentiable partition of unity [7].

Let then M be a manifold and let $p \in M$. A **tangent vector** at p is a $\mathbf{R}$-linear map $v_p : \mathcal{F}(M) \to \mathbf{R}$ such that for all $f, h \in \mathcal{F}(M)$:

$$v_p(f.h) = h(p)v_p(f) + f(p)v_p(h). \tag{1.2.1}$$

The notion of tangent vector is local in character, in the sense that, given $f, h \in \mathcal{F}(M)$ such that f and h coincide in some open subset U of M, and given $p \in U$, we have $v_p(f) = v_p(h)$ for every tangent vector v_p to M at p.

Given a coordinate chart (U, ϕ) of M and $p \in U$ for every $f \in \mathcal{F}(M)$ the map $f \circ \phi^{-1}$ is a differentiable map from $\phi(U)$ to $\mathbf{R}$; denoting by $(x^1, ..., x^n)$ the coordinates in $\mathbf{R}^n$, we can therefore compute

$$\frac{\partial}{\partial x^i} f \circ \phi^{-1}(\phi(p)).$$

This allows us to define for $1 \leq i \leq n$ a tangent vector at p, which we denote by $\partial_{i|p}$, by:

$$\partial_{i|p} f = \frac{\partial}{\partial x^i}(f \circ \phi^{-1})(\phi(p)), \tag{1.2.2}$$

for all $f \in \mathcal{F}(M)$ [50].

The set of all tangent vectors to M at p is denoted by $T_p(M)$ and is called the **tangent space** to M at p. $T_p(M)$ is a real vector space and for any given chart (U, ϕ) of M at p (i.e. such that $p \in M$) the set $(\partial_{1|p}, ..., \partial_{n|p})$ defined above is a basis of $T_p(M)$; we call this basis of $T_p(M)$ the **natural** basis associated with the chart (U, ϕ); (in particular we see that $T_p(M)$ is of the same dimension as M).

Let now M, N be manifolds and let $\phi : M \to N$ be a differentiable map. Given $f \in \mathcal{F}(N)$, the map $fo\phi$ is an element of $\mathcal{F}(M)$; consequently, given any point $p \in M$ and any $v_p \in T_p(M)$, we can compute $v_p(f \circ \phi)$.

Consider then the map :

$$\phi_{*p}(v_p) : \mathcal{F}(N) \to \mathbf{R},$$

defined for every $f \in \mathcal{F}(N)$ by:

$$\phi_{*p}(v_p)(f) = v_p(f \circ \phi). \tag{1.2.3}$$

Then $\phi_{*p}(v_p)$ is a tangent vector to N at $\phi(p)$ [50]. This allows us to define a map :

$$\phi_{*p} : T_p(M) \rightarrow T_{\phi(p)}(N),$$

which is linear. The map thus defined is called the **derivative** of ϕ at p. Given a chart (U, ψ) of M at p and a chart (V, ρ) of N at $\phi(p)$ it is to prove that with respect to the natural basis of $T_p(M)$ and $T_{\phi(p)}(N)$ associated with the above charts the matrix of ϕ_{*p} is the matrix of the derivative of the map $\rho \circ \phi \circ \psi^{-1}$ at $\psi(p)$ in the canonical basis of $\mathbf{R}^n$ and $\mathbf{R}^m$ [7]. The chain rule holds for differentiable maps on manifolds [7].

With these definitions and constructions, classical notions such as that of **rank** of a map,and classical results of differential calculus, such as the implicit function, inverse function and constant rank theorems, hold on manifolds.

We end this section with some definitions [16].

Given manifolds M, N a differentiable map $\phi : M \rightarrow N$ and a point p on M, we say that ϕ is an **immersion** (resp. **submersion**) (resp. **local diffeomorphism**) at p if and only if ϕ_{*p} is one-to-one (resp. onto) (resp. onto and one-to-one). ϕ is said to be an **immersion** (resp. **submersion**) if and only if it is an immersion (resp. submersion) at every point of M.

Given manifolds M, N with $N \subset M$, we say that N is a **submanifold** of M if and only if the natural injection $i : N \rightarrow M$ is an immersion.

We say that N is a **regular submanifold** of M if, furthermore, the manifold topology of N coincides with the topology induced on N by the topology of M.

1.3.Vector fields; distributions.

Let M be a n-dimensional manifold; a **vector field** X on M is a map:

$$X : \mathcal{F}(M) \rightarrow \mathcal{F}(M),$$

which is $\mathbf{R}$-linear and such that, for all $f, h \in \mathcal{F}(M)$:

$$X(f.h) = h.X(f) + f.X(h). \tag{1.3.1}$$

The set of vector fields on M is naturally a real vector space and a module over the ring $\mathcal{F}(M)$ and is denoted by $\mathcal{D}_1(M)$.

The behaviour of a vector field is local in character, that is, if $f, h \in \mathcal{F}(M)$ coincide in the neighbourhood of a point p of M then $X(f)(p) = X(h)(p)$ [50].

If $X \in \mathcal{D}_1(M)$ and U is an open subset of M, X defines naturally a vector field on U, called the **restriction** of X to U and denoted by $X_{|U}$, in the following way: given any point p on U and $f \in \mathcal{F}(U)$, we can find an open neighbourhood V of p in U and $\hat{f} \in \mathcal{F}(M)$ such that the restriction of $\hat{f}$ to V is equal to the restriction of f to V (this is a consequence of the theorem of partitions of unity, (cf. NOTE.1.2.1)); we define then $X_{|U}(f)(p) = X(\hat{f})(p)$.

Given a vector field X and a point p on M we can associate with X the element of $T_p(M)$, denoted by X_p, given by:

$$X_p(f) = (X(f))(p).$$

This shows that, given a coordinate chart (U, ψ) of M and a vector field X on M there exist unique functions $X^1, .., X^n$ from U to $\mathbb{R}$ such that for all $p \in U$ and all $f \in \mathcal{F}(M)$:

$$X_p(f) = X^i(p)\partial_{i|p}f. \tag{1.3.2}$$

The functions X^i are in fact elements of $\mathcal{F}(U)$ as follows from the definition of vector field. The expression (1.3.1) is called the **local expression** of X in the chart (U, ψ).

Each ∂_i itself. is a vector field in the coordinate domain U.

NOTE.1.3.1. Expression (1.3.1) above gives, when $X = \partial_i$:

$$X(f) = \partial_i f;$$

In order to lighten the notation we shall, in general, write $f_{,i}$ instead of $\partial_i f$.

Given vector fields X, Y on M we can define a new vector field, denoted by $[X, Y]$ and called the **Lie bracket** of X and Y, by:

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{1.3.3}$$

for all $f \in \mathcal{F}(M)$. Clearly the map [,] is bilinear and antisymmetric; it satisfies the **Jacobi identity**:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \tag{1.3.4}$$

for all vector fields X, Y, Z on M. $\mathcal{D}_1(M)$ is, therefore, a **Lie-algebra**.

If (U, ϕ) is a coordinate chart of M, and X, Y are vector fields on U, using the local expressions of X and Y given by (1.3.1), we get for $[X, Y]$ the local expression (cf. NOTE.1.3.1.):

$$[X, Y] = (X^i Y^j{}_{,i} - Y^i X^j{}_{,i})\partial_j. \tag{1.3.5}$$

It follows from the definition of tangent vector that, given a manifold M, a point p in M and $v \in T_p(M)$, there exists a curve $\gamma : I \to M$ such that $0 \in I$, $\gamma(0) = p$ and, 1 denoting the natural basis of $T_t(I)$ in the chart (I, id):

$$\gamma_{*0}.1 = v;$$

(in general we write $\gamma_{*s}.1 = \frac{d\gamma}{dt}_{|s}$).

Let then $X \in \mathcal{D}_1(M)$, $p \in M$. The **Cauchy problem** for X with initial condition $(0, p)$ consists in finding an open interval I of $\mathbb{R}$, with $0 \in I$, and a curve $\gamma : I \to M$ such that:

$$\gamma(0) = p,$$

and

$$\frac{d\gamma}{dt}_{|t} = X_{\gamma(t)},$$

for all t in I; The curve γ is then called an **integral curve** of X through p. Using charts around p, this problem reduces to the classical Cauchy problem for ordinary differential equations; the smoothness of the functions involved guarantees, then, that it always has a solution [50].

Given a point $p_o \in M$ and $X \in \mathcal{D}_1(M)$, it follows from well known results of the classical theory of ordinary differential equations [7] that there exists an open neighbourhood U of p_o in M and an open interval $I(p_o)$ of $\mathbb{R}$, with $0 \in I(p_o)$, such that, for all $p \in U$, the Cauchy problem with initial condition$(0, p)$ has a solution with domain $I(p_o)$. The solution thus obtained for the Cauchy problem is denoted by $t \to \phi_t(p)$, so that $\phi_0(p) = p$. The maps ϕ_t thus defined are maps from U to M for all $t \in I(p_o)$. It can be shown that each ϕ_t is differentiable and that, for every $s, t \in I(p_o)$ such that $s + t \in I(p_o)$ and every $p \in U$ such that $\phi_t(p), \phi_s(p) \in U$:

$$\phi_t \circ \phi_s(p) = \phi_s \circ \phi_t(p) = \phi_{s+t}(p). \tag{1.3.6}$$

It follows immediatly from this relation, the fact that $\phi_0 = \mathrm{id}_U$ and the inverse function theorem that each ϕ_t is a local diffeomorphism at each point of U. The set $\{\phi_t : t \in I(p_o)\}$ is called the **local one-parameter group of local transfomations** of M associated with X at p_o.

A vector field is said to be **complete** if and only if $I(p) = \mathbf{R}$ for all $p \in M$. In such case, $\{\phi_t : t \in \mathbf{R}\}$ is a true one-parameter group and it can be proved [7] that each ϕ_t is actually a transformation of M.

Let M, N be manifolds and $\Phi : M \to N$ be a differentiable map. Given $X \in \mathcal{D}_1(M)$ and $X' \in \mathcal{D}_1(N)$, we say that X, X' are **Φ-related** [7] if and only if for every $f \in \mathcal{F}(N)$:

$$X(fo\Phi) = (X'(f))o\Phi.$$

It can be proved [7] that if $X, Y \in \mathcal{D}_1(M)$ and $X', Y' \in \mathcal{D}_1(N)$ are such that X, X' and Y, Y' are Φ-related, then $[X, Y]$ and $[X', Y']$ are Φ-related. If, in particular, Φ is a diffeomorphism, then the vector field $\Phi_* X$ defined on N by:

$$(\Phi_* X)_q = \Phi_{*\Phi^{-1}(q)} X_{\Phi^{-1}(q)},$$

is Φ-related to X. We call it the **direct image** of X under Φ.

Given a chart (U, ϕ) of M, the vector fields $\partial_1, ..., \partial_n$ on U are such that:

$$[\partial_i, \partial_j] = 0,$$

for all $1 \le i, j \le n$. Conversely, we have [7]:

THEOREM 1.3.1. *Let M be an n-dimensional manifold, U be an open subset of M and $X_1, ..., X_n$ be a family of vector fields on M with the following properties:*
(a). There exists a point p in U such that $X_{1|p}, ..., X_{n|p}$ span $T_p(M)$;
(b). There exists a neighbourhood V of p such that for all $1 \le i, j \le n$: $[X_i, X_j]_{|V} = 0$.
Then there exists a chart (W, ϕ) of M around p such that $X_{i|W} = \partial_i$, for $1 \le i \le n$.

Another result that we shall use frequently in the sequel is the following [7]:

THEOREM 1.3.2. *Let $X \in \mathcal{D}_1(M)$ and $p \in M$; if $X_p \neq 0$ there exists a chart (U, ϕ) around p such that : $X = \partial_1$ in U.*

Let now M be a n-dimensional manifold, U be an open subset of M and assume the existence of submanifolds $N_1, ..., N_l$ of U such that $U = N_1 \times .... \times N_l$. We call the family $L = (N_1, ..., N_l)$ a **decomposition** of U (or a **local decomposition** of M). Given such a decomposition L, we define, for $1 \leq k \leq l$, $\pi^k : U \to N_k$ as being the natural projection, and for each $p \in U$, $\mathrm{i}_k(p) : N_k \to U$ as being the natural injection $q \mapsto (\pi^1(p), .., \pi^{k-1}(p), q, \pi^{k+1}(p), .., \pi^l(p))$. Each π^k (resp. i_k) is a submersion (resp. immersion) and, for every $p \in U$, we have :

$$T_p(M) = T_{\pi^1(p)}(N_1) \oplus .... \oplus T_{\pi^l(p)}(N_l).$$

Given $X \in \mathcal{D}_1(N_k)$ we can define a vector field $\mathrm{i}_k(X)$ on U by:

$$\mathrm{i}_k(X)_{|p} = (\mathrm{i}_k(p))_{*\pi^k(p)} X_{\pi^k(p)},$$

for all $p \in U$.

On the other hand, given a vector field X on M, we can define, for each $1 \leq k \leq l$, a new vector field on M, denoted by $\pi^k(X)$, and given by:

$$\pi^k(X)_{|p} = (\pi^k)_{*p} X_p,$$

for all $p \in U$. Obviously, for every $p \in U$, $\pi^k(X)_p$ is an element of the subspace $T_{\pi^k(p)}(N_k)$ of $T_p(M)$ but, in general, $\pi^k(X)$ cannot be considered as a vector field on the manifold N_k.

We say that a vector field X on U is L-**projectable** if and only if for $1 \leq k \leq l$ there exists $X_k \in \mathcal{D}_1(N_k)$ such that for every $p \in U$:

$$X_p = \mathrm{i}_1(X_1)_{|\pi^1(p)} \oplus .... \oplus \mathrm{i}_l(X_l)_{|\pi^l(p)},$$

for all $p \in U$.

The following is a characterization of L-projectable vector fields:

THEOREM 1.3.3. *$X \in \mathcal{D}_1(U)$ is L-projectable if and only if for $1 \leq k, j \leq l$, $k \neq j$, and every $Y \in \mathcal{D}_1(N_j)$:*

$$[\pi^k(X), \mathrm{i}_j(Y)] = 0.$$

Sketch of proof. Let us analyse the case when $l = 2$. In this case, we can choose in U coordinates of the form $(x^1, ..., x^q, y^1, ..., y^{n-q})$, where $q = \dim N_1$ and $q - n = \dim N_2$, such that $(x^1, ..., x^q)$ are coordinates in N_1 and $(y^1, ..., y^{n-q})$

are coordinates in N_2. If X_1 (resp. X_2) is a vector field on N_1 (resp. N_2), there exist then functions $f^i \in \mathcal{F}(N_1)$ and $h^j \in \mathcal{F}(N_2)$ such that

$$X_1 = \sum_{1 \le i \le q} f^i \frac{\partial}{\partial x^i};$$

$$X_2 = \sum_{1 \le j \le (n-q)} h^j \frac{\partial}{\partial y^j}.$$

These will also be the expressions of the vector fields $i_1(X_1)$ and $i_1(X_2)$, that is, if $p = (a^1, .., a^q, b^1, .., b^{n-q})$ is a point in U and we set $p^1 = (a^1, .., a^q)$ and $p^2 = (b^1, .., b^{n-q})$, then:

$$i_1(X_1)_p = \sum_{1 \le i \le q} f^i(a^1, .., a^q) \frac{\partial}{\partial x^i};$$

and a similar expression for $i_2(X_2)$.

If $X \in \mathcal{D}_1(U)$, there exist functions m^i, n^j in $\mathcal{F}(U)$ such that:

$$X = \sum_{1 \le i \le q} m^i \frac{\partial}{\partial x^i} + \sum_{1 \le j \le (n-q)} n^j \frac{\partial}{\partial y^j},$$

and one has:

$$\pi^1(X) = \sum_{1 \le i \le q} m^i \frac{\partial}{\partial x^i},$$

and

$$\pi^2(X) = \sum_{1 \le j \le (n-q)} n^j \frac{\partial}{\partial y^j}.$$

The theorem is then just a statement of the fact that if X is projectable then the functions m^i (resp. n^j) do not depend on the variables y^j (resp. x^i). The theorem follows for any value of l by induction ■

A **distribution** on a manifold M is a map Δ that associates with each $p \in M$ a subspace $\Delta(p)$ of $T_p(M)$. We shall always assume that the dimension of $\Delta(p)$ is constant; if this dimension is k, we say that Δ is a k-dimensional distribution. Given a k-dimensional distribution Δ on M and a vector field X defined on some open subset U of M, we say that X **belongs** to Δ, and we write $X \in \Delta$, if and only if for all $p \in U$, $X_p \in \Delta(p)$. Δ is said to be differentiable if for every point $p \in M$ we can find an open neighbourhood U of p in M and k vector fields on U, $X_1, ..., X_k$, such that, for every $q \in U$, $\Delta(q)$ is spanned by $X_{1|q}, ..., X_{k|q}$. An **integral manifold** of Δ through a point p of M is a submanifold S_p of M such that $p \in S_p$ and such that for every point $q \in S_p$ the tangent space to S_p at q is exactly $\Delta(q)$. A distribution is said to be **integrable** if and only if for every point $p \in M$ there exists an integral manifold of Δ through p. The classical theorem of Frobenius characterizes integrable distributions [7]:

THEOREM 1.3.4. *A differentiable distribution Δ on a manifold M is integrable if and only if for every pair of vector fields X, Y belonging to Δ, $[X, Y]$ also belongs to Δ.*

Given an integrable distribution Δ on M and given $m \in M$, we say that a submanifold S of M is a **maximal integral** submanifold of Δ through m if S is an integral submanifold of Δ through m and if given any other submanifold S' of M with the same properties, then $S' \subset S$.

Given an integrable distribution Δ on M and any point $m \in M$ there exists a unique maximal integral submanifold of Δ through m [7]. In general the maximal integral submanifolds of an integrable distribution are not regular submanifolds of M - the classical example being the Denjoy curves of the torus T^2 [4]. However, given any point $m \in M$, one can always find an open neighbourhood U of m in M such that the maximal integral submanifold of the restriction of Δ to U through m is a regular submanifold of M [7].

In this setting, the following result will also be usefull [7]:

THEOREM 1.3.5. *Let Δ be a k-dimensional integrable distribution on a manifold M of dimension n; let U be an open subset of M and $X_1, ..., X_k$ be vector fields on U such that $X_i \in \Delta$ for $q \leq i \leq k$ and $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Then, for every $p \in U$ there exists a coordinate chart (V, ϕ) of M around p such that $X_i = \partial_i$ in V, for $1 \leq i \leq k$.*

1.4. Tensor fields; differential forms; the Lie derivative.

Let M be a n-dimensional manifold. A **differential form of degree one** on M (we shall say, for short, **1-form**) is a $\mathcal{F}(M)$-linear map:

$$w : \mathcal{D}_1(M) \to \mathcal{F}(M).$$

As for tangent vectors and vector fields, the notion of 1-form is local in character, in the sense that if two vector fields X, Y coincide in some open subset U of M

then, for every $p \in U$, $w(X)(p) = w(Y)(p)$ [50]. This allows us to define, as for vector fields, the restriction of a 1-form to an open subset U of M.

Given a 1-form w on M and $p \in M$, we can define a linear map:

$$w_p : T_p(M) \to \mathbb{R},$$

as follows; if $v \in T_p(M)$, there exists a vector field X on M such that $X_p = v$; define

$$w_p(v) = w(X)(p);$$

(It can be proved that the above definition does not depend on the vector field X, only on its value at p). This shows that at any given point p of M a 1-form determines a unique element of the space ${T^*}_p(M)$ dual to $T_p(M)$.

Given a chart (U, ϕ) of M (with $\phi(p) = (x^1, ..., x^n)$), the maps:

$$dx^i : \mathcal{D}_1(U) \to \mathcal{F}(U),$$

defined for $X = X^i \partial_i$, by:

$$dx^i(X) = X^i,$$

are 1-forms over U; for every 1-form w on U there exist unique $w_1, ..., w_n \in \mathcal{F}(U)$ such that:

$$w_{|U} = w_i dx^i. \tag{1.4.1}$$

This expression is called the **local expression** of w in the chart (U, ϕ).

A particular example of 1-form is the **absolute differential** of an element of $\mathcal{F}(M)$; given $f \in \mathcal{F}(M)$, this 1-form is denoted by df and is defined as being the unique 1-form w such that for every $X \in \mathcal{D}_1(M)$:

$$w(X) = X(f). \tag{1.4.2}$$

In the domain of a chart (U, ϕ), it is given by:

$$df = f_{,i} dx^i. \tag{1.4.3}$$

As follows from the definition, the set of 1-forms, denoted $\mathcal{D}^1(M)$, is the $\mathcal{F}(M)$-module dual to $\mathcal{D}_1(M)$.

Let now (r, s) be integers; a **tensor field of type** (r, s) is an element of the tensor product of r copies of $\mathcal{D}_1(M)$ and s copies of $\mathcal{D}^1(M)$, that is, a multilinear map on the product of r copies of $\mathcal{D}^1(M)$ with s copies of $\mathcal{D}_1(M)$ with values in

$\mathcal{F}(M)$. This is, again, a notion local in character, and so, very much in the same way as for vector fields and 1-forms, we can define the restriction of a tensor field of type (r, s) to an open subset of M [50].

In particular, if (U, ϕ) is a chart of M (with $\phi(p) = (x^1, ..., x^n)$), for every choice of $a_1, ..., a_s$, $b_1, ..., b_r$ in $\{1, ..., n\}$ we define a tensor field of type (r, s) on U,

$$E_{a_1...a_r}{}^{b_1...b_s} = \partial_{a_1} \otimes ... \otimes \partial_{a_r} \otimes dx^{b_1} \otimes ... \otimes dx^{b_s},$$

by requiring it to be the unique $\mathcal{F}(M)$-linear map over the product of r copies of $\mathcal{D}_1(U)$ with s copies of $\mathcal{D}_1(U)$ such that, for any choice $c_1, ..., c_s$, $d_1, ..., d_r$ in $\{1, ..., n\}$:

$$E_{a_1...a_r}{}^{b_1...b_s}(dx^{c_1}, ..., dx^{c_r}, \partial_{d_1}, ..., \partial_{d_s}) = \delta_{a_1}{}^{c_1} ... \delta_{a_r}{}^{c_r} \delta^{b_1}{}_{d_1} ... \delta^{b_s}{}_{d_s},$$

where the $\delta_i{}^j$ are Kronecker deltas.

If T is tensor field of type (r, s) on M, one can then prove [50] that there exist unique functions $T^{a_1...a_r}{}_{b_1...b_s} \in \mathcal{F}(U)$ such that the restriction of T to U is given by:

$$T_{|U} = T^{a_1...a_r}{}_{b_1...b_s} E_{a_1...a_r}{}^{b_1...b_s}. \tag{1.4.4}$$

This expression is called the **local expression** of T in the chart (U, ϕ).

The set of tensor fields of type (r, s) over M is denoted by $\mathcal{T}_s^r(M)$; it is a module over the ring $\mathcal{F}(M)$ and a real vector space; obviously, with this definition, vector fields are just tensor fields of type $(1, 0)$, and 1-forms are tensor fields of type $(0, 1)$. We define as tensor fields of type $(0, 0)$ the elements of $\mathcal{F}(M)$. We can then construct the graded algebra:

$$\mathcal{T}(M) = \bigoplus_{r,s \geq 0} \mathcal{T}_s^r(M),$$

which is called the **tensor algebra** of M.

It is well known that the tensor product of modules over the same ring is associative and commutative [60]; using this fact we identify for all pairs (r, s), (r', s') of integers the tensor product $\mathcal{T}_s^r(M) \otimes \mathcal{T}_{s'}^{r'}(M)$ with $\mathcal{T}_{(s+s')}^{(r+r')}(M)$.

A **metric** over a manifold M is a tensor field g of type $(0, 2)$ on M which is symmetric (i.e. $g(X, Y) = g(Y, X)$ for all $X, Y \in \mathcal{D}_1(M)$) and such that for every point $m \in M$, the tensor g_m on $T_m(M)$ is nondegenerate and its signature does not depend on m.

A **pseudo-riemannian** manifold is a manifold M equiped with a metric g; we denote it by (M, g). We say that (M, g) is a **riemannian** manifold if g_m is positive definite ($m \in M$); we say that (M, g) is a **lorentzian** manifold if g_m ($m \in M$) has signature $(n-1, 1)$ (n being the dimension of M).

It is well known that, under the topological conditions imposed at the beginning of this chapter, for every manifold M there exists a metric g on M such that (M, g) is a riemannian manifold [49]; again under those topological conditions, it is well known that a necessary and sufficient condition for a structure of lorentzian manifold to exist on M is that it admits a globally defined 1-dimensional differentiable distribution [49].

If (M, g) is a pseudo-riemannian manifold, for every $m \in M$ the tensor g_m can be regarded as a "canonical" isomorphism from $T_m(M)$ onto $T^*_m(M)$ (cf. §1.1). The operations of raising and lowering of indices described in §1.1 as well as the classifications of subspaces of $T_m(M)$ described in that section can therefore be carried to each tangent space of M. Given any tensor field T on a chart of M we can at every point $m \in U$ perform the above operations (if they have any sense); the result is of course a new tensor field on U; notice that if T is of type (r, s) then the tensor one obtains from T by raising (resp. lowering) an index is of type $(r+1, s-1)$ (resp. $(r-1, s+1)$).

Given a tensor field T of type (r, s), with $r, s \geq 1$, it admits in any coordinate domain the local expression given in (1.4.4); for $1 \leq i \leq r$ and $1 \leq j \leq s$, we define then the **contraction** of indices i, j, which we denote by C^i_j as the map that associates with T the tensor field $C^i_j(T)$ of type $(r-1, s-1)$ given in the same coordinates by:

$$C^i_j(T)^{a_1 \ldots a_{r-1}}{}_{b_1 \ldots b_{s-1}} = T^{a_1 .. a_{i-1} k a_i .. a_{r-1}}{}_{b_1 .. b_{j-1} k b_j .. b_{s-1}}. \tag{1.4.5}$$

NOTE.1.4.1. In the sequel, given a pseudo-riemannian manifold (M, g) and a tensor field T on M, whenever no confusion arises, we shall denote by the same symbol T all the tensor fields one obtains from T by raising, lowering or contracting indices.

Let $T \in \mathcal{T}^r_0(M)$; we define then two new tensor fields on M, denoted $\mathcal{S}T$ and $\mathcal{A}T$, and called, respectively, the **symmetrized** and **antisymmetrized** of T, as follows; let $\mathcal{P}_r$ be the group of permutations of $\{1,...,r\}$ and for $\sigma \in \mathcal{P}_r$ let $\epsilon(\sigma)$ be its signature; then the tensors $\mathcal{S}T$ and $\mathcal{A}T$ are given for all $X_1,...,X_r \in \mathcal{D}_1(M)$ by :

$$\mathcal{S}T(X_1,...,X_r) = \frac{1}{r!}\sum_{\sigma\in\mathcal{P}_r} T(X_{\sigma^{-1}(1)},....,X_{\sigma^{-1}(r)}); \tag{1.4.6}$$

and

$$\mathcal{A}T(X_1,...,X_r) = \frac{1}{r!}\sum_{\sigma\in\mathcal{P}_r} \epsilon(\sigma)T(X_{\sigma^{-1}(1)},....,X_{\sigma^{-1}(r)}). \tag{1.4.7}$$

It is obvious how to define the symmetrisation and antisymmetrisation processes described above but operating only on some of the variables.

As an example take $r = 2$ and choose a coordinate chart (U,ϕ) of M so that :

$$T = T_{ij}dx^i \otimes dx^j.$$

The components of the tensors $\mathcal{S}T$ and $\mathcal{A}T$ in the above chart are then given by:

$$(\mathcal{S}T)_{ij} = \frac{1}{2}(T_{ij} + T_{ji});$$

and

$$(\mathcal{A}T)_{ij} = \frac{1}{2}(T_{ij} - T_{ji}).$$

When using coordinate notation we will denote the components of the symmetrised and antisymmetrised tensor fields of a given tensor field T of type $(0,r)$ by:

$$(\mathcal{S}T)_{a_1...a_r} = T_{(a_1...a_r)}; \tag{1.4.8}$$

and

$$(\mathcal{A}T)_{a_1...a_r} = T_{[a_1...a_r]}. \tag{1.4.9}$$

The above mentioned "partial" symmetrisation or antisymmetrization processes give then tensor fields :

$$T_{a_1...(a_{i_1}...a_{i_2}|a_{i_2+1}...a_{i_{(k-1)}}|a_{i_{(k-1)}+1}...a_{i_k})a_{i_k+1}...a_r};$$

and

$$T_{a_1...[a_{i_1}...a_{1_2}|a_{i_2+1}...a_{i_{(k-1)}}|a_{i_{(k-1)}+1}...a_{i_k}]a_{i_k+1}...a_r};$$

where the symmetrisation and antisymmetrisation have been performed only over the indices $a_{i_1}..a_{i_2}a_{i_3}..a_{i_4}....a_{i_{k-1}}..a_{i_k}$.

A tensor field T of of type $(0,r)$ is said to be **symmetric** if $\mathcal{S}T = T$.

An element T of $\mathcal{T}_0^p(M)$ is called a p-**form** (or **differential form of degree** p) if it is alternate, that is, if $\mathcal{A}T = T$.

The set of p-forms over M is denoted by $\mathcal{D}^p(M)$; it is a module over $\mathcal{F}(M)$ and a real vector space. We define in the usual way the **exterior product** $w \wedge v$, of two forms w, v [55]. If w is a s-form and v a p-form, $w \wedge v$ is a $(s+p)$-form and one has the relation:

$$v \wedge w = (-1)^{ps} w \wedge v. \tag{1.4.10}$$

In a coordinate domain (U, ϕ), the p-forms :

$$dx^{a_1} \wedge ... \wedge dx^{a_p}, \tag{1.4.11}$$

where $1 \leq a_1 < a_2 < .. < a_p \leq p$ define a local basis for $\mathcal{D}^p(M)$, in the sense that any p-form on M, when restricted to U, is a linear combination of the above p-forms with coefficients in $\mathcal{F}(U)$.

NOTE.1.4.2. We consider the elements of $\mathcal{F}(M)$ as 0-forms on M; for $f \in \mathcal{F}(M)$ and a p-form w, we define:

$$f \wedge w = f.w.$$

Given a p-form w, we can define a $(p+1)$-form on M, denoted dw and called the **exterior differential** of w, as follows [50]; For all $X_1, ..., X_{p+1} \in \mathcal{D}_1(M)$, dw is given by:

$$dw(X_1, ..., X_{p+1}) = \frac{1}{(p+1)} \sum_{1 \leq i \leq (p+1)} (-1)^{i+1} X_i(w(X_1, ..., \hat{X}_i, ..., X_{p+1}))$$

$$+\frac{1}{(p+1)} \sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1}), \tag{1.4.12}$$

where the symbol $\hat{}$ indicates that the vector field under it has been removed from the expression. One has then [50]:

THEOREM 1.4.1. *The operator d thus defined is the unique $\mathbb{R}$-linear operator over p-forms with the following properties:*

(a). If U is an open subset of M, and if $d_{|U}$ denotes the operator of exterior differentiation on U, then for every p-form w in M : $d_{|U}(w_{|U}) = (dw)_{|U}$;

(b). For every $f \in \mathcal{F}(M)$, df is the absolute differential of f ;

(c). $dod = 0$;

(d). If w is an r-form and v is an s-form : $d(w \wedge v) = (dw) \wedge v + (-1)^r w \wedge (dv)$.

A p-form w on M is said to be **closed** if $dw = 0$; it is said to be **exact** if there exists a $(p-1)$-form v on M such that $dv = w$. Obviously, as follows from (c), every exact form is closed. Conversely we have the [58]:

THEOREM 1.4.2.(Poincaré lemma). *Let U be the open unit ball of $\mathbf{R}^n$; then every closed form on U is exact.*

Let (M, g) be a pseudo-riemannian manifold . A vector field X on M is said to be **hypersurface orthogonal** if the 1-form $g(X, .)$ is parallel to a closed 1-form.

Let now M, N be manifolds and $\Phi : M \to N$ be a differentiable map. Let w be a p-form on N.

Take $X_1, ..., X_p \in \mathcal{D}_1(M)$; for any given $p \in M$, $z_i = \Phi_{*p}.X_{i|p}$ is an element of $T_{\Phi(p)}(N)$. We can therefore compute:

$$w_{\Phi(p)}(z_1, ..., z_p).$$

We define the **pullback** of w by Φ , and we denote it by $\Phi^* w$, as being the p-form on M given by:

$$(\Phi^* w)_q(X_{1|q}, ..., X_{p|q}) = w_{\Phi(q)}(\Phi_{*q} X_{1|q}, ..., \Phi_{*q} X_{p|q}).$$

If Φ is a diffeomorphism, we can define the direct image of a p-form on M by taking its pullback with Φ^{-1}.

NOTE.1.4.3. It follows immediately from the above definitions that if w is a p-form on N and $\Phi : M \to N$ is differentiable , then :

$$d(\Phi^* w) = \Phi^*(dw). \qquad (1.4.13)$$

A consequence of this and the Poincaré lemma is that, if M is a manifold diffeomorphic to the unit ball of $\mathbf{R}^n$ (for some n) then every closed form on M is exact.

Obviously, if T is any tensor field of type $(0, s)$ on M, we can in the same way define its pullback by any differentiable map $\Phi : N \to M$, where N is some other manifold.

Let then $(M,g),(M',g')$ be pseudo-riemannian manifolds and let $\Phi : M \to N$ be a differentiable map. We say that Φ is an **isometry** of M onto N if it is a diffeomorphism of M onto N and, furthermore : $\Phi_{*}g' = g$. If there exists an isometry from (M,g) onto (M',g') we say that these pseudo-riemannian manifolds are **isometric.**

Let now w be a 1-form on M, X be a vector field on M and let p be a point in M. Each ϕ_t, $t \in I(p)$, (cf. §1.3), is then a local diffeomorphism at p, hence we can compute the direct image of w under each one of the ϕ_t. We define then the **Lie derivative** of w along X as being the 1-form, denoted $\mathcal{L}_X w$, given at p by:

$$(\mathcal{L}_X w)_p = -\lim_{t\to 0}\frac{1}{t}[((\phi_t)_{*}w)_p - w_p]. \tag{1.4.14}$$

Since the maps ϕ_t are local diffeomorphisms, defining for a vector field Y (resp. a function f on M) the Lie derivative along X as being $\mathcal{L}_X Y = [X,Y]$ (resp. $\mathcal{L}_X f = X(f)$), this definition can be extended to any tensor field of type (r,s). One has then [50]:

THEOREM 1.4.3. *The Lie derivative has the following properties:*

(a). $\mathcal{L}_X$ is **R**-*linear;*

(b). For all tensor fields T,S: $\mathcal{L}_X(S\otimes T) = (\mathcal{L}_X S)\otimes T + S\otimes(\mathcal{L}_X T)$;

(c). For every tensor field T, $\mathcal{L}_X T$ is of the same type as T;

(d). $\mathcal{L}_X$ commutes with the contraction operators;

(e). If w is a p-form on N and $\Phi : M \to M$ is differentiable: $\mathcal{L}_X\Phi^*(w) = \Phi^*(\mathcal{L}_X w)$;

(f). For all vector fields X,Y on M: $\mathcal{L}_{[X,Y]} = \mathcal{L}_X o \mathcal{L}_Y - \mathcal{L}_Y o \mathcal{L}_X$

If (U,ϕ) is a coordinate chart of M, one easily finds using (b), (c) and (d) that, if $X = X^i\partial_i$ is a vector field on U, then :

$$\mathcal{L}_X dx^j = X^j{}_{,i}dx^i, \tag{1.4.15}$$

from where it follows that if $w = w_j dx^j$ is a 1-form on U :

$$\mathcal{L}_X w = (X^i w_{k,i} + w_i X^i{}_{,k})dx^k. \tag{1.4.16}$$

Let now T be a tensor field of type (r,s) on a manifold M; we say that T is **conformally invariant** under a vector field X if and only if there exists a function $\phi \in \mathcal{F}(M)$ such that:

$$\mathcal{L}_X T = \phi T. \tag{1.4.17}$$

We say that T is **homothetically invariant** (resp. **invariant**) under X if it is conformally invariant under X and the function ϕ is a real constant (resp. the zero function).

It follows immediately from T.1.4.3.(f) that:

THEOREM 1.4.4. *For every tensor field T on M, the set of vector fields on M under which T is conformally invariant (resp. homothetically invariant, resp. invariant) is a Lie-subalgebra of $\mathcal{D}_1(M)$.*

1.5. Lie Groups; Lie groups acting on a manifold; fibre bundles.

A **Lie group** G is a group (whose composition law we denote multiplicatively and whose unity we denote by e) which is simultaneously a manifold and such that the map from the product manifold $G \times G$ to G, given by $(a, b) \mapsto a.b^{-1}$, is differentiable.

Given a Lie group G and $a \in G$, the map $L_a : G \to G$ (resp. $R_a : G \to G$) given by $L_a(b) = ab$ (resp. $R_a(b) = ba$) for all $b \in G$ is a transformation of G. We call it the **left translation** (resp.**right translation**) associated with a.

A vector field X on a Lie group G is said to be **left** (resp. **right**) **invariant** if and only if for every $a \in G$:

$$(L_a)_* X = X, \tag{1.5.1}$$

$$(\text{resp.} (R_a)_* X = X.) \tag{1.5.2}$$

It is well known [7] that a left invariant vector field of a Lie group G is a complete vector field of G and that it is entirely determined by its value at the unity element of G. Conversely, every vector tangent to G at e defines in a unique way a left invariant vector field on G [7]. The same results also hold for right invariant vector fields. Furthermore, as follows immediately from the remarks made previously concerning Φ-related vector fields (cf. p.12), the set of left invariant vector fields of a Lie group G is a Lie algebra, which we denote by g. The above comments show that this Lie algebra is finite dimensional and of the same dimension as G. The same holds for the set of right invariant vector fields, and in fact one can show that these two Lie algebras are naturally isomorphic [7]. The Lie algebra of left invariant vector fields is called the **Lie algebra of** G.

Given Lie groups G, H, a **homomorphism** of G into H is a group homomorphism that is differentiable for the underlying manifold structures. We define the concept of **isomorphism** of Lie groups in the obvious way. A **Lie subgroup** H of a Lie group G is a Lie group which is also a submanifold of G and such that the canonical injection is a Lie group homomorphism. Obviously, in such case $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, we have [17]:

THEOREM 1.5.1. *Let G be a Lie group and let $\mathfrak{a}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a unique connected Lie subgroup H of G such that : $\mathfrak{h} = \mathfrak{a}$.*

If two Lie groups G, H are isomorphic it is clear (again by the properties of Φ-related vector fields) that their Lie algebras are also isomorphic. (A homomorphism of Lie algebras is just a linear map which preserves the bracket). An obvious consequence of this is the following [17]:

THEOREM 1.5.2. *Given Lie groups G_1, G_2 let $G = G_1 \times G_2$. Then G is, for the product structure, a Lie group and its Lie algebra is isomorphic to the Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$.*

The converse of this result is in general false (see [17] for some examples). However, it holds locally in the following sense; given Lie groups G and H define a **local homomorphism** of G into H as being a map f from an open neighbourhood U of the unity of G to an open neighbourhood V of the unity of H, which is differentiable and such that, whenever $a, b \in U$ are such that $a.b \in U$, then $f(a.b) = f(a).f(b)$. We say then that f is a **local isomorphism** if, furthermore, f is a homemorphism of U onto V, in which case one can prove that h^{-1} is also a local homomorphism [16]. We have then [17]:

THEOREM 1.5.3. *If the Lie algebras of two given Lie groups are isomorphic, the Lie groups are locally isomorphic. If, furthermore, both Lie groups are connected and simply connected then they are isomorphic.*

It can also be shown that [18]:

THEOREM 1.5.4. *Given a real Lie-algebra $\mathfrak{h}$ of finite dimension, there exists a Lie group G such that $\mathfrak{g} = \mathfrak{h}$.*

An **action** on the right of a Lie group G on a manifold M is an action Φ of G on M in the sense of §1.1 such that Φ is differentiable when considered as a map from the product manifold $M \times G$ to M. Let then Φ be an action on the right of a Lie group G on a manifold M. For every $p \in M$ the map $\Phi_p : G \to M$, $a \mapsto \Phi(p,a)$ is differentiable . Given then an element of $T_e(G)$, we can define a vector field v^* on M by setting, for all $p \in M$:

$$v_p^* = (\Phi_p)_{*e} v. \tag{1.5.5}$$

The vector field v^* is called a **fundamental** vector field. The set of fundamental vector fields is denoted [7] by $\mathcal{R}(G,M)$; it is a Lie algebra naturally homomorphic to the Lie algebra g' of right invariant vector fields of G [7] (in fact this two Lie algebras are even isomorphic, provided that Φ is effective, [7]). The completeness of the right invariant vector fields of G implies then the following [7]:

THEOREM 1.5.5. *$\mathcal{R}(G,M)$ is a Lie algebra of complete vector fields on M.*

The converse of this result is due to Palais [66]:

THEOREM 1.5.6. *Let $\mathcal{A}$ be a finite dimensional Lie algebra of complete vector fields on a manifold M. Then exists there a connected Lie group G which acts effectively on M and such that $\mathcal{A} = \mathcal{R}(G,M)$.*

Related with these notions is the concept of **principal fibre bundle** that we now define. Let M be a manifold, G a Lie group; a principal fibre bundle $P(M,G)$ over M with structure group G consists of a manifold P, together with an action on the right of G on P:

$$\Phi : P \times G \to P,$$

satisfying the following conditions:

(F1). M is the quotient space of P resulting from the equivalence relation " $p \sim q$ if and only if $q \in \mathcal{O}_p$ ", and the natural projection $\pi : P \to M$ is differentiable;

(F2). P is locally trivial in the sense that for every $m \in M$ we can find an open neighbourhood U of m in M such that there exists a diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times G,$$

such that $\psi(u) = (\pi(u), \phi(u))$, where $\phi : \pi^{-1}(U) \to G$ satisfies : $\phi(\Phi(u,a)) = \phi(u).a$, for all $a \in G$ and all $u \in \pi^{-1}(U)$.

P is then called the **total space**, M the **base space**, G the **structure group** and π the **projection**. For each $m \in M$, $\pi^{-1}(m)$ is a closed submanifold of P, called the **fibre** over m; it is diffeomorphic to G. Given $A \in$ g, we denote again by A^* the fundamental vector field on P associated with A.

The fundamental example of a fibre bundle, needed in this work, is the **frame bundle** of a manifold which we now define.

Given a manifold M and $m \in M$, a **linear frame** of M at m is an ordered basis of the tangent space to M at m. We denote by $L(M)$ the disjoint reunion of all linear frames of M at all points of m. It can be shown [55] that the manifold structure of M naturally induces a manifold structure on $L(M)$. Take now a linear frame $(X_1, .., X_n)$ at some point $m \in M$, and let $G = GL(n)$, the linear group of $\mathbf{R}^n$. Identifying each element of G to its matrix in the canonical basis of $\mathbf{R}^n$, consider $(a^i_j) \in G$; define then $\Phi((X_1, ..., X_n), (a^i_j))$ as being the linear frame $(Y_1, ..., Y_n)$ at m given by :

$$Y_i = a^j_i X_j.$$

It can be shown [55] that the map Φ thus defined is an action of G on $L(M)$ on the right and that it satisfies the fibre bundle axioms (F1) and (F2). With this fibre bundle structure we call $L(M)$ the **frame bundle** of M.

If $P(M, G)$ is a principal fibre bundle and F is a manifold on which G acts on the left by $\psi : G \times F \to F$, we can define an action Θ of G on $P \times F$ on the right by $\Theta((u, \zeta), a) = (\Phi(u, a), \psi(a^{-1}, \zeta))$. Denoting by E the quotient space of $P \times G$ by this action, one can prove that there exists a manifold structure on E for which the mapping $\pi_E : E \to M$, $(u, \zeta) \mapsto \pi(u)$ is differentiable [55]. $E = E(M, F, G, P)$ is then called a **bundle** over the base M with fibre F and structure group G associated with the principal fibre bundle $P(M, G)$.

If one denotes by $T^r_s(m)$ the vector space of tensors of type (r, s) over the space $T_m(M)$, where $m \in M$ and M is manifold, and by $T^r_s(M)$ the disjoint reunion of all these spaces, for all $m \in M$, one gets examples of bundles associated with the frame bundle of M [55].

Given any such bundle E, a **section** of E is a mapping $\sigma : E \to M$ such that $\pi_E o \sigma$ is the identity mapping of M. Every tensor field of type (r, s) is a section of the bundle $T^r_s(M)$ [55].

1.6.Linear connections; fundamental tensors; the Levi-Civita connection.

Let $P(M,G)$ be a principal fibre bundle; for $u \in P$ denote by G_u the subspace of $T_u(P)$ tangent to the fibre through u.

A **connection** Γ on P is a differentiable distribution Q on P such that, writing $Q_u = Q(u)$ for $u \in P$:

(C1). For all $u \in P$ $T_u(P) = G_u \oplus Q_u$;

(C2). For all $u \in P$ and all $a \in G$: $Q_{\Phi(u,a)} = \Phi_{*a} Q_u$.

We say then that G_u (resp. Q_u) is the **vertical** (resp. **horizontal**) subspace of $T_u(P)$. Given a vector field X on P it follows from (C1) and the differentiability of Q that there exist unique vector fields vX and hX on P such that $X = vX \oplus hX$ and for every $u \in P$: $(vX)_u \in G_u$ and $(hX)_u \in Q_u$; we call vX (resp. hX) the **vertical** (resp. **horizontal**) component of X. Obviously, for every $A \in \mathrm{g}$, A^* is a vertical vector field (i.e. $hA^* = 0$).

Given a connection Γ on P we can define a 1-form on P, with values in g as follows; X being a vector field on P, we define $w(X)$ as being the unique $A \in \mathrm{g}$ such that $A^*_u = (vX)_u$, for all $u \in P$. The 1-form w thus defined is called the **connection form** of Γ. Obviously, $w(X) = 0$ if and only if X is horizontal (i.e. $vX = 0$).

Let $\gamma : I \to M$ be a curve; a **horizontal lift** of γ to P is a curve $\gamma^* : I \to P$ such that: $\pi \circ \gamma^* = \gamma$. One can prove, [55], that if γ is piecewise differentiable (in the sense that there exists a finite sequence $a_1 < ... < a_k$ of points of I such that γ is differentiable at all other points of I) then for every $s,t \in I$, with $s \leq t$, and every $u \in \pi^{-1}(\gamma(s))$ there exists a unique curve $\tau_u : [s,t] \to P$ which is a horizontal lift of γ and is such that $\tau_u(s) = u$. This allows us to define a map $\tau^\gamma_{st} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$ by $\tau^\gamma_{st}(u) = \tau_u(t)$. This map is called the **parallel transport** from $\gamma(s)$ to $\gamma(t)$ along γ. It can be shown that it defines an isomorphism between the fibres at $\gamma(s)$ and $\gamma(t)$ [55]. If we define for a curve $\gamma : [a,b] \to M$ the curve $\hat{\gamma} : [a,b] \to M$, $t \mapsto \gamma(a+b-t)$, and if we define for curves $\gamma, \delta : [a,b] \to M$ such that $\gamma(b) = \delta(a)$, the curve $\delta.\gamma : [a,b] \to M$, by $\delta.\gamma(t) = \gamma(2t-a)$ if $a \leq t \leq \frac{(a+b)}{2}$ and $\delta.\gamma(t) = \delta(2t-b)$ if $\frac{(a+b)}{2} \leq t \leq b$, then it can be shown that, setting $\alpha = \hat{\gamma}$ and $\beta = \delta.\gamma$ and $c = \frac{(a+b)}{2}$, then [55]:

$$\tau^\alpha_{st} = (\tau^\gamma_{st})^{-1};$$

and

$$\tau_{ab}^{\beta} = \tau_{cb}^{\delta} o \tau_{ac}^{\gamma}.$$

This notion of parallel transport can be extended to any bundle associated with $P(M,G)$ [55]. Given one such bundle E and a section σ of E, let $m \in M$ and let $v \in T_m(M)$, γ be a curve on M such that $\gamma(t) = m$. We define then the **covariant derivative** $\nabla_v \sigma$ of σ at m in the direction of v by:

$$\nabla_v \sigma = lim \frac{1}{h} \{ \tau_{t(t+h)}^{\gamma}(\sigma(\gamma(t+h))) - \sigma(\gamma(t)) \}. \tag{1.6.1}$$

A **linear connection** on a manifold M is a connection on the frame bundle $L(M)$ of M. From now on, all connections considered are linear connections.

Given one such connection, it follows from the above remarks that we can, for every $m \in M$ and every $v \in T_m(M)$ and every tensor field defined on some open neighbourhood of m in M, define the covariant derivative $\nabla_v T$ of T at m in the direction of v; if X is a vector field we define then the **covariant derivative**, $(\nabla_X T)_m$, of T in the direction of X at $m \in M$ as being

$$(\nabla_X T)_m = \nabla_{X_m} T.$$

It follows then that the mapping $m \mapsto (\nabla_X T)_m$ defines a tensor field of the same type as T. One has, furthermore [55]:

THEOREM 1.6.1. *Let $X, Y \in \mathcal{D}_1(M)$. Then the covariant differentiation has the following properties:*

(a).∇_X is a derivation of the tensor algebra of M (cf. §1.4);

(b).For every $f \in \mathcal{F}(M)$: $\nabla_X f = X(f)$;

(c).$\nabla_{X+Y} = \nabla_X + \nabla_Y$;

(d).$\nabla_{fX} = f\nabla_X$, for all $f \in \mathcal{F}(M)$.

NOTE.1.6.1. Conversely, if over a given manifold M is defined a "rule" ∇ which associates with each $X \in \mathcal{D}_1(M)$ a mapping $\nabla_X : \mathcal{D}(M) \to \mathcal{D}(M)$ with the above properties, then there exists a unique linear connection Γ over M whose covariant derivative is ∇ [55].

Let Γ be a linear connection on a manifold M. For a tensor field K of type (r,s) we define a new tensor ∇K, called the **covariant derivative** of K, as follows; for every $m \in M$, K_m can be considered as a multilinear mapping from $T_m(M) \times .. \times T_m(M)$ (s times) to $T^*_m(M) \times .. \times T^*_m(M)$ (r times); given then $X, X_1, ..., X_s \in \mathcal{D}_1(M)$ we define ∇K by:

$$(\nabla K(X_1, ..., X_s; X))_m = (\nabla_X K)_m(X_1, ..., X_s). \tag{1.6.2}$$

A tensor field K is said to be **parallel** (or **covariantly constant**) if and only if :

$$\nabla K = 0. \tag{1.6.3}$$

Given a linear connection Γ on a manifold M we define the **torsion** tensor $\mathbf{T}$ and the **curvature** (or **Riemann**) tensor $\mathbf{R}$ of Γ as being the tensor fields defined for all $X, Y, Z \in \mathcal{D}_1(M)$ by:

$$\mathbf{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]; \tag{1.6.4}$$

and

$$\mathbf{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.6.5}$$

It is obvious from the definition that for all $X, Y, Z \in \mathcal{D}_1(M)$:

$$\mathbf{R}(Y,X)Z = -\mathbf{R}(X,Y)Z. \tag{1.6.6}$$

A linear connection is said to be **torsion free** if and only if its torsion tensor is zero. From now on all linear connections considered are supposed to be torsion free. For such connections one has [55]:

THEOREM 1.6.2. *If Γ is a (torsion free) linear connection on a manifold M, its curvature tensor has the following properties:*
(a).(Bianchi's 1^{st} identity) For all $X, Y, Z \in \mathcal{D}_1(M)$:

$$\mathbf{R}(X,Y)Z + \mathbf{R}(Z,X)Y + \mathbf{R}(Y,Z)X = 0.$$

(b).(Bianchi's 2^{nd} identity) For all $X, Y, Z \in \mathcal{D}_1(M)$:

$$(\nabla_X \mathbf{R})(Y,Z) + (\nabla_Z \mathbf{R})(X,Y) + (\nabla_Y \mathbf{R})(Z,X) = 0.$$

Let now (U,ϕ) be a chart of M (with $\phi(m) = (x^1, ..., x^n)$) and set, for $1 \leq i \leq n$, $X_i = \partial_i$. For every $1 \leq i,j \leq n$ $\nabla_{X_i}X_j$ is a vector field on U, and so, there exist unique functions $\Gamma^k_{ij} \in \mathcal{F}(U)$ such that:

$$\nabla_{X_i}X_j = \Gamma^k_{ij}X_k. \tag{1.6.7}$$

The functions thus defined are called the **Christoffel symbols** of the connection Γ with respect to the chart (U,ϕ).

NOTE 1.6.2. It should be noticed that these functions are not the components of a tensor field on M, as it is readily shown from the relationship between the Christoffel symbols relative to two charts whose domains overlap (see, for instance [55]).

If V is a vector field in U, $V = V^iX_i$, then, using the properties of the covariant derivative (cf. T.1.6.2), one finds that, in U:

$$\nabla_{X_i}V = (V^k_{,i} + V^j\Gamma^k_{ij})X_k. \tag{1.6.8}$$

If $w = w_i dx^i$ is a 1-form on U we find :

$$\nabla_{X_i}w = (w_{i,j} - w_k\Gamma^k_{ij})dx^j. \tag{1.6.9}$$

More generally, if $K = T^{a_1...a_r}{}_{b_a...b_s}E_{a_1...a_r}{}^{b_1...b_s}$ is a tensor field on U, one finds that:

$$\nabla_{X_i}K = \{K^{a_1...a_r}{}_{b_1...b_s,k} + K^{ia_2...a_r}{}_{b_1...b_s}\Gamma^{a_1}_{ki} + ... + K^{a_1...a_{r-1}i}{}_{b_1...b_s}\Gamma^{a_r}_{ki} \tag{1.6.10}$$

$$- K^{a_1...a_r}{}_{jb_2...b_s}\Gamma^j_{kb_1} - ...... - K^{a_1...a_r}{}_{b_1...b_{s-1}j}\Gamma^j_{kb_s}\}E_{a_1...a_r}{}^{b_1...b_sk}$$

For every tensor field K of type (r,s) on M we define $K^{a_1...a_r}{}_{b_1...b_s;k}$ as being the expression which appears between curly brackets in (1.6.10). It is obvious from the definition that these are precisely the components of the tensor field ∇K in the above chart. With this notation convention we have therefore:

$$\nabla K = K^{a_1...a_r}{}_{b_1...b_s;k}\ E_{a_1...a_r}{}^{b_1...b_sk}, \tag{1.6.11}$$

and

$$\nabla_V K = [V^k K^{a_1 \dots a_r}{}_{b_1 \dots b_s;k}] E_{a_1 \dots a_r}{}^{b_1 \dots b_s}. \tag{1.6.12}$$

Define now $\mathbf{T}(X_i, X_j) = \mathbf{T}^k{}_{ij} X_k$ and $\mathbf{R}(X_k, X_l)X_j = \mathbf{R}^i{}_{jkl} X_i$ (notice the change in the respective positions of j, k, l); one has then [55]:

THEOREM 1.6.3. *The components of the torsion and curvature tensors are given in the chart (U, ϕ) by:*
(a). $\mathbf{T}^i{}_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$;
(b). $\mathbf{R}^i{}_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}$.

Hence, Γ is torsion free if and only if the Christoffel symbols are symmetirc in the lower indices. The following result is a consequence of the definition of the Riemann tensor [55]:

THEOREM 1.6.4. (Ricci Identity) *For every vector field Z on U:*

$$Z^i{}_{;k;l} - Z^i{}_{;l;k} = \mathbf{R}^i{}_{jkl} Z^j.$$

NOTE.1.6.3. More generally, if S is a tensor field of type (r, s), one has the Ricci identity [58]:

$$S^{a_1 \dots a_r}{}_{b_1 \dots b_s;cd} - S^{a_1 \dots a_r}{}_{b_1 \dots b_s;dc} = -S^{a_1 \dots a_r}{}_{b_1 .. b_{k-1} i b_{k+1} .. b_s} \mathbf{R}^i{}_{b_k dc} + S^{a_1 .. a_{l-1} j a_{l+1} .. a_r}{}_{b_1 \dots b_s} \mathbf{R}^{a_l}{}_{jdc}$$

NOTE.1.6.4. We shall write $Z^i{}_{;kl}$ for $Z^i{}_{;k;l}$ (and we extend this convention to all tensor fields).

As for the Bianchi identities (cf. T.1.6.2) they are in the chart (U, ϕ):

$$\mathbf{R}^i{}_{jkl} + \mathbf{R}^i{}_{ljk} + \mathbf{R}^i{}_{klj} = 0; \tag{1.6.13}$$

and

$$\mathbf{R}^i{}_{jkl;m} + \mathbf{R}^i{}_{jmk;l} + \mathbf{R}^i{}_{jlm;k} = 0 \tag{1.6.14}$$

or, using the notation convention for the antisymmetrisation process described in §1.4:

$$\mathbf{R}^i{}_{[jkl]} = 0,$$

and

$$\mathbf{R}^{i}{}_{j[kl;m]} = 0.$$

Relation (1.6.6) reads, of course:

$$\mathbf{R}^{i}{}_{jlk} = -\mathbf{R}^{i}{}_{jkl}. \tag{1.6.15}$$

If Γ is a (torsion free) linear connection on a manifold M, we define its **Ricci** tensor as the tensor of type $(0,2)$ (still denoted $\mathbf{R}$, cf.NOTE.1.4.1)on M defined locally by:

$$\mathbf{R}_{ij} = \mathbf{R}^{k}{}_{ikj}. \tag{1.6.16}$$

It is well known that [55]:

THEOREM .1.6.5. *Given a pseudo-riemannian manifold (M,g) there exists over M a unique torsion free linear connection Γ such that:*

$$\nabla g = 0.$$

This connection is called the **Levi-Civita connection** associated with g. If (U,ϕ) is a chart of M, the Christoffel symbols of this connection are given in U by [55]:

$$\Gamma^{k}_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}). \tag{1.6.17}$$

Using the metric to raise and lower indices (cf. §1.4) one proves then that [49]:

$$\mathbf{R}_{ijkl} = \mathbf{R}_{klij}. \tag{1.6.18}$$

An immediate consequence of this relation is that for the Levi-Civita connection the Ricci tensor is symmetric [49].

The tensor $\mathbf{R}$ given by:

$$\mathbf{R} = \mathbf{R}^{i}{}_{i}, \tag{1.6.19}$$

is called the **curvature scalar** of (M,g).

Relations (1.6.13), (1.6.15) and (1.6.18) can be used to show that for a pseudo-riemannian manifold of dimension n the Riemann tensor has at most $\frac{n^2(n^2-1)}{12}$ algebraically independent components; of these, $\frac{n(n+1)}{2}$ are determined by the Ricci tensor; it follows that if $n \leq 3$ the Ricci tensor completely determines the

Riemann tensor [49]. For $n > 3$ the remaining components of the Riemann tensor can be described by the **Weyl** tensor $\mathbf{C}$ whose components are locally given by [49]:

$$\mathbf{C}_{ijkl} = \mathbf{R}_{ijkl} + \frac{2}{(n-2)}\{g_{i[l}\mathbf{R}_{k]j} + g_{j[k}\mathbf{R}_{l]i}\} + \frac{2}{(n-1)(n-2)}\mathrm{R}g_{i[k}g_{l]j}. \quad (1.6.20)$$

It follows immediately from this relation that the Weyl tensor satisfies relations (1.6.13), (1.6.15) and (1.6.18); furthermore it is trace free, in the sense that:

$$\mathbf{C}^k{}_{ikj} = 0. \quad (1.6.21)$$

It is well known that [49] if $g, \hat{g}$ are metrics on a manifold M (of dimension > 2) such that $\hat{g} = \Omega^2 g$ for some nowhere zero $\Omega \in \mathcal{F}(M)$, then the Weyl tensors of g and $\hat{g}$ coincide.

A pseudo-riemannian manifold (M, g) is said to be **flat** if and only if its Riemann tensor is identically zero. In such case, for every point $p \in M$ there exists an open neighbourhood U of p in M such that $(U, g_{|U})$ is isometric to an open subset of $\mathbf{R}^n$ (where n is the dimension of M), together with the metric induced on it by the pseudo-euclidean metric on $\mathbf{R}^n$ of the same signature as g [55]. A pseudo-riemannian manifold (M, g) is said to be **pseudo-euclidean** if it is isometric to some open subset of $\mathbf{R}^n$ ($n = \dim M$) together with the metric induced on it by the pseudo-euclidean metric of the same signature as g. In such case it is well known that the Riemann tensor of (M, g) is identically zero on M. It is a well known result [55] that if (M, g) is a pseudo-riemannian manifold wich is flat, connected, simply connected and geodesically complete then it is isometric to $\mathbf{R}^n$ ($n = \dim M$) together with its pseudo-euclidean metric of the same signature as g.

A pseudo-riemannian manifold (M, g) is said to be **conformally flat** if and only if its Weyl tensor is identically zero.

NOTE.1.6.5. It follows from T.1.6.1 that most of the formulas we have got previously using partial differentiation can also be expressed using covariant differentiation. In particular, we have the following local expressions, where X, Y are vector fields, w is a 1-form and $f \in \mathcal{F}(M)$:

$$[X, Y] = (X^j Y^i{}_{;j} - Y^j X^i{}_{;j})\partial_i; \quad (1.6.22)$$

$$\mathcal{L}_X f = X^i f_{;i};\tag{1.6.23}$$

$$\mathcal{L}_X w = (X^i w_{j;i} + w_i X^i{}_{;j})dx^j;\tag{1.6.24}$$

$$dw = w_{[i;j]}dx^i dx^j.\tag{1.6.25}$$

1.7. Geodesics; normal coordinates.

Let (M, g) be a pseudo-riemannian manifold, Γ be its Levi-Civita connection and ∇ be its covariant differentiation operator. A vector field $X \in \mathcal{D}_1(M)$ is called **geodesic** if and only if $\nabla_X X = \lambda X$ for some $\lambda \in \mathcal{F}(M)$. In such case it can be proved [55] that for every point $p \in M$ there exists an open neighbourhood U of p in M and $f \in \mathcal{F}(U)$ such that $\nabla_{fX} fX = 0$. In local terms, X is geodesic if and only if :

$$X^i X^j{}_{;i} = \lambda X^j,\tag{1.7.1}$$

for some function λ.

Given a curve $\gamma : I \to M$ we say that γ is a **geodesic** of M if and only if $\nabla_{\gamma_{*t}\cdot 1}\gamma_{*t}\cdot 1$ is parallel to $\gamma_{*t}\cdot 1$ for all $t \in I$. If X is a geodesic vector field, every integral curve of X is a geodesic. If γ is a geodesic, we can find a reparametrization of γ such that:

$$\nabla_{\gamma_{*t}\cdot 1}\gamma_{*t}\cdot 1 = 0.\tag{1.7.2}$$

Any parameter for which this relation holds is called an **affine parameter** (for γ); it can be shown [55] that if s, t are affine parameters for γ then $s = at + b$ for some real numbers a, b. The condition for γ to be a geodesic is, in local terms, that there exists a parameter t such that [55]:

$$\frac{d^2\gamma^l}{dt^2} + \frac{d\gamma^i}{dt}\frac{d\gamma^k}{dt}\Gamma^l_{ik} = 0.\tag{1.7.3}$$

This is a system of ordinary differential equations; the smoothness of the functions involved guarantees then that given any $m \in M$ and any $v \in T_m(M)$ there exists a geodesic $\gamma : I \to M$ of M such that $\gamma(0) = m$ and $\gamma_{*0}\cdot 1 = v$. Furthermore, this geodesic can be chosen **maximal**, in the sense that if $\gamma' : I' \to M$ is any other geodesic with the same properties then $I' \subset I$ and $\gamma_{|I'} = \gamma'$. We denote this maximal geodesic by γ_v, and its domain by I_v. Using the properties of the solutions of (1.7.2) it is easy to prove [55] that if $c \in I_v$ then for every $t \in I_{cv}$ we have $ct \in I_v$ and $\gamma_{cv}(t) = \gamma_v(ct)$. If we consider, for $m \in M$, $T_m(M)$ as a manifold (by identification with $\mathbf{R}^n$, ($\dim M = n$)), we have, furthermore [55]:

THEOREM 1.7.1.*Let (M, g) be a pseudo-riemannian manifold, $m \in M$. Then there exits an open neighbourhood N_0 of 0 in $T_m(M)$ such that for every $v \in N_0$ $1 \in I_v$. The map $v \mapsto \gamma_v(1)$ from N_0 to M is a local diffeomorphism at $0 \in T_m(M)$.*

It follows from this result that there exists an open neighbourhood W_0 of $0 \in T_m(M)$ such that the above map restricted to W_0 is a diffeomorphism of W_0 onto its image. The map $v \mapsto \gamma_v(1)$ thus defined is called the **exponential** of (M, g) at m, and it is denoted exp_m.

Clearly $(exp_m(W_0), exp_m^{-1})$ is a chart of M; we call it a **normal coordinate chart** of M at m. If $(e_1, .., e_n)$ is a basis of $T_m(M)$ and we define $\phi^i = e^i o exp_m^{-1}$, $1 \leq i \leq n$, then it is well known [55] that for every $v \in T_m(M)$ and every $t \in I_v$:

$$\phi^i o \gamma_v(t) = a^i t, \tag{1.7.3}$$

where $v = a^i e_i$, $1 \leq i \leq n$. The choice of a given basis $(e_1, .., e_n)$ of $T_m(M)$ corresponds therefore to a choice of coordinates in U; any such coordinate system is called a **normal coordinate system** of M with origin m. It can be shown [55] that the Christoffel symbols Γ^k_{ij} of Γ with respect to this coordinates satisfy: $\Gamma^k_{ij}(m) = 0$.

An open neighbourhood U of $m \in M$ is said to be **convex** if for any $m_0, m_1 \in U$ there exists a geodesic $\gamma : [a, b] \to M$ of M such that $\gamma(a) = m_0, \gamma(b) = m_1$ and $\gamma([a, b]) \subset U$. One has then the following [55]:

THEOREM 1.7.2.*Let $(x^1, .., x^n)$ be a system of normali coordinates with origin $m_0 \in M$ and let $U(m_0; \rho)$ be the neighbourhood of m_0 defined by $\sum_i (x^i)^2 < \rho^2$. Then there exists a real number $a > 0$ such that for all $0 < \rho < a$:*
(a).$U(m_0; \rho)$ is convex;
(b).Each point of $U(m_0; \rho)$ has a normal neighbourhood containing $U(m_0; \rho)$.

We say that a pseudo-riemannian manifold (M, g) is **geodesically complete** if every geodesic of M can be extended for arbitrary large values of its affine parameter.

1.8.Holonomy groups; reducibility; decomposability.

In all that follows (M, g) is a n-dimensional pseudo-riemannian manifold , Γ its Levi-Civita connection.

A **loop** at $m \in M$ is a piecewise differentiable curve on M starting and ending at m. For a loops γ, δ we can construct, as in §1.6 for curves, the loops $\hat{\gamma}$ and $\gamma.\delta$. We denote by $C(m)$ the set of loops at m; by $C^0(m)$ we denote the set of all loops at m that are homothopic to zero.

If γ is a loop at m then the parallel transport along γ from m to m defines an automorphism of the fibre $\pi^{-1}(m)$; the properties of the parallel transport show that the set of all this automorphisms of $\pi^{-1}(m)$ is in fact a group : the **holonomy group** of Γ with reference point m; we denote it by $\Psi(m)$.

If one restricts the loops to belong to $C^0(m)$ we still obtain a group which, obviously, is a subgroup of the preceding ; this new group is denoted $\Psi^0(m)$ and is called the **restricted holonomy group** of Γ with reference point m.

These groups can in fact be realised as subgroups of the structure group $G = GL(n)$ of $L(M)$. To see this, take a point $u \in \pi^{-1}(m)$ (i.e. a frame of M at m). Each $\gamma \in C(m)$ determines an element $a \in G$ such that , denoting by τ the parallel transport along γ, $\tau(u) = \Phi(u, a)$ (where Φ is the action of G on $L(M)$). The set of all such $a \in G$ forms a group isomorphic to $\Psi(m)$ [55]. The same constructions can be performed for the restricted holonomy group. The groups above described are denoted respectively by $\Psi(u)$ and $\Psi^0(u)$. One has:

THEOREM 1.8.1.$\Psi(u)$ *and* $\Psi^0(u)$ *are both Lie subgroups of* G*; furthermore,* $\Psi^0(u)$ *is connected and a normal subgroup of* $\Psi(u)$ *and* $\Psi(u)/\Psi^0(u)$ *is countable.*

If M is connected the Lie groups $\Psi(m)$ (resp. $\Psi^0(m)$) and $\Psi(m')$ (resp. $\Psi^o(m')$) are isomorphic for all $m, m' \in M$ [55].

Given a point $m \in M$ we can for every open neighbourhood U of m consider $(U, g_{|U})$ as a pseudo-riemannian manifold and therefore define its own holonomy group with reference point $u \in \pi^{-1}(m)$; we denote this group by $\Psi(U; u)$. The **local holonomy group** $\Psi^*(u)$ of Γ with reference point u is then defined as the intersection $\bigcap \Psi(U; u)$ where U runs over all connected open neighbourhoods of m. One has [55]:

THEOREM 1.8.2. *$\Psi^*(u)$ is a connected Lie subgroup of G which is contained in $\Psi^0(u)$. Furthermore, if $\Psi^*(u)$ has constant dimension on $L(M)$, then $\Psi^0(u) = \Psi^*(u)$ for all $u \in L(M)$.*

We now look at the Lie algebras of these groups. One has [55]:

THEOREM 1.8.3. *The Lie algebra $d\Psi(m)$ of $\Psi(m)$ is equal to the subspace of linear endomorphisms of $T_m(M)$ spanned by the maps that take $z \in T_m(M)$ to*

$$\tau^{-1}(\mathbf{R}(\tau(x), \tau(y))\tau(z))$$

where $x, y \in T_m(M)$ and τ denotes parallel transport along a loop at m.

Consider now, for all $v, w \in T_m(M)$ the tensors of type $(1,1)$ on $T_m(M)$ given by $\mathbf{R}(v,w)$, $\nabla_{V_1}\mathbf{R}(v,w)$,..., $\nabla_{V_1}...\nabla_{V_k}\mathbf{R}(v,w)$, where the V_i are vector fields defined in a neighbourhood of m , $k = 1, 2, ...$ and v, w run over all elements of $T_m(M)$. It can be proved that every one of these endomorphisms of $T_m(M)$ is in fact an element of the Lie algebra $d\Psi^*(m)$ of $\Psi^*(m)$. Denote by $d\Psi'(m)$ the subspace of $d\Psi^*(m)$ spanned by all these endomorphisms. Then [58] $d\Psi'(m)$ is a Lie subalgebra of $d\Psi^*(m)$; consequently (cf. T.1.5.1) there exists a unique connected Lie subgroup $\Psi'(m)$ of $\Psi^*(m)$ whose Lie algebra is precisely $d\Psi'(m)$. This Lie group is called the **infinitesimal holonomy group** of Γ at m. One has [58]:

THEOREM 1.8.4. *If the dimension of $\Psi'(m)$ is constant on M then:*

$$\Psi^0(m) = \Psi^*(m) = \Psi'(m)$$

for all $m \in M$.

Since Γ is the Levi-Civita connection, parallel transport with respect to it preserves orthogonality; thus given any orthonormal basis of $T_m(M)$ (with respect to g_m) the parallel transport of this basis along a loop at m yields another orthonormal basis at m. It follows that the elements of $\Psi(m)$ can be considered as members of the orthogonal group of g_m. Conversely, if Γ is a (torsion free) linear connection on a manifold M and $\Psi(m)$ ($m \in M$) leaves invariant a nondegenrate symmetric bilinear form b on $T_m(M)$ then there exists a metric h on M of the same signature as b such that Γ is the metric connection of h. This result is due to Schmidt [71].

A subgroup G of the linear group $GL(n)$ is said to be **reducible** if there exists a non-trivial subspace E of $\mathbf{R}^n$ such that $u(E) = E$ for all $u \in G$; otherwise the group G is said to be **irreducible.** A (connected) pseudo-riemannian manifold (M, g) is said to be **reducible** if $\Psi(m)$ is reducible.

Let (M, g) be a connected reducible pseudo-riemannian manifold , $m \in M$ and let E_m be a non-trivial subspace of $T_m(M)$ invariant by $\Psi(m)$. Let then $m' \in M$ and γ be a curve on M from m to m', τ be the parallel transport along this curve from m to m' and denote by $E_{m'}$ the subspace of $T_{m'}(M)$ obtained by parallel transport of E_m along γ: $E_{m'} = \tau(E_m)$. It can be proved [55] that $E_{m'}$ is independent of the choice of the curve γ; it follows that $m' \mapsto E_{m'}$ is a well defined distribution on M. This distribution is differentiable [55].

We introduce a new definition: a submanifold S of M is said to be **totally geodesic** at $m \in S$ if given any $v \in T_m(S)$, the geodesic γ_v (cf. §1.7) lies in S for small values of its affine parameter. If S is totally geodesic at each of its points we say that it is **totally geodesic.**

One has then [55]:

THEOREM 1.8.5. *The distribution $m \mapsto E_m$ is integrable; furthermore, if $m \in M$ and S is the maximal integral submanifold of E through m, then S is totally geodesic.*

As every element of $\Psi(m)$ is an orthogonal transformation of $T_m(M)$ (with respect to g_m) it follows that if E_m is invariant by $\Psi(m)$ then so is $E_m^\perp$ (cf. §1.1). The above result can therefore be applied to the distribution $m \mapsto E_m^\perp$, which is thus integrable and whose maximal integral submanifolds are totally geodesic.

If (M, g) is riemannian one has $E_m \bigcap E_m^\perp = \{0\}$, and so $T_m(M) = E_m \oplus E_m^\perp$. Take then a point $m \in M$ and let U be a sufficiently small open neighbourhood of m in M. Denoting by S_m and $S_m^\perp$ the integral submanifolds of E and $E^\perp$ through m in U, define h and $h^\perp$ as the metrics induced on these manifolds by g. It can then be proved [58] that one can choose coordinates $(x^1, ..., x^k, y^1, ..., y^{(n-k)})$ in U (where $k = \dim E$, $n = \dim M$) such that $(x^1, ..., x^k)$ are coordinates on S_m and $(y^1, ..., y^{(n-k)})$ are coordinates in $S_m^\perp$. In these coordinates, using greek letters to denote indices in $\{1, .., k\}$ and latin letters to denote indices in $\{1, ..., n-k\}$, one has:

$$h = h_{\alpha\beta} dx^\alpha \otimes dx^\beta; \qquad (1.8.1)$$

and

$$h^{\perp} = h^{\perp}_{ab} dy^a \otimes dy^b, \tag{1.8.2}$$

where the $h_{\alpha\beta}$ and the $h^{\perp}_{ab}$ are elements of $\mathcal{F}(U)$. We have:

$$g = h \oplus h^{\perp}. \tag{1.8.3}$$

A simple calculation [58] shows then that in fact the $h_{\alpha\beta}$ depend only on the variables x^δ whilst the $h^{\perp}_{ab}$ depend only on the variables y^c.

This leads to a new definiton; a pseudo-riemannian manifold (M, g) is said to be **locally decomposable** if for every point $m \in M$ there exist an open neighbourhood U of m in M and a family (S_i, h_i), $1 \leq i \leq p$, of pseudo-riemannian manifolds such that $(U, g_{|U})$ is isometric to the product $(S_1 \times ... \times S_p, h_1 \oplus ... \oplus h_p)$. We say that (M, g) is **decomposable** if in the above definition U can be replaced by M. The manifolds S_i above will be called, in what follows, the **leaves of the decomposition**. One has then, from the above observations [58]:

THEOREM 1.8.6. *A reducible riemannian manifold is locally decomposable, the leaves of any local decomposition being the integral submanifolds of the distributions on M defined by the subspaces of the tangent space which are invariant under the holonomy group.*

A fundamental result in this sense is then [55]:

THEOREM 1.8.7.(de Rham)*A reducible,connected, simply connected and geodisically complete riemannian manifold is decomposable, the leaves of the decomposition being the maximal integral submanifolds of the distributions defined by the subspaces invariant under the holonomy group .*

For general pseudo-riemannian manifolds (i.e. whose metric is not necessarily positive definite) the results are different, due to the fact that not all subspaces of the tangent space are in direct sum with their orthogonals (cf. §1.1). So we introduce another concept [80]: a reducible pseudo-riemannian manifold (M, g) is said to be **non-degenerately reducible** if its holonomy group leaves invariant a non-null subspace of the tangent space (cf. §1.1); otherwise it will be said to be **degenerately reducible**.

If (M,g) is a non-degenerately reducible pseudo-riemannian manifold, the arguments presented above, for riemannian manifolds, hold without modification, and so one gets:

THEOREM 1.8.8. *Every non-degenerately reducible pseudo-riemannian manifold is locally decomposable, the leaves of any local decomposition being the integral submanifolds of the distributions defined by the subspaces invariant under the holonomy group.*

The theorem of de Rham can also be extended to general pseudo-riemannian manifolds [80]:

THEOREM 1.8.9. (Wu) *A non-degenerately reducible, connected, simply connected and geodesically complete pseudo-riemannian manifold is decomposable, the leaves of the decomposition being the maximal integral submanifolds of the distributions defined by the subspaces invariant under the holonomy group.*

If a pseudo-riemannian manifold is degenerately reducible these results do not hold. Theorem 1.8.5, however, holds in all cases.

Suppose now that (M,g) is a reducible pseudo-riemannian manifold, and let again E denote the distribution defined above; We can then define a tensor field h on M by:

$$h_m = g_{m|E_m}.$$

At each point m of M, h_m coincides with the evaluation at m of the restriction of g to the integral submanifold of E through m. An immediate consequence of the fact that these submanifolds are totally geodesic is then that (see [56]) :

THEOREM 1.8.10. *The tensor field h is covariantly constant, that is:*

$$\nabla h = 0.$$

NOTE.1.8.1. We shall see later, in Ch.4, that, conversely, if (M,g) is a pseudo-riemannian manifold and it admits a symmetric tensor field of type $(2,0)$ which is covariantly constant, then (M,g) is reducible [33].

NOTE.1.8.2. Of the above defined, the infinitesimal holonomy group, due to the form of the spanning elements of its Lie algebra, is the easier to handle. This has lead to the study of such holonomy groups in general relativity, the aim being that of establishing a new classification of 4-dimensional Lorentzian manifolds as well as that of establishing the relationship between such a classification and others already known. This study was initiated by Schell [70] and Goldberg and Kerr [24], culminating with the work of Hall and Kay [44] whose results will be the subject of more detailed explanation in the sequel.

1.9.Symmetries.

In this section all manifolds are supposed connected

Let (M, g) and (M', g') be pseudo-riemannian manifolds of the same dimension n, and let $f : M \to M'$ be a diffeomorphism. Given $m \in M$ and a linear frame (cf. §.1.5) $(X_1, ..., X_n)$ of M at m, the family $(f_{*m}X_1, ..., f_{*m}X_n)$ is a frame of M' at $f(m)$; thus, f defines a natural map $\hat{f} : L(M) \to L(M')$; this map is differentiable [55]. This allows the definition of a 1-form $\hat{w}$ on M as follows; denote by w (resp. w') the connection form of the Levi-Civita connection of (M, g) (resp. (M', g')); then we set $\hat{w} = f^*w'$. We say that f is an **affine diffeomorphism** of M onto M' if and only if :

$$\hat{w} = w.$$

When $M' = M$, we call any such map an **affine transformation** of M. It is known [55] that, for the compact open topology of $L(M)$, the set of affine transformations of a n-dimensional pseudo-riemannian manifold (M, g) is a Lie group whose dimension is at most $n(n+1)$. Moreover, if the dimension of this Lie group is $n(n+1)$ then the pseudo-riemannian manifold (M, g) is flat (cf. §.1.6). We denote this Lie group by $\mathbf{A}(M, g)$.

NOTE.1.9.1. In fact the notion of affine transformation requires only the existence of a connection on the manifold M [55]. The presentation chosen here is due to the fact that we shall not need such generality.

Let now (M, g) be a pseudo-riemannian manifold . Let X be a vector field on M, $m \in M$ and denote by $(\phi_t)_{|t|<\epsilon}$ the local 1-parameter group of X at m (cf. §.1.3). Denoting by U an open neighbourhood of M sufficiently small for the ϕ_t

to be defined on it, set $U_t = \phi_t(U)$. Consider then both U and U_t as equiped with the pseudo-riemannian structure induced by that of M. We say then that X is an **affine vector field** if the maps ϕ_t above defined are affine diffeomorphisms of U onto U_t , for all $m \in M$. The set of affine vector fields of (M, g) is denoted by $\mathcal{A}(M, g)$. It is a Lie subalgebra of $\mathcal{D}_1(M)$ as it is readily shown by the following [55]:

THEOREM 1.9.1 *Let* $(M.g)$ *be a pseudo-riemannian manifold . For* $X \in \mathcal{D}_1(M)$ *the following statements are equivalent:*
(a).X is an affine vector field;
(b).For all $Y \in \mathcal{D}_1(M)$: $\mathcal{L}_X \circ \nabla_Y - \nabla_Y \circ \mathcal{L}_X = \nabla_{[X,Y]}$;
(c). If $A_X = \mathcal{L}_X - \nabla_X$, *then, for all* $Y \in \mathcal{D}_1(M)$: $\nabla_Y \circ A_X = \mathbf{R}(X, Y)$.

The following result can also be found in [55]:

THEOREM 1.9.2.*Let* (M, g) *be a pseudo-riemannian manifold of dimension* n. *Then:*
(a).$\mathcal{A}(M, g)$ *is a finite dimensional Lie algebra, whose dimension is* $\leq n(n+1)$. *Moreover, if* $\dim\ \mathcal{A}(M, g) = n(n+1)$ *then* (M, g) *is flat.*
(b).If (M, g) *is geodesically complete, every affine vector field of* M *is complete.*

The following result can be gathered from [55] and [81]

THEOREM 1.9.3 *Let* (M, g) *be a pseudo-riemannian manifold and* X *an affine vector field of* (M, g). *Let also* $p \in M$ *and let* exp_p *be the exponential map at* p *(cf. §.1.7). Then,*
(a).If σ_t *is the local one-parameter group of* X *at* p, *then* $\sigma_t o exp_p = exp_{\sigma_t(p)} o \sigma_{t*}(p)$;
(b).If $f_{ab} = \frac{1}{2}(X_{a;b} - X_{b;a})$, *then* $f_{ab;c} = X^d \mathbf{R}_{abcd}$.

Given now a pseudo-riemannian manifold (M, g), denote by $\mathbf{I}(M, g)$ its set of isometries (cf. §.1.4). Then $\mathbf{I}(M, g)$ is also a Lie group for the compact open topology of $L(M)$ [55]. We can define, in a fashion similar to that of affine vector fields, the concept of isometric vector field: this is a vector field such that (preserving the notation for the local 1-parameter group used above) each ϕ_t is an isometry from U onto U_t. These vector fields receive the name of **Killing vector fields**. They form a Lie subalgebra of $\mathcal{D}_1(M)$, which we denote by $\mathcal{I}(M, g)$. We have then [55]:

THEOREM 1.9.4*If (M, g) is a pseudo-riemannian manifold of dimension n, then $\mathcal{I}(M, g)$ is a finite dimensional Lie-algebra whose dimension is at most $n(n+1)/2$. Moreover, if* dim $\mathcal{I}(M, g) = n(n+1)/2$, *then (M, g) is a pseudo-riemannian manifold of constant curvature, that is, its Riemann tensor satisfies:*

$$\mathbf{R}_{ijkl} = K g_{i[k} g_{l]j},$$

where K is a real constant.

It is well known that $X \in \mathcal{D}_1(M)$ is a Killing vector field if and only if :

$$\mathcal{L}_X g = 0. \tag{1.9.1}$$

This equation is known as **Killing's equation.** A vector field X on M is said to be a **homothetic vector field** if and only if the metric g is homothetically invariant (cf. §.1.4) by X. The set of homothetic vector fields of (M, g) forms a Lie subalgebra of $\mathcal{D}_1(M)$ (cf. T.1.4.5) and is denoted by $\mathcal{H}(M, g)$; we have [55]:

THEOREM 1.9.5*Let (M, g) be a pseudo-riemannian manifold of dimension n. Then the Lie algebra $\mathcal{H}(M, g)$ is finite dimensional and its dimension is bounded by $1 + n(n+1)/2$. Moreover, if* dim $\mathcal{H}(M, g) = 1 + n(n+1)$ 2, *then (M, g) is flat.*

It is obvious that every Killing vector field is a homothetic vector field; therefore, we say that a homothetic vector field is a **proper homothetic vector field** if it is not a Killing vector field.

It can be shown that every homothetic vector field is an affine vector field [55]. Thus, we say that an affine vector field is a **proper affine vector field** if it is not a homothetic vector field.

A vector field X on M is said to be a **conformal vector field** if the metric g is conformally invariant under X (cf. §.1.4). The set of all such vector fields forms a Lie subalgebra of $\mathcal{D}_1(M)$ (cf. T.1.4.5) which we denote by $\mathcal{C}(M, g)$. It is well known that [55]:

THEOREM 1.9.6.*Let (M, g) be a pseudo-riemannian manifold of dimension $n \geq 3$. Then $\mathcal{C}(M, g)$ is a finite dimensional Lie algebra whose dimension is bounded by $(n+1)(n+2)/2$. Moreover, if* dim $\mathcal{C}(M, g) = (n+1)(n+2)/2$, *then (M, g) is conformally flat.*

NOTE.1.9.2. It is well known that this result does not hold in dimension 2. In this case in fact, it can be proved that a vector field is conformal if and only if its components in an isothermal coordinate system (see [79] or §.5.2) satisfy the Cauchy-Riemann equations.

As follows from the definition, every homothetic vector field is a conformal vector field. A conformal vector field that is not a homothetic vector field is called a **proper conformal vector field**. It is well known that a proper conformal vector field is not an affine vector field [55].

Finally, a vector field X on M is said to be a **curvature collineation** if the Riemann tensor of (M, g) is invariant under X. As follows from T.1.4.5, the set of curvature collineations forms a Lie subalgebra of $\mathcal{D}_1(M)$ which we denote by $\mathcal{K}(M, g)$. This Lie algebra may be infinite dimensional as it is readily shown by the case of flat pseudo-riemannian manifolds. It is well known [81] that every affine vector field is a curvature collineation. A curvature collineation that is not an affine vector field is said to be a **proper curvature collineation**.

NOTE.1.9.3. The notion of curvature collineation was first introduced by Katzin et al, [52].

2.SPACE-TIMES.

2.1.Space-times.

In the sequel we shall use the mathematical model of space-time and the main postulates of general relativity as they have been proposed by Hawking and Ellis [49].

We take as a model for **space-time**, that is, the collection of all events, a 4-dimensional lorentzian manifold (M, g), the underlying topological space being, for physical reasons, connected and Hausdorff. The first of these conditions is natural in the sense that we would have no knowledge of any other connected component of space-time but our own; the second, rather more contentious, seems anyway to be in agreement with normal experience.

NOTE.2.1.1. The existence of a lorentz metric on M together with the Hausdorff condition makes of M a paracompact manifold [23]; this implies that M has a countable basis of open sets; hence, the conditions imposed at the beginning of Ch.1 to any topological space for it to be a manifold are satisfied.

As pointed out by the above authors, the differentiability class is not significant from the physical point of view, as all measurements contain errors, so we take it to be C^∞.

On M various fields are defined such as the electromagnetic field the neutrino field, etc; these fields describe the matter content of space-time and so the theory one obtains depends on the nature of the fields it incorporates. These fields will obbey equations which can expressed as tensorial relations on M, in which all covariant derivatives considered are taken with respect to the metric connection of M.

Given a space-time (M, g) and a point $m \in M$, the tangent space $T_m(M)$ is isomorphic to Minkowski space (cf. §1.1); thus any $v \in T_m(M)$ can be classified as null, timelike or spacelike. This notions can therefore be extended in a pointwise fashion to any vector field on M.

We now describe the fundamental postulates of general relativity.

Let (M, g) be a space-time, and let U be a convex normal neighbourhood of M (cf. §1.7). Given points $p, q \in U$, the equations governing the matter fields maust be such that a signal can be sent in U from p to q if and only if p, q can be joined by a curve γ whose tangent vector is never spacelike; this postulate is known as the postulate of **local causality.** It can be expressed more precisely in the terms of the Cauchy problem for the matter fields [49].

The governing equations for the matter fields are such that there exists a $(0,2)$ symmetric tensor field T on M, called the **energy-momentum tensor**, which depends on the fields, their covariant derivatives and the metric and satisfies:

(E1).Given any open subset V of M, then $T_{|V} = 0$ if and only if all the matter fields vanish in V;

(E2).T satisfies the equation $T^{ab}{}_{;b} = 0$

The postulate of local causality makes possible the measurement of the metric up to a conformal factor at each point of M, the second postulate relates this measurements at different points [49].

The above postulates do not tell us how to construct the energy-momentum tensor for a given set of fields or whether it is unique; in practice this tensor field is constructed by relying on our intuitive knowledge of what such a tensor should be.

In special relativity one takes the metric g to be flat (cf. §1.6). In general relativity one has to give some prescription for the curvature of space-times. That is, one has to solve the problem of finding field equations relating the metric to the description of matter. These equations (which should involve our knowledge of the matter content of space-time only through the energy-momentum tensor) are the Einstein equations:

$$\mathbf{R}_{ij} - \frac{1}{2}\mathbf{R}g_{ij} + \Lambda g_{ij} = 8\pi T_{ij}, \tag{2.1.1}$$

where Λ is a constant, the **cosmological constant.** In all that follows we shall assume that $\Lambda = 0$.

Thus, the third postulate of general relativity is that this equation must hold on M.

NOTE.2.1.2.A discussion and justification of the Einstein equations can be found in [49].

The predictions of these equations are in close agreement with the observations that have been made so far [49].

In the actual universe, the energy-momentum tensor will be made up of contributions of a large number of different matter fields and so it would be extremely complicated, to describe such a tensor. Nevertheless, it is plausible to assert that any candidate to be energy-momentum tensor of space-time has to satisfy certain conditions. These conditions, known as the **energy conditions** have been proposed by Hawking and Ellis [49]:

(WEC).(Weak Energy Condition). At each $p \in M$ the energy-momentum tensor satisfies:

$$T_{ij}w^i w^j \geq 0; \tag{2.1.2}$$

for any timelike vector w at p.

(DEC).(Dominant Energy Condition). For every timelike vector $w \in T_p(M)$:

$$T^{ij}w_i w_j \geq 0, \tag{2.1.3}$$

and $T^i j w_i$ is not spacelike.

The first of these conditions can be interpreted as a statement that to any observer the energy density appears non-negative; the second goes further in imposing that the local energy flow vector is non-spacelike [49].

2.2.Algebraic classifications.

Let (E, h) be a pseudo-euclidean space, r, s integers and consider the vector space $T^r_s(E)$ (cf. §.1.1). Each $u \in O(E, h)$ defines naturally an endomorphism ${}^t u^{-1}$ of E^*, the dual of E. On the other hand, each $T \in T^r_s(E)$ can be thought of as a multilinear map on the product of r copies of E^* with s copies of E. Thus, it is natural, in this sense, the consideration of the map given for $w^1, ..., w^r \in E^*$ and $v_1, ..., v_s \in E$ (cf. §.1.1) by:

$$\Omega(T,u)(w^1,..,w^r,v_1,..,v_s) = T(({}^t u)^{-1}(w^1),..,({}^t u)^{-1}(w^r),u^{-1}(v_1),..,u^{-1}(v_s)). \tag{2.2.1}$$

Ω is a group action on the right of $O(E, h)$ in $T^r_s(E)$. The classification problem it leads to (cf. §.1.1), when solved, leads to properties of the elements of $T^r_s(E)$ which are invariant under the group $O(E, h)$ of the "fundamental" tensor h.

Given a space-time (M,g) and a point $m \in M$, $(T_m(M), g_m)$ is a pseudo-euclidean space, and so the above remarks can be applied to it.

In this section we describe the solutions to the classification problems that arise in this way for certain spaces of tensors in general relativity.

To simplify the notation, we set $E = T_m(M)$ and $h = g_m$.

Let (e_i) be a basis of E and T be a tensor of type $(0,2)$ over E, so that:

$$T = T_{ij} e^i \otimes e^j. \tag{2.2.2}$$

We then define a map $\Delta(T) : E \to E^*$ by:

$$\Delta(T)(v^i e_i) = T_{ij} v^j e^i; \tag{2.2.3}$$

(that is, $\Delta(T)$ acts by "lowering indices with T "). The map Δ thus defined from $T_2^0(E)$ to $\mathcal{L}(E, E^*)$ is clearly linear and an isomorphism.

Consider then the map $L(T) = G^{-1} o \Delta(T)$ (cf. §1.1 for definition of G); it is an endomorphism of E and the map L thus defined is clearly linear and an isomorphism from $T_2^0(E)$ onto $\mathcal{L}(E, E)$.

A simple computation shows that we have, for all $T \in T_2^0(E)$ and all $u \in O(E,h)$:

$$L(\Omega(T,u)) = \Psi(L(T), u^{-1}), \tag{2.2.4}$$

where Ψ is the action associated with the Jordan equivalence problem (cf. §1.1). This, together with the fact that $O(E,h)$ is a subgroup of $GL(E)$ shows that the classification problem posed by Ω can be solved by using the solution to the Jordan equivalence problem. Obviously (cf. §1.1) every orbit of $T_2^0(E)$ under Ω is a subset of an orbit of $T_2^0(E)$ under Ψ; the converse is false.

Denote by $\mathcal{S}_2^0(E)$ (resp $\mathcal{A}_2^0(E)$) the vector subspace of $T_2^0(E)$ of the symmetric (resp. antisymmetric) tensors. The action Ω defined above, when restricted to $\mathcal{S}_2^0(E)$ or $\mathcal{A}_2^0(E)$ defines an action of $O(E,h)$ on the right on these subspaces. We describe now the complete solutions to the respective classification problems that have been obtained by Hall [28], [29], [30].

The Symmetric Case.

Given $w \in \mathcal{L}(E)$, let W be the $(0,2)$ tensor such that $L(W) = w$. One can then prove that, if w has Segre type $\{4\}$ or $\{2,2\}$ (cf. §.1.1), then the condition that W is symmetric is incompatible with a metric h with lorentz signature (in fact one can prove that the above conditions imply $\det(h) \geq 0$ [29]). Similarly, one proves that $L(T)$ can at most have a complex pair of (conjugate) eigenvalues and that in such case the only possible Segre type is $\{z, \overline{z}, 11\}$. Thus one is left with the possible Segre types $\{31\}$, $\{211\}$ and $\{1,111\}$ and $\{z, \overline{z}, 11\}$. To study these one considers a null tetrad (cf. §1.1) (l, n, x, y) (where l, n are the null vectors) and one expands T with respect to this basis as [28]:

$$T_{ij} = 2T^1 l_{(i}n_{j)} + T^2 l_i l_j + T^3 n_i n_j + 2T^4 l_{(i}x_{j)} + 2T^5 l_{(i}y_{j)}$$
$$+ 2T^6 n_{(i}x_{j)} + 2T^7 n_{(i}y_{j)} + 2T^8 x_{(i}y_{j)} + T^9 x_i x_j + T^{10} y_i y_j. \tag{2.2.5}$$

If u is a null rotation about l (cf. §1.1), then the components of $\Omega(T, u)$ with respect to the above null tetrad can be computed in terms of the components of T; these components can also be interpreted as the components of T in the null tetrad $(u(l), u(n), u(x), u(y))$. Then, considering separately the cases when T has or has not a null eigenvector, it is possible to achieve a complete classification of the elements of $\mathcal{S}_2^0(E)$ according to their Segre type and, furthermore, one can get canonical forms for T [28]. These results are resumed in the:

THEOREM 2.2.1.(Hall) *The possible Segre types for a symmetric $(0,2)$ tensor T on E are $\{211\}$, $\{31\}$,$\{(1,1)11\}$ and $\{z, \overline{z}, 11\}$. Furthermore:*
(a).If $L(T)$ has a null eigenvector l, then the possible Segre types are $\{211\}$,$\{31\}$ and $\{(1,1)11\}$ and their degeneracies, and for each of these types there exists a null tetrad (l, n, x, y) such that, with respect to this null tetrad T has the form:

$$T_{ij} = 2\alpha l_{(i}n_{j)} \pm l_i l_j + \beta x_i x_j + \gamma y_i y_j \ - \ \text{Segre type } \{211\};$$

$$T_{ij} = 2\alpha l_{(i}n_{j)} + 2\beta l_{(i}x_{j)} + \alpha x_i x_j + \gamma y_i y_j \ - \ \text{Segre type } \{31\};$$

$$T_{ij} = 2\alpha l_{(i}n_{j)} + \beta x_i x_j + \gamma y_i y_j \ - \ \text{Segre type } \{(1,1)11\};$$

where, in the second expression, $\beta \neq 0$.
(b).If $L(T)$ has no null eigenvectors, then the possible Segre types are $\{1,111\}$ and $\{z, \overline{z}, 11\}$, the possible degeneracies being $\{1,1(11)\}$, $\{1,(111)\}$, and $\{z\overline{z},(11)\}$. For each of these types there exists a null tetrad (l, n, x, y) with respect to which T has the form:

$$T_{ij} = 2\alpha l_{(i}n_{j)} + \beta(l_i l_j + n_i n_j) + \gamma x_i x_j + \delta y_i y_j \ - \ \text{Segre type } \{1,111\};$$

$$T_{ij} = 2\alpha l_{(i}n_{j)} + \beta(l_i l_j - n_i n_j) + \gamma x_i x_j + \delta y_i y_j \quad - \quad \text{Segre type } \{z\bar{z},(11)\}.$$

where $\beta \neq 0$ and, in the first, $\gamma \neq \alpha - \beta \neq \delta$.

This classification applies at each point in a space-time to any symmetric tensor field of type $(0,2)$. In particular, it applies to the energy-momentum tensor. The energy conditions (cf. §.2.1) imply then some restrictions on the allowed Segre types [31]:

THEOREM 2.2.2. *Let (M,g) be a space-time and T be its energy-momentum tensor. If T satisfies the dominant energy condition then for every point $p \in M$:*
(a).T_m cannot have Segre type $\{31\}$ or $\{z\bar{z},11\}$;
(b).If T_m has Segre type $\{211\}$ then it satisfies the dominant energy condition if and only if in its expression given in T.2.2.1 the optional sign is positive and $\alpha \leq \beta(\gamma) \leq -\alpha$.
(c).If T_m has Segre type $\{1,111\}$ then it satisfies the dominant energy condition if and only if in its expression given in T.2.2.1 we have $\alpha \leq 0$, $\beta \geq 0$ and $(\alpha - \beta \leq \gamma(\delta) \leq (\beta - \alpha)$.

As mentioned in §.2.1, the energy momentum tensor depends on the matter fields of space-time. Its Segre type depends therefore on the nature of the matter fields in question; it can be shown [41] [46] that the Segre type of non-null electromagnetic fields (i.e. electromagnetic fields whose bivector is non-null - see the next subsection) is $\{(1,1)(11)\}$;the Segre type of null electromagnetic fields or pure radiation fields is $\{(2,1,1)\}$ and that the Segre type of perfect fluids is $\{1,(111)\}$.

NOTE.2.2.1. Churchill [9] has been the first to consider the problem of algebraic classification of symmetric tensor fields over a pseudo-euclidean space. Subsequently, other classifications have been obtained by Plebanski [69] Penrose [67] and Ludwig and Scanlon [59]. The solution presented here is due to Hall and can be found in [28]. Hall [41] has shown that the classifications obtained by these authors are equivalent.

The Antisymmetric Case.

We study now the classification of 2-forms; we shall, from now on, use the term **bivector** instead of 2-form.

If $F \in \mathcal{A}^0_2(E)$, its matrix in any given basis of E is an antisymmetric matrix and it is a classical result of linear algebra that such matrices always have even rank. This provides us with a preliminary classification of a non-zero bivector as **simple** (resp. **non-simple**) if its rank is 2 (resp. 4).

In the case when F is simple, the kernel of $L(F)$ is invariant under $L(F)$. Due to the antisymmetry of F, the subspace orthogonal to the kernel of $L(F)$ is also invariant under $L(F)$; we call it the **blade** of F. F can then be classified according to the type of its blade (cf. T.1.1.1). Thus a simple bivector F is said to be **spacelike, timelike** or **null** according to wether its blade is spacelike, timelike or null.

If F is a simple bivector, there exists a null tetrad (l, m, x, y) and a real number $\lambda \neq 0$ such that if F is spacelike:

$$F_{ab} = 2\lambda x_{[a}y_{b]};$$

if F is timelike:

$$F_{ab} = 2\lambda l_{[a}m_{b]};$$

and if F is null:

$$F_{ab} = 2\lambda l_{[a}x_{b]}.$$

Obviously, $span(x, y)$ (resp. $span(l, m)$) (resp. $span(n, x)$) is then the blade of F, whilst $span(l, m)$ (resp. $span(x, y)$) (resp. $span(l, y)$) is the kernel of $L(F)$.

In the case when F is non-simple, one can use the solution of the Jordan classification problem to prove the existence of a 2-dimensional subspace H of E which is invariant under $L(F)$ [30]. H is non-null [30] and the antisymmetry of F guarantees then that the subspace orthogonal to H is also invariant under $L(F)$. This shows that there exists a null tetrad (l, m, x, y) and non-zero real numbers α, β such that:

$$F_{ab} = 2\alpha l_{[a}m_{b]} + 2\beta x_{[a}y_{b]}.$$

The Segre types corresponding to the above classes are [30] $\{z, \overline{z}, (1,1)\}$ (simple spacelike), $\{1, 1(11)\}$ (simple timelike), $\{(31)\}$ (simple null) and $\{z, \overline{z}, 11\}$ (non-simple).

Given any basis (e_i) of E we define a symmetric bilinear form on $\mathcal{A}_2^0(E)$ by requiring it to be the unique such form that satisfies:

$$\mathcal{G}(e^i \wedge e^j, e^k \wedge e^l) = 2g^{i[k}g^{l]j}.$$

This bilinear form is non-degenerate and its signature is $(3,3)$. One has then the following important result [21]:

THEOREM 2.2.3. *Let (e_i) be an orthonormal basis of E and let $u = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. Then for every bivector F there exists a unique bivector F^* such that for every bivector S:*

$$F \wedge S = \mathcal{G}(F^*, S).u.$$

The bivector F^* is called the **dual** (or Hodge dual) of F and it can be shown that $(F^*)^* = -F$.

It can be proved that it is simple (resp. non-simple) when F is simple (resp. non-simple). Moreover, if F is simple spacelike (resp. timelike) (resp. null) then F^* is simple timelike (resp. spacelike) (resp.null); in every case, the kernel of $L(F^*)$ is precisely the blade of F [31]. In particular, we see that if (l, m, x, y) is a null tetrad and we define:

$$F_{ab} = 2l_{[a}m_{b]};$$

$$G_{ab} = 2l_{[a}x_{b]};$$

and

$$H_{ab} = 2m_{[a}x_{b]};$$

then we have:

$$F^*_{ab} = 2x_{[a}y_{b]};$$

$$G^*_{ab} = 2l_{[a}y_{b]};$$

and

$$H^*_{ab} = 2m_{[a}y_{b]},$$

so that (F, F^*, G, G^*, H, H^*) is a basis of $\mathcal{A}_2^0(E)$.

The Petrov Classification of the Weyl Tensor.

Keeping the notation of the preceding subsections we consider now the space of tensors of type $(1,3)$ on E which have the symmetries of the Weyl tensor, that is, which satisfy relations (1.6.13), (1.6.15), (1.6.18) and (1.6.21). Due to these relations, if $\mathbf{C}$ is any such tensor it has associated with it an endomorphism of $\mathcal{A}_2^0(E)$, $W(\mathbf{C})$ given by $F \mapsto G = W(\mathbf{C})(F)$, where the bivector G is given by:

$$G_{ab} = \mathbf{C}_{ab}{}^{cd} F_{cd}.$$

If we consider $\mathcal{A}_2^0(E)$ together with the metric $\mathcal{G}$ then we can consider the problem of classifying $W(\mathbf{C})$ as a tensor of type $(1,1)$ on $\mathcal{A}_2^0(E)$ for the action Ω defined on this space by $\mathcal{G}$. It can be proved that this classification problem is equivalent to the classification problem for $\mathbf{C}$ [75]. The classification obtained in this way is due to Petrov [68]. An equivalent way of classifying this type of tensors has been found by Geheniau [22]. We now explain his method following Synge [75].

Let (e_i) be an orthonormal basis of E ordered in such a way that e_4 is its timelike element. For $F \in \mathcal{A}_2^0(E)$ we define:

$$\Theta(F) = (F_{14} - iF_{23}, F_{24} + iF_{13}, F_{34} - iF_{12});$$

where the F_{ij} are the components of F with respect to the above basis. This relation defines a linear map from $\mathcal{A}_2^0(E)$ to $\mathbf{C}^3$ (considered as a real vector space): this map is obviously a bijection.

Now we consider $\mathbf{C}^3$ as equiped with the bilinear form τ given in its canonical basis by $\tau((z_1, z_2, z_3), (z_1', z_2', z_3')) = z_1 z_1' + z_2 z_2' + z_3 z_3'$. It is then possible to define, as for real symmetric bilinear forms (cf. §1.1) the concept of orthogonal map for τ. These maps obviously form a group wich is isomorphic to the subgroup of the $O(E,h)$ of all those u such that $\det(u) = 1$ and $e^4(u(e_4)) > 0$ [75]. The map Θ being an isomorphism one can then consider the endomorphism of $\mathbf{C}^3$ defined by:$\Theta \circ W(\mathbf{C}) \circ \Theta^{-1}$. Its matrix with respect to the canonical basis of $\mathbf{C}^3$, $M(\mathbf{C})$, is given by [75]:

$$\begin{pmatrix} \mathbf{C}_{14}{}^{14} + i\mathbf{C}_{23}{}^{14} & \mathbf{C}_{14}{}^{24} + i\mathbf{C}_{23}{}^{24} & \mathbf{C}_{14}{}^{34} + i\mathbf{C}_{23}{}^{34} \\ \mathbf{C}_{24}{}^{14} - i\mathbf{C}_{13}{}^{14} & \mathbf{C}_{24}{}^{24} - i\mathbf{C}_{13}{}^{24} & \mathbf{C}_{24}{}^{34} - i\mathbf{C}_{13}{}^{34} \\ \mathbf{C}_{34}{}^{14} + i\mathbf{C}_{12}{}^{14} & \mathbf{C}_{34}{}^{24} + i\mathbf{C}_{12}{}^{24} & \mathbf{C}_{34}{}^{34} + iWe_{12}{}^{34} \end{pmatrix}$$

Furthermore, it can be proved that the initial classification problem is equivalent to the problem of classifying $M(\mathbf{C})$ for the Jordan action of $\mathbf{C}^3$ (the group being restricted to be the orthogonal group of τ) [75]. The fact that $\mathbf{C}$ is traceless (1.6.21) implies then that $M(\mathbf{C})$ is a traceless matrix. This restricts the number of possible Segre types for $M(\mathbf{C})$ and so leads us to the classification of $\mathbf{C}$ by its Petrov type; these types are:

$$\text{TYPE 0}: \mathbf{C} = 0;$$

$$\text{TYPE I}: M(\mathbf{C}) \text{ has Segre type } \{111\};$$

$$\text{TYPE D}: M(\mathbf{C}) \text{ has Segre type } \{(11)1\};$$

$$\text{TYPE II}: M(\mathbf{C}) \text{ has Segre type } \{21\};$$

$$\text{TYPE N}: M(\mathbf{C}) \text{ has Segre type } \{(21)\};$$

$$\text{TYPE III}: M(\mathbf{C}) \text{ has Segre type } \{3\}.$$

Type I Weyl tensors are said to be algebraically general whilst the remaining types are said to be algebraically special.

A useful set of criteria to decide the Petrov type of the Weyl tensor has been found by Bel [6]. Define ϵ_{abcd} as the signature of the permutation $\{abcd\}$ of $\{1234\}$ and set:

$$\mathbf{C}^*_{abcd} = \frac{1}{2}\epsilon_{cdef}\mathbf{C}_{ab}{}^{ef};$$

then:

THEOREM 2.2.4.(Bel's criteria)*Let C be a tensor on E with the symmetries of the Weyl tensor. Then:*

(a).C has Petrov type N if and only if there exists a (necessarily null) vector $l \in E$ which is unique up to multiples, such that $l^a C_{abcd} = 0$.

(b).C has Petrov type III if and only if there exists a null vector $l \in E$ which is unique up to multiples, such that $l^a l^c C_{abcd} = l^a l^c C^*_{abcd} = 0$ and $l^a C_{abcd} \neq 0$;

(c).C has Petrov type II if and only if there exists a (necessarily null) vector $l \in E$ which is unique up to multiples and real numbers α and β such that $l^a l^c C_{abcd} = \alpha l_b l_d$, $l^a l^c C^*_{abcd} = \beta l_b l_d$ and $\alpha^2 + \beta^2 \neq 0$;

(d). C has Petrov type D if and only if there exist (necessarily null) vectors $l, m \in E$, which are unique up to order, and real numbers α and β such that

$\alpha^2 + \beta^2 \neq 0$ and $l^a l^c C_{abcd} = \alpha l_b l_d$, $l^a l^c C^*_{abcd} = \beta l_b l_d$, $m^a m^c C_{abcd} = \alpha m_b m_d$ and $m^a m^c C^*_{abcd} = \beta m_b m_d$.

Let (l, n, x, y) be a null tetrad and define $m = (x + iy)/\sqrt{2}$ and $\overline{m} = (x - iy)/\sqrt{2}$. Define then the follwing complex quantities:

$$C_1 = 2\mathbf{C}_{abcd} n^a n^c m^b m^d;$$

$$C_2 = -\mathbf{C}_{abcd} n^a m^b (l^c n^d + m^d \overline{m}^c);$$

$$C_3 = 2\mathbf{C}_{abcd} n^a m^b l^c \overline{m}^d;$$

$$C_4 = -\mathbf{C}_{abcd} l^a \overline{m}^b (l^c n^d + \overline{m}^c m^d);$$

$$C_5 = 2\mathbf{C}_{abcd} l^a l^c \overline{m}^b \overline{m}^d.$$

Then one has the following [74]:

THEOREM 2.2.5. *Let C be a tensor field of type $(1,3)$ on a space time (M, g). with the symmetries of the Weyl tensor and let $m \in M$. Then there exists a null vector l and a null tetrad (l, n, x, y) at m with the following properties (where $C1, ..., C_5$ are the real numbers defined by the above relations with respect to this null tetrad):*

(a).If C has Petrov type 0 at m then all the C_i vanish;

(b).If C has Petrov type N at m then $C_2 = C_3 = C_4 = C_5 = 0$ and $C_{abcd} l^a = 0$;

(c).If C has Petrov type III at m then $C_3 = C_4 = C_5 = 0$ and $l_{[b} C_{a]cde} l^c = 0$;

(d).If C has Petrov type II or D at m then $C_4 = C_5 = 0$ and $l_{[b} C_{a]cde} l^c l^d = 0$;

(e).If C has Petrov type I at m then $C_5 = 0$ and $l_{[b} C_{a]cd[e} l_{f]} l^c l^d = 0$.

NOTE.2.2.2. In fact, the null tetrad (l, n, x, y) can be so chosen that, besides the relations above, one has $C_1 \in \mathbf{R}$ for type N; $C_2 = 0$ and $C_1 \in \mathbf{R}$ for type II; $C_1 = 0$ and $C_2 \in \mathbf{R}$ for type III; $C_1 = C_2 = 0$ for type D.

2.3.Holonomy classification.

In this section we describe a classification based on the infinitesimal holonomy group.

Given a space-time (M, g) and a point $m \in M$ we shall consider the infinitesimal holonomy group $\Psi'(m)$. As mentioned in §.1.8, $\Psi'(m)$ is a connected subgroup of the Lorentz group; thus we shall begin by considering a classification of the connected subgroups of the Lorentz group.

Given a Minkowski space (E, h) we consider the group $O(E, h)$. As mentioned in §.1.1, each element of $O(E, h)$ can be classified according to the type and dimension of the subspaces of E that it leaves invariant. This leads to the consideration of those subgroups of $O(E, h)$ that leave a certain subspace of E invariant.

Another approach is possible; in fact, as follows from T.1.5.1, the connected subgroups of $O(E, h)$ are in 1:1 correspondence with the subalgebras of the Lie algebra $o(E, h)$ of $O(E, h)$. On the other hand, as pointed out by Shaw [72], the elements of the Lie algebra of the Lorentz group can be regarded as bivectors on E, the Lie product being given, for $\eta, \phi \in o(E, h)$ by $[\eta, \phi] = L^{-1}(L(\eta) \circ L(\phi) - L(\phi) \circ L(\eta))$, where L is the map that associates with each bivector the endomorphism of E obtained from it by raising one index with h.

Using these observations, a complete classification of the connected Lie subgroups of $o(E, h)$ can be achieved. Such a classification can be found in the article of Shaw mentioned above. Traditionally, the subalgebras of $o(E, h)$ (15 classes in total, of which, two (R_5 and R_12 are in fact 1-parameter families)) are labeled, following Schell [70], $R_1, .., R_{15}$, where $R_1 = \{0\}$ is the Lie algebra of the trivial subgroup and R_{15} is the full Lie algebra $o(E, h)$. This classification can be found in the first two columns of TABLE.1

NOTE.2.3.1. This table has been taken from [44].

We assume from now on that we are in an open connected subset U of a space-time (M, g) such that in U the dimension of the infinitesimal holonomy group is constant, so that, because of T.1.8.4, in U the restricted, local and infinitesimal holonomy groups coincide.

As follows from the definition of the infinitesimal holonomy group, the Lie algebra of this group is spanned by all elements of the form $\mathbf{R}^a{}_{bcd}x^c y^d$, $\mathbf{R}^a{}_{bcd;e}x^c y^d z^e$, etc. One can therefore compute its spanning elements at any given point of U by gathering all bivectors resulting from the computation of the above expressions.

NOTE.2.3.2. We can describe this gathering procedure as operating by steps: in the first step one varies the vectors x, y and computes the ensuing bivectors from $\mathbf{R}^a{}_{bcd}x^c y^d$; the second step consists of the same procedure, but now taking

$\mathbf{R}^a{}_{bcd;e}$ and contracting it with $x^c y^d z^d$ for all possible choices of x, y and z in E. It is obvious how to describe the n^{th} step. At each step one has then to answer the question of wether all the spanning elements of $d\Psi'(m)$ have already been obtained or not. Clearly if the elements already in our possesion do not span a Lie algebra then the procedure must continue. Conversely, Hall and Kay [44] have found a useful criteria that tells us when to stop: if at the n^{th} step the bivectors one obtains have all been obtained in previous steps then no other step will provide us with new bivectors and the procedure can therefore be stopped; the bivectors found form necessarily a sub-algebra.

It is clear that if $\Psi'(m)$ is one of the groups $R_2, ..., R_{14}$ (with the exception of R_5 which cannot be the infinitesimal holonomy group of a space-time [24], [44]), then certain restrictions are imposed on the Riemann tensor; these in turn imply restrictions on the Ricci tensor; these restrictions can be seen as restrictions on the Segre type of this tensor. The definition of the Weyl tensor shows then that restrictions occur also for the Petrov type of this tensor. Thus, we have a relationship between the algebraic classifications obtained in the previous chapter and the classification by the infinitesimal holonomy group. This relationship was originally studied by Schell [70] and Goldberg and Kerr [24] for the vacuum case (their work can in fact be simplified as it was shown by Hall [36]). A complete solution to this problem has been obtained by Hall and Kay [44] and is contained in TABLE.1 which can be found in the article mentioned above. Collinson and Smith [13] studied this problem in the context of electromagnetic fields.

A related question which has been analysed in recent years is the so called question of "uniqueness of the metric from the curvature" which we now explain. Suppose (M, g) is a space-time and that there exists on M another metric h of the same signature as g whose (contravariant) Riemann tensor coincides with the Riemann tensor of g. What is the relationship between these two metrics? Since it is clear that for any given non-zero real number λ the metric λg defines exactly the same Riemann tensor as g, the above question can be reformulated as "to what extent does the Riemann tensor determine the metric it comes from (up to a constant conformal factor)?" This question has been analysed by several authors, among them Ihrig [51], Collinson and Vaz [14], McIntosh and Halford [61], Hall and McIntosh [45], Hall [35], Hall and Kay [44].

We describe now the main results that have been obtained in this setting.

Suppose then that h is a metric on a space-time (M, g) defining the same Riemann tensor as g. Then, as follows from the Ricci identity (cf. NOTE.1.6.3) and T.1.6.5, one has:

$$h_{e(a}\mathbf{R}^{e}{}_{b)cd} = 0. \tag{2.3.1}$$

Thus any given candidate to be a metric on M with the same Riemann tensor as g must satisfy this equation.

McIntosh and Halford [61] have found a complete solution to the problem of determining all the possible solutions to the above equation at a point in M. Starting from a more geometrical point of view, Hall and McIntosh [45] obtained the same solution. The global problem was solved by Hall [35].

In this sense an important observation is that, as follows from (1.6.15) and (1.6.18), for any given bivector F on M the tensor field H given by:

$$H_{ab} = \mathbf{R}_{abcd}F^{cd},$$

is also a bivector. Thus, the Riemann tensor can be considered at every point of M as an endomorphism of the space of bivectors at that point. One can, in particular, talk about the **rank** of the Riemann tensor (at the point in question) as the rank of this linear map. Since the space of bivectors has dimension 6 the Riemann tensor has at most rank 6. The rank of the Riemann tensor is the diemnsion of the subspace of the space of bivectors spanned by the elements of the form $\mathbf{R}^{a}{}_{bcd}x^{c}y^{d}$. These bivectors are said to **span the curvature** since the Riemann tensor can be written then, denoting these bivectors by F, G, etc :

$$\mathbf{R}_{abcd} = \alpha F_{ab}F_{cd} + \beta(F_{ab}G_{cd} + F_{cd}G_{ab}) + ...$$

The results of Hall and McIntosh are then the following [45]:

THEOREM 2.3.1. *Let h be any symmetric tensor on $T_m(M)$ of type $(0,2)$ which is a solution of (2.3.1). Then:*

(a).If the rank of the Riemann tensor is ≥ 4, there exists a real number λ such that $h = \lambda g_m$;

(b).If the rank of the Riemann tensor is 2 or 3 at m and the bivectors that span the curvature all have a common eigenvector u, with zero eigenvalue, then there exist real constants α, β such that:

$$h_{ab} = \alpha g(m)_{ab} + \beta u_a u_b.$$

(c).If the rank of the Riemann tensor is 2 and it is spanned by a pair of simple bivectors of the form $2l_{[a}m_{b]}$ *and* $2x_{[a}y_{b]}$*, where* (l, m, x, y) *is a null tetrad, then there exist real constants* α *and* β *such that:*

$$h_{ab} = 2\alpha l_{(a}m_{b)} + \beta(x_a x_b + y_a y_b);$$

(d).If the rank of the Riemann tensor is 1 the bivectors that span it are multiples of a bivector F *which is necessarily simple. In this case there exist real constants* α*,* β*,* γ *and* δ *such that :*

$$h_{ab} = \alpha g(m)_{ab} + \beta u_a u_b + 2\gamma u_{(a}v_{b)} + \delta v_a v_b,$$

where u, v *are orthogonal to the blade of* F*.*

Furthermore, we have the following [45]:

THEOREM 2.3.2. *Consider the equation:*

$$\mathbf{R}^a{}_{bcd}z^d = 0.$$

Then in case (d) of the above theorem there are exactly two linearly independent solutions of this equation : u *and* v*. In case (b) there exists exactly one independent solution to this equation :* u*. In the remaining cases this equation has no non-zero solutions.*

If U is some open connected subset of M such that the rank of the Riemann tensor is constant on it, then the above results apply at all points of U; one can then reformulate the above theorems in terms of vector fields and differentiable functions, as it has been shown by Hall [35].

TABLE.1, as mentioned above, lists the relation between the several classifications presented in this work. It has been taken from [44]. In this table, (l, n, x, y) denotes a null tetrad, $\{x, y, z\}$ an orthonormal triad of spacelike vectors; in the case of R_{10}, u is a timelike vector, z a spacelike vector; they are orthogonal to each other and to x. The second row of the table contains the spanning bivectors of the Lie subalgebra in question.

NOTE.2.3.3. It should be noticed here that whilst this table is a complete classification at any given point m of a space-time M according to the infinitesimal holonomy group at that point, it only provides us with a holonomy classification of M (and not of M at m) if M is assumed to be such that its infinitesimal holonomy group has constant dimension, in which case it coincides with the holonomy group (cf. T.1.8.4). A classification of space-times according to the holonomy group without this assumption has been obtained by Hall [40].

Alg.	Bivectors	Curv. rank	Segre	Petrov
R_2	$l \wedge n$	1	[(1,1)(11)]	D
R_3	$l \wedge x$	1	[(211)]	N
R_4	$x \wedge y$	1	[(1,1)(11)]	D
R_5	$l \wedge n + \rho(x \wedge y)$	Impossible		
R_6	$l \wedge n,\ l \wedge y$	1 $(l \wedge n)$	[(1,1)(11)]	D
R_6	$l \wedge n,\ l \wedge y$	2	[(31)]	III
R_6	$l \wedge n,\ l \wedge y$	2	[(211)]	II
R_7	$l \wedge n,\ x \wedge y$	2	[(1,1)(11)] or [(1,111)]	D or O
R_8	$l \wedge x,\ l \wedge y$	$1(l \wedge x)$	[(211)]	N
R_8	$l \wedge x,\ l \wedge y$	2	[(211)]	N or O
R_9	$l \wedge n,\ l \wedge x, l \wedge y$	1 $(l \wedge n)$	[(1,1)(11)]	D
R_9	$l \wedge n,\ l \wedge x,\ l \wedge y$	2 $(l \wedge n,\ l \wedge x)$	[(31)]	III
R_9	$l \wedge n,\ l \wedge x,\ l \wedge y$	2 $(l \wedge n,\ l \wedge x)$	[2(11)]	II
R_9	$l \wedge n,\ l \wedge x,\ l \wedge y$	3	[(31)]	III
R_9	$l \wedge n,\ l \wedge x,\ l \wedge y$	3	[2(11)]	II or D
R_9	$l \wedge n,\ l \wedge x,\ l \wedge y$	3	[(1,1)(11)]	II or D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	1 $(l \wedge n)$	[(1,1)(11)]	D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	1 $(l \wedge x)$	[(211)]	N
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	1 $(x \wedge z)$	[(1,1)(11)]	D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(l \wedge n,\ l \wedge x)$	[(31)]	III
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(l \wedge n,\ l \wedge x)$	[2(11)]	II
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(l \wedge n,\ n \wedge x)$	[(1,1)11] or $[z\,\bar{z}\,11]$ or $[z\,\bar{z}\,(11)]$	I
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(l \wedge n,\ n \wedge x)$	[(1,1)11] or [1,1(11)]	I or D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(l \wedge n,\ n \wedge x)$	[211]	II
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(u \wedge x,\ u \wedge z)$	[1,111]	I
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(u \wedge x,\ u \wedge z)$	[(1,1)(11)]	D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	2 $(u \wedge x,\ u \wedge z)$	[(1,1)11] or [1,1(11)]	I or D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[(1,11)1]	O
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[1,111] or [(1,1)11] or [1,1(11)] or $[z\,\bar{z}\,11]$ or $[z\,\bar{z}\,(11)]$	I
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[(1,1)11] or [1,1(11)] or [(1,1)(11)] or [1,(111)] or [(1,11)1]	D
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[211] or [(21)1] or [2(11)]	II
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[31] or [(31)]	III
R_{10}	$l \wedge n,\ l \wedge x,\ n \wedge x$	3	[(21)1]	N
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	1 $(x \wedge y)$	[(1,1)(11)]	D
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	2 $(l \wedge x,\ x \wedge y)$	[(31)]	III
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	2 $(l \wedge x,\ x \wedge y)$	[2(11)]	II or D
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	3	[(31)]	III
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	3	[2(11)]	II or D
R_{11}	$l \wedge x,\ l \wedge y,\ x \wedge y$	3	[(1,1)(11)]	II or D
R_{12}	$l \wedge x,\ l \wedge y,$ $l \wedge n + \rho(x \wedge y)$	≥ 2	[(31)]	III
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	1 $(x \wedge y)$	[(1,1)(11)]	D
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	2 $(x \wedge y,\ y \wedge z)$	[1,111]	I
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	2 $(x \wedge y,\ y \wedge z)$	[1,1(11)]	D
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	3	[1,111] or [(1,1)11]	I
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	3	[1,1(11)] or [(1,1)(11)] or [(1,11)1]	D
R_{13}	$x \wedge y,\ y \wedge z,\ x \wedge z$	3	[1,(111)]	O
R_{14}	$l \wedge n,\ x \wedge y,$ $l \wedge x,\ l \wedge y$	≥ 2	[(1,1)11] or [211] or [31] or a degeneracy of one of these	Alg. special

3.HOMOTHETIC VECTOR FIELDS WITH FIXED POINTS

Let (M, g) be a pseudo-riemannian manifold and X be a vector field on M. A point $p \in M$ is said to be a **fixed point** of X if $X_p = 0$.

It is well known [55] that, if (M, g) is a remannian manifold and it admits a proper homothetic vector field X whith a fixed point, then there exists an open neighbourhood U of p in M such that $(U, g_{|U})$ is flat. In this chapter we analyse the case of space-time, for which the situation is quite different.

Although most of the work and results in this chapter are due to Alekseevski [3] and Hall [38], some different approaches and techniques are discussed and more details are given.

3.1. The geometric structure at the fixed point of a proper homothetic vector field.

Given two pseudo-riemannian manifolds (M, g) and (M', g'), a **proper homothetic map** from M to M' is a diffeomorphism $f : M \to M'$ such that, for all $p \in M$ and all $v, w \in T_p(M)$

$$g'_{f(p)}(f_{*p}v, f_{*p}w) = c^2 g_p(v, w),$$

where c is a non-zero real number which does not depend on p, v and w. The constant c is called the **homothetic constant** of f.

A differentiable map $f : M \to M'$ is said to be **locally proper homothetic** at $p \in M$ if there exists an open neighbourhood U of p in M such that $f_{|U} : U \to f(U)$ is proper homothetic.

Given a locally proper homothetic map $f : M \to M$, we say that a point $p \in M$ is a **fixed point** of f if

$$f(p) = p.$$

We have then the following result due to Beem [5]

THEOREM 3.1.1. *Let (M, g) be a space-time and let $f : M \to M$ be a differentiable map, $p \in M$ be such that $f(p) = p$. If f is locally proper homothetic at p and if all the eigenvalues of the linear map f_{*p} are in absolute value < 1, there exists an open neighbourhood U of p in M such that $(U, g_{|U})$ is flat.*

If X is a proper homothetic vector field of (M, g), given any point $p \in M$ the elements of the one parameter local group of X in some open neighbourhood of p are locally proper homothetic maps and it is clear that a fixed point of X gives rise to a fixed point of each of these maps. Thus, the above result can be used in the study of the fixed points of X.

Let then X be a proper homothetic vector field of a space-time (M, g). There exists then a real constant λ such that (cf. §.1.9)

$$\mathcal{L}_X g = 2\lambda g. \tag{3.1.1}$$

We shall call λ the homothetic constant of X. We introduce also, for an arbitrary vector field Z, the tensor field f^Z given locally by

$$f^Z_{ab} = \frac{1}{2}(Z_{a;b} - Z_{b;a}), \tag{3.1.2}$$

and we call it the bivector of Z. When no confusion arises the superscript Z will be dropped.

Let p be a fixed point of the proper homothetic vector field X, so that $X_p = 0$. Consider then an open neighbourhood D of 0 in $T_p(M)$ (for the natural topology), such that D is a normal coordinate domain (cf. §.1.7), and consider the exponential map

$$exp_p : D \to U,$$

and, choosing a basis (e_a) of $T_p(M)$, consider the normal coordinate system associated with it.

Going now back to the vector field X, we have, with the above notation,

$$X_{a;b} = \lambda g_{ab} + f_{ab}. \tag{3.1.3}$$

Consider then the endomorphism M of $T_p(M)$, given in the normal coordinates above, by

$$M(\partial_{a|p}) = (\lambda \delta^b{}_a + f^b{}_a(p))\partial_{b|p}. \tag{3.1.4}$$

Consider the vector field Y in D defined, for every vector $v \in D$, by

$$Y_v = M(v). \tag{3.1.5}$$

From T.1.9.3.(a) one deduces then that [3], [38]

THEOREM 3.1.2. *The local one parameter group of Y at 0 is the local expression of the local one-parameter group of X at p in the normal coordinate system above.*

Consider the metric h defined on the open neighbourhood D defined above by $h_{ab} = g_{ab}(p)$. Then the vector field Y defined above is a proper homothetic vector field of the flat manifold (D, h). Combining this result with the preceding theorem and the observation that the elements of the one-parameter local group of Y are locally proper homothetic maps we get

THEOREM 3.1.3. *Let (M, g) be a pseudo-riemannian manifold and X be a proper homothetic vector field of (M, g). Let $p \in M$ and let σ_t be the elements of the one-parameter local group of X at p. Let $t \in \mathbb{R}$ and $q \in M$ be such that q lies in the domain of definition of σ_t. Then σ_t is locally proper homothetic at q and its homothetic constant is $e^{\lambda t}$, where λ is the homothetic constant of X.*

Let now $(\mathcal{M}, \eta)$ denote Minkowski space.

As follows immediately from T.1.9.3.(b) and the fact that Minkowski space is flat, the bivector of a homothetic vector field in Minkowski space is a constant bivector. Thus, in Minkowski space a proper homothetic vector field X with homothetic constant λ $(\neq 0)$ is given by

$$X_{a,b} = \lambda\eta_{ab} + f_{ab}, \tag{3.1.6}$$

that is, in matrix form

$$(X_{a,b}) = \lambda\mathcal{E} + \mathcal{F}, \tag{3.1.7}$$

where $\mathcal{E}$ and $\mathcal{F}$ denote the matrices of η and f respectively. From the preceding theorem and from the above relations we get therefore, identifying the points of Minkowski space to vectors

$$X_p = (\lambda I + \mathcal{F}')(p) + k, \tag{3.1.8}$$

for all $p \in \mathcal{M}$, where I denotes the identity map and $\mathcal{F}'$ represents the linear map one obtains by raising the first index in f_{ab}. k is obviously the value of X at the origin of the coordinate system. Let us denote by $\mathcal{N}$ the linear map $\lambda I + \mathcal{F}'$. Then we have

Lemma 3.1.5. *X has a fixed point if and only if k lies in the image of $\mathcal{N}$. In particular, X has a unique fixed point if $\det \mathcal{N} \neq 0$.*

This is obvious.

This results lead directly to the consideration of the bivector f of X at a fixed point.

Let p be a fixed point of X and choose normal coordinates (x^i) with origin at p and such that the vector fields ∂_i form a null-tetrad.

In these coordinates

$$\mathcal{E} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1.9}$$

The coordinate system above can of course be chosen to be adapted to the bivector f (in the sense that the null tetrad associated with the coordinates is such that with respect to it, f takes one of the canonical forms described in Ch.2. p.53).

Suppose first that the bivector f is null at the fixed point. Then the null-tetrad above can be so chosen that $f = edx^2 \wedge dx^4$, that is,

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \\ 0 & -e & 0 & 0 \end{pmatrix},$$

so that, raising indices, we get

$$\mathcal{N} = \begin{pmatrix} \lambda & 0 & 0 & e \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & -e & 0 & \lambda \end{pmatrix}. \tag{3.1.10}$$

It follows that $\det \mathcal{N} = \lambda^4 \neq 0$, and so p is the unique fixed point of X. The same result holds in the case when $f = 0$ (i.e. $e = 0$).

If f is simple-spacelike the null tetrad can be so chosen that $f = bdx^3 \wedge dx^4$. A simple calculation shows then that

$$\mathcal{N} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & b \\ 0 & 0 & -b & \lambda \end{pmatrix}. \tag{3.1.11}$$

It follows that $\det \mathcal{N} = \lambda^2(\lambda^2 + b^2) \neq 0$, and so p is the unique fixed point of X.

Finally, let us consider the simple-timelike and the non-simple cases together. In these cases the null tetrad above can be so chosen that

$$f = adx^1 \wedge dx^2 + bdx^3 \wedge dx^4$$

(with $b = 0$ in the simple-timelike case), so

$$\mathcal{N} = \begin{pmatrix} \lambda - a & 0 & 0 & 0 \\ 0 & \lambda + a & 0 & 0 \\ 0 & 0 & \lambda & b \\ 0 & 0 & -b & \lambda \end{pmatrix}. \tag{3.1.12}$$

This gives $\det \mathcal{N} = (\lambda^2 - a^2)(\lambda^2 + b^2)$. Thus, when $\lambda \neq \pm a$ the origin of the coordinates is the unique fixed point. We are left with the cases $\lambda = \pm a$. Let us analyse the case $\lambda = a$ (the case $\lambda = -a$ leads to identical results). In this case $\mathcal{N}$ has zero determinant and rank 3, so all points in its kernel (spanned by the null vector ∂_1) are fixed points of X. Notice that in this case, the set of fixed points of X is a null geodesic.

Now as the origin of the coordinate system is the fixed point p we have in the coordinates (x^i) described precedingly

$$X_p = \mathcal{N}(p).$$

This shows that the differential equation which defines the local one parameter group of X around p is given by

$$\frac{d\psi}{dt}_{|t} = \mathcal{N}(\psi(t)).$$

As $\mathcal{N}$ is constant, the solutions of the above equation are given by

$$\psi(t,p) = e^{t\mathcal{N}}(p), \tag{3.1.13}$$

where $e^{t\mathcal{N}}$ is the exponential matrix of $\mathcal{N}$. Now, the special form of $\mathcal{N}$ shows that

$$e^{t\mathcal{N}} = e^{\lambda t} e^{t\mathcal{F}'}. \tag{3.1.14}$$

One can then use the canonical forms of the matrix $\mathcal{F}'$ corresponding to the possible types of the bivector f to study the eigenvalue structure of the elements of the local 1 parameter group of X at p.

In the null case one finds that, in the appropriate null tetrad used before, we have

$$e^{t\mathcal{F}'} = \begin{pmatrix} 1 & -\frac{e^2t^2}{2} & 0 & et \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -et & 0 & 1 \end{pmatrix}, \tag{3.1.15}$$

and this shows the eigenvalues of this matrix are all equal to 1. Thus the eigenvalues of $e^{t\mathcal{N}}$ are all equal to $e^{\lambda t}$. For values of t for which $\lambda t < 0$ these are in absolute value < 1. This applies also to the case when $f = 0$.

In the simple spacelike case a simple calculation yields

$$e^{t\mathcal{N}} = \begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 \\ 0 & e^{\lambda t} & 0 & 0 \\ 0 & 0 & e^{\lambda t}\cos tb & e^{\lambda t}\sin tb \\ 0 & 0 & -e^{\lambda t}\sin tb & e^{\lambda t}\cos tb \end{pmatrix}, \tag{3.1.16}$$

and so the eigenvalues are in this case $e^{(\lambda \pm ib)t}$ and $e^{\lambda t}$ (double). We see that whenever $\lambda t < 0$ these eigenvalues are in absolute value < 1.

In the simple-timelike and non-simple cases, using the form of $\mathcal{N}$ introduced above, we find that

$$e^{t\mathcal{N}} = \begin{pmatrix} e^{(\lambda-a)t} & 0 & 0 & 0 \\ 0 & e^{(\lambda+a)t} & 0 & 0 \\ 0 & 0 & e^{\lambda t}\cos tb & e^{\lambda t}\sin tb \\ 0 & 0 & -e^{\lambda t}\sin tb & e^{\lambda t}\cos tb \end{pmatrix}. \tag{3.1.17}$$

This matrix has eigenvalues $e^{(\lambda-a)t}$, $e^{(\lambda+a)t}$ and $e^{(\lambda\pm ib)t}$. Without loss of generality we may assume from now on that $\lambda > 0$ and $a > 0$.

Then, when $\lambda > a$, all the eigenvalues are < 1 in absolute value (for some value of t).

When $\lambda = a$, ∂_1 is an eigenvector with eigenvalue 1.

When $0 < \lambda < a$, there is, for all possible values of t, an eigenvalue which is in absolute value > 1.

This analysis has been made for proper homothetic vector fields in Minkowski space. In particular this study applies to the vector field Y of the pseudo-riemannian manifold (D, h) defined previously in p.64.

It follows then from T.3.1.2 that these results can be applied to the general case. The above results on the eigenvalues, combined then with T.3.1.1, give [3], [38]

THEOREM 3.1.6.*Let (M, g) be a space-time, X be a proper homothetic vector field of (M, g). Then*

(a). The set $F(X)$ of fixed points of X is either a discrete subset of M or (part of) a null geodesic of M. Furthermore, this last case occurs only when the bivector of X at a fixed point is either simple-timelike or non-simple with $\lambda = \pm a$;

(b). If at a fixed point p of X the bivector of X is zero, simple null, simple spacelike, or simple timelike or non-simple with $\lambda > a$, then there exists an open neighbourhood U of p such that $(U, g_{|U})$ is flat.

This theorem leaves to be analysed the cases when the bivector f at a fixed point is either simple-timelike or non-simple with (assuming $\lambda > 0$ and $a > 0$) $\lambda \leq a$. We shall study the cases $a = \lambda$ and $a > \lambda$ separately in the next sections.

3.2.General comments. Segre and Petrov types.

Let the (M, g) be a space-time and X be a proper homothetic vector field of (M, g) with a fixed point p, such that at p the bivector f of X is either simple-timelike or non-simple.

We assume in the sequel that X has been so scaled that its homothetic constant is $\lambda = 1$.

Let U be a normal coordinate neighbourhood of p in M (cf. §.1.7). We identify U with $exp_p^{-1}(U)$. We have then, by T.3.1.2

$$\sigma_t(q) = \sigma_t^{Y}(q),$$

wher σ_t is the local one-parameter group of X at p and σ_t^{Y} is the local one-parameter group of the vector field Y (defined in (3.1.5)) at 0.

In the sequel we do not distinguish these local one-parameter groups and so we shall drop the superscript Y.

By choosing then an appropriate null tetrad (l, n, z, w) (so as to have the bivector in canonical form) we have, as shown in the preceding section

$$\sigma_t(p) = e^t e^{t\mathcal{F}'}(q). \tag{3.2.1}$$

Now as we assume that f is simple-timelike or non-simple, this gives, by (3.1.17), writing $q = ul + vn + xz + yw$

$$\sigma_t(q) = (e^{(1-a)t}u, e^{(1+a)t}v, e^t(Cx + Dy), e^t(-Dx + Cy)), \tag{3.2.2}$$

where we have set $C = \cos tb$ and $D = \sin tb$. It follows that in the above coordinates (u, v, x, y) the vector field X has the following canonical form

$$X_{(u,v,x,y)} = (1-a)u\partial_u + (1+a)v\partial_v + (x + by)\partial_x + (y - bx)\partial_y. \tag{3.2.3}$$

Now, as each σ_t is a proper homothetic map (with homothetic constant e^t, cf. T.3.1.3) we have for $q \in U$ and $\nu, \mu \in T_q(M)$

$$g_{\sigma_t(q)}(\sigma_{t*q}\nu, \sigma_{t*q}\mu) = e^{2t} g_q(\nu, \mu). \tag{3.2.4}$$

This reads in matrix form

$$^t(e^{t\mathcal{F}'})G(\sigma_t(q))e^{t\mathcal{F}'} = G(q), \tag{3.2.5}$$

where $G(q)$ is the matrix of g_q.

Now (3.2.5) gives, using (3.1.17)

$$\begin{aligned} e^{-2at}g_{11}o\sigma_t &= g_{11}, \\ g_{12}o\sigma_t &= g_{12}, \\ e^{2at}g_{22}o\sigma_t &= g_{22}, \end{aligned} \tag{3.2.6}$$

$$\begin{aligned} Cg_{13}o\sigma_t - Dg_{14}o\sigma_t &= e^{at}g_{13}, \\ Dg_{13}o\sigma_t + Cg_{14}o\sigma_t &= e^{at}g_{14}, \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} Cg_{23}o\sigma_t - Dg_{24}o\sigma_t &= e^{-at}g_{23}, \\ Dg_{23}o\sigma_t - Cg_{24}o\sigma_t &= e^{-at}g_{24}, \end{aligned} \tag{3.2.8}$$

and

$$\begin{aligned} C^2g_{33}o\sigma_t - 2CDg_{34}o\sigma_t + D^2g_{44}o\sigma_t &= g_{33}, \\ DCg_{33}o\sigma_t + (C^2 - D^2)g_{34}o\sigma_t - CDg_{44}o\sigma_t &= g_{34}, . \\ D^2g_{33}o\sigma_t + 2CDg_{34}o\sigma_t + C^2g_{44}o\sigma_t &= g_{44} \end{aligned} \tag{3.2.9}$$

On the other hand, we have

$$\partial_{j|q}(g_{ab}o\sigma_t) = \partial_{k|\sigma_t(q)}g_{ab}.\partial_{j|q}(\sigma_t)^k.$$

Since

$$\partial_{j|q}(\sigma_t)^k = (e^{t\mathcal{N}})_j{}^k,$$

we get

$$\partial_{j|q}(g_{ab}o\sigma_t) = (e^{t\mathcal{N}})_j{}^k\partial_{k|\sigma_t(q)}g_{ab}. \tag{3.2.10}$$

Consider now the Ricci tensor. One has $\mathcal{L}_X Ricci = 0$ (cf. §.1.9) and this reads

$$X^c\mathbf{R}_{ab;c} + X^c{}_{;a}\mathbf{R}_{cb} + X^c{}_{;b}\mathbf{R}_{ac} = 0. \tag{3.2.11}$$

Thus, at the fixed point p of X we have, using (3.1.3)

$$2\mathbf{R}_{ab} + f^c{}_a\mathbf{R}_{cb} + f^c{}_b\mathbf{R}_{ac} = 0. \tag{3.2.12}$$

Let then (l, n, x, y) be a principal null tetrad of the bivector f, so that we have

$$f_{ab} = 2al_{[a}n_{b]} + 2bx_{[a}y_{b]}. \qquad (3.2.13)$$

Expanding the Ricci tensor with respect to this null tetrad (see Ch.2, (2.5.5)) we have

$$\mathbf{R}_{ab} = 2R^1 l_{(a}n_{b)} + R^2 l_a l_b + R^3 n_a n_b + 2R^4 l_{(a}x_{b)} + 2R^5 l_{(a}y_{b)} \qquad (3.2.14)$$
$$+ 2R^6 n_{(a}x_{b)} + 2R^7 n_{(a}y_{b)} + 2R^8 x_{(a}y_{b)} + R^9 x_a x_b + R^{10} y_a y_b.$$

Replacing in (3.2.13) we get, after performing some contractions, the following result due to Hall [38]

THEOREM 3.2.1. *Let (M, g) be a space-time and X be a proper homothetic vector field of (M, g), with homothetic constant $\lambda > 0$. Let f be the bivector of X at a fixed point p of X and (l, n, x, y) be a null tetrad such that*

$$f_{ab} = 2al_{[a}n_{b]} + 2bx_{[a}y_{b]}.$$

Then
(a).If $a = \lambda$ the Ricci tensor of (M, g) at p is either zero or its Segre type is $\{(211)\}$ with zero eigenvalue. In this last case the Ricci tensor at p is given by $\mathbf{R}_{ab} = Al_a l_b$;
(b).If $a > \lambda$ but $a \neq 2\lambda$ or if $b \neq 0$ the Ricci tensor vanishes at p;
(c).If $a = 2\lambda$ and $b = 0$ at p the Ricci tensor is either zero or its Segre type is $\{(31)\}$ with zero eigenvalue. In this last case by performing a rotation in the (x, y)-plane the null tetrad can be so chosen that $R_{ab} = 2Al_{(a}x_{b)}$ at p.

We give, as an example a proof of (c).

Assume that $\lambda = 1$.

Starting from (3.2.12), (3.2.13) and (3.2.14), we get, contracting with l^a

$$2R^1 l_b + 2(1 + a)R^3 n_b + ((2 + a)R^6 - bR^7)x_b + (bR^6 + (2 + a)R^7)y_b = 0,$$

and so we have $R^1 = R^3 = R^6 = R^7 = 0$. Using this fact and contracting then (3.2.12) with n^a we get

$$2(1 - a)R^2 l_b + ((2 - a)R^4 - bR^5)x_b + (bR^4 + (2 - a)R^5)y_b = 0.$$

As $a = 2$ in this case we have therefore $R^2 = 0$.

Contracting then (3.2.12) with x^a we get $R^8 = R^9 = 0$. Finally, contracting with y^a we get $R^{10} = 0$. This shows that

$$\mathbf{R}_{ab} = 2R^4 l_{(a}x_{b)} + 2R^5 l_{(a}y_{b)},$$

and proves (c).

A similar type of argument can be used to study the Petrov type of the Weyl tensor at the fixed point. In this case however, such a process is very long. Hall [38] studied instead the eigenvalues of the Weyl and the Ricci tensors and proved that they all must vanish at the fixed point. Thus (cf. §.2.2) the possible Petrov types are either 0, N or III. More precisely, we have the following result due to Hall [38]

THEOREM 3.2.2. *With the notations of the preceding theorem*
(a).If $a = \lambda$ and $b = 0$ then the Petrov type at p is either 0 or N;
(b).If $a = \lambda$ and $b \neq 0$ then the Petrov type at p is 0;
(c).If $a > \lambda$ but $a \neq 2\lambda$ or if $b \neq 0$, the Petrov type at p is 0;
(d).If $a = 2\lambda$ and $b = 0$,then the Petrov type at p is either III or 0.

NOTE.3.2.1. A proof of (a) and (b) of the above theorem, which differs slightly from that of Hall [38], will be given in the next section. A proof of (c), different from that of Hall [38], will be given in §.3.4. As for (d), the proof can be gathered from [38].

Let us introduce a definition. Let p be a fixed point of the homothetic vector field X, and let $q \in M$. We say that q **converges** to p if the maximal integral curve of X through q gets arbitrarily close to p. A subset S of M will be said to **converge** to p if all its points converge to p. Finally a subset S of M will said to be X-**convergent** if every point q of S converges to some fixed point of X.

It has been proved by Hall [38] that if a point q converges to a fixed point p of X then the eigenvalues of both the Ricci and Weyl tensors at q are zero.

In the case when $a = \lambda(= 1)$, take a point $q = (u, v, x, y)$ in the normal coordinate neighbourhood U of a fixed point p. Then we have, from (3.2.2)

$$\sigma_t(u, v, x, y) = (u, e^{2t}v, e^t(Cx + Dy), e^t(-Dx + Cy)).$$

Assuming U to be convex (cf. §.1.7), we see that for every $t < 0$ the point $\sigma_t(q)$ lies in U. Taking then limits as $t \to -\infty$ we get

$$\lim_{t\to-\infty} \sigma_t(u, v, x, y) = (u, 0, 0, 0),$$

which, as follows from (3.2.3) is a fixed point of X. U is, therefore, a X-convergent set.

In the case when $a > \lambda(=1)$, (3.2.2) gives, setting $\rho = \lambda - a < 0$,

$$\sigma_t(u,v,x,y) = (e^{\rho t}u, e^{(1+a)t}v, e^t(Cx+Dy), e^t(-Dx+Cy)),$$

and so we see that in this case the X-convergent sets are the hypersurface $S = \{q \in U : u = 0\}$ and the line $L = \{q \in U : v = x = y = 0\}$.

Thus in the first case the eigenvalues of both the Ricci and Weyl tensors are identically zero in U. In the second case this holds in the line L and in the hypersurface S.

NOTE.3.2.2. It should be noticed, however, that both the Segre and the Petrov types may change in the X-convergent sets (but not along the integral curves of a homothetic vector field [38]). This happens, for instance, in the first example at the end of this chapter; in this example the fixed point is isolated and both the Ricci and Weyl tensors at the fixed point are zero. However, along the line $L' = \{q \in U : u = v = y = 0, x > 0\} \subset S$ the Ricci tensor is of Segre type $\{(211)\}$ and the Weyl tensor is of Petrov type N.

3.3.The case $a = 1$ (null geodesic of fixed points).

In this case, using the same notation as in the preceding sections, we have $\sigma_t(q) = e^{t\mathcal{N}}(q)$, and it is clear that (cf. (3.2.2)) for every q in U $e^{t\mathcal{N}}(q)$ lies in U for all values of $t < 0$.

Take then the limit as $t \to -\infty$ of the third relation in (3.2.6) and of relations (3.2.9). We get

$$g_{22} = g_{23} = g_{24} = 0.$$

Consider now the second relation in (3.2.6) and the relations in (3.2.8) and (3.2.9). They are of the form

$$e^{\rho t} A^{ab} g_{ab} o \sigma_t = g_{cd},$$

where the A^{ab} stand for $C, D, 1$ and $\rho = \pm 1, 0$. From (3.2.2) we get

$$e^{\rho t} A^{ab} g_{ab}(u, e^{2t}v, e^t(Cx+Dy), e^t(-Dx+Cy)) = g_{cd}(u,v,x,y).$$

A derivation with respect to v gives

$$e^{(\rho+2)t} A^{ab} g_{ab,2} o \sigma_t = g_{cd,2},$$

with $\rho+2>0$. Taking limits as $t \to -\infty$ we see that $g_{ab,2}=0$, except in the case when $(a,b)=(1,1)$.

For (3.2.9) we have $\rho=0$, so a derivation with respect to x or y shows, taking limits as $t \to -\infty$, that $g_{33,c}=g_{34,c}=g_{44,c}=0$, whenever $c=3,4$. It follows that g_{33}, g_{34} and g_{44} depend only on the variable u. The same in fact applies also to g_{12}. For (3.2.7) we have to perform two derivations to obtain a similar result, that is, we have $g_{13,cd}=g_{14,cd}=0$, whenever $c,d=3,4$. For g_{11} we get $g_{11,cde}=0$, whenever $c,d,e=3,4$, and also $g_{11,22}=0$. Combining these results we get

THEOREM 3.3.1. *In the case $a=1$ the matrix of g with respect to the normal coordinate system defined by the principal null tetrad of the bivector f is of the form*

$$g=\begin{pmatrix} \alpha v+\beta & \nu & \gamma x+\delta y+\pi & \mu x+\rho y+\theta \\ \nu & 0 & 0 & 0 \\ \gamma x+\delta y+\pi & 0 & \epsilon & \tau \\ \mu x+\rho y+\theta & 0 & \tau & \sigma \end{pmatrix}$$

where $\alpha,\gamma,\delta,\pi,\mu,\rho,\sigma,\ \theta,\tau$ and ϵ are differentiable functions of u and β depends an u,x,y and is given by

$$\beta(u,x,y)=A(u)x^2+B(u)xy+C(u)y^2,$$

where A,B,C are differentiable functions of u.

This leads then to the following result [38]

THEOREM 3.3.2. *Let (M,g) be a space-time. Assume (M,g) admits a proper homothetic vector field X which admits a null geodesic γ of fixed points. Then, given $p\in\gamma$ there exists an open neighbourhood U of p in M such that $(U,g_{|U})$ admits a null covariantly constant vector field l. Furthermore, $(U,g_{|U})$ is (part of) a generalized plane wave space-time* [19] [56].

NOTE.3.3.1. A much shorter and elegant proof of this fact can be found in [38] which contains further details about this case. It is well known [19] that these generalized pp-wave space-times if non-vacuum are of Segre type $\{(211)\}$ with zero eigenvalue and that their Petrov type is either 0 or N. This confirms (a) of T.3.2.1.

In the case when f is non-simple one can in fact prove that in T.3.3.1 we have $\pi=\theta=0$, $\mu=-\delta$, $\rho=\gamma$, $\sigma=\epsilon$, $\tau=0$ and that $b(u,x,y)=\beta(u)(x^2+y^2)$. Thus in this case the metric reduces to

$$g = \begin{pmatrix} \alpha v + \beta(x^2+y^2) & \nu & \gamma x + \delta y & -\delta x + \gamma y \\ \nu & 0 & 0 & 0 \\ \gamma x + \delta y & 0 & \epsilon & 0 \\ -\delta x + \gamma y & 0 & 0 & \epsilon \end{pmatrix},$$

This shows then that [38]

THEOREM 3.3.3. *In the non-simple case, the vector field given in the normal coordinates defined by the principal null tetrad of f at a fixed point, by*

$$Y = y\partial_x - x\partial_y,$$

is a Killing vector field. Moreover, (M, g) is conformally flat.

NOTE 3.3.2. That (M, g) is conformally flat in this case can be proved by direct calculation of the Weyl tensor or by using the results in [19]. This theorem confirms (b) of T.3.2.2.

3.4.The cases when $a > 1$ (isolated fixed point case).

In this case, as noticed in §.3.1, for all t, σ_{*t} always has an eigenvalue in absolute value > 1. Because of this fact there exist points in U which when transformed by the σ_t eventually get out of U as t grows in absolute value (as shown in §.3.2). Thus,in this case, we cannot take limits as $t \to \pm\infty$ as we have done in the preceding case. This fact makes the study of this case much harder to handle that the preceding one. The results of Hall [38] mentioned in §.3.2, concern both the Segre and Petrov types at the fixed point and in some given submanifolds of M (the hypersurface S and the line L) through the isolated fixed point, as we have seen. Unfortunately, these results do not tell us very much about the global structure of the space-times that admit such homothetic vector fields.

In the sequel we give a proof of T.3.2.1.(c) and of T.3.2.2.(c) which differs from that given by Hall in [38], being closer to Alekseevski's method [3].

We will end the section with two examples intended to show that there exist families of physically significant space-times (in the sense that they satisfy the energy conditions introduced in Ch.2) which admit such vector fields.

In the normal coordinates (u,v,x,y) the proper homothetic vector field X takes the form

$$X = (1-a)u\partial_u + (1+a)v\partial_v + (x+by)\partial_x + (y-bx)\partial_y. \qquad (3.4.1)$$

Thus, after some simple computations, the equation $\mathcal{L}_X g = g$ (recall that we assume X scaled so as to have $\lambda = 1$) reduces to the following system of partial differential equations where a comma denotes partial differentiation

$$\begin{aligned} X^c g_{11,c} &= 2a g_{11}; \\ X^c g_{12,c} &= 0, \\ X^c g_{22,c} &= -2a g_{22}; \end{aligned} \qquad (3.4.2)$$

$$\begin{aligned} X^c g_{13,c} &= a g_{13} + b g_{14}, \\ X^c g_{14,c} &= -b g_{13} + a g_{14}; \end{aligned} \qquad (3.4.3)$$

$$\begin{aligned} X^c g_{23,c} &= -a g_{23} + b g_{24}, \\ X^c g_{14,c} &= -b g_{23} - a g_{24}; \end{aligned} \qquad (3.4.4)$$

and,

$$\begin{aligned} X^c g_{33,c} &= 2b g_{34}, \\ X^c g_{34,c} &= -b g_{33} + b g_{44}, \\ X^c g_{44,c} &= -2b g_{34}. \end{aligned} \qquad (3.4.5)$$

Using these equations it is easy to prove that if $b \neq 0$ then the first and second derivatives of the g_{ab} with respect to all variables vanish at the fixed point of X.

As an example take the second of these systems. Differentiating the first relation with respect to the third variable we get

$$X^c g_{13,c3} + X^c{}_{,3} g_{13,c} = a g_{13,c} + b g_{14,c}.$$

Since $X^1{}_{,3} = X^2{}_{,3} = 0$ and $X^3{}_{,3} = 1$ and $X^4{}_{,4} = -b$, this relation gives, at the fixed point

$$(a-1) g_{13,3} + b g_{13,4} + b g_{14,3} = 0.$$

Similar computations with the remaining equation lead to the following linear system where all functions have been evaluated at the fixed point

$$\begin{aligned} (a-1) g_{13,3} + b g_{13,4} + b g_{14,3} &= 0, \\ -b g_{13,3} + (a-1) g_{13,4} + b g_{14,4} &= 0, \\ -b g_{13,3} + (a-1) g_{14,3} + b g_{14,4} &= 0, \\ -b g_{13,4} - b g_{14,3} + (a-1) g_{14,4} &= 0. \end{aligned}$$

Since the matrix of this system has non-zero determinant we deduce that, at the fixed point we have

$$g_{13,3} = g_{13,4} = g_{14,3} = g_{14,4} = 0.$$

A similar process can be applied to the remaining equations, thus giving

Lemma 3.4.1 *Let (M,g) be a space-time and X a proper homothetic vector field of M which admits an isolated fixed point p. If at p the bivector of X is non-simple the Riemann tensor vanishes at p.*

Let us assume from now on that $b = 0$. In such case, the above equations can all be written under the form

$$X^c g_{ab,c} = \nu a g_{ab}, \tag{3.4.6}$$

where $\nu = 2$ if $(a,b) = (1,1)$, $\nu = 1$ if $(a,b) = (1,3)$ or $(1,4)$, $\nu = 0$ if $(a,b) = (1,2), (3,3), (3,4)$ or $(4,4)$, $\nu = -1$ if $(a,b) = (2,3)$ or $(2,4)$ and $\nu = -2$ if $(a,b) = (2,2)$. Setting now $(u,v,x,y) = (x^1, x^2, x^3, x^4)$ and defining $k_1 = (1-a)$, $k_2 = (1+a)$, $k_3 = k_4 = 1$ and using the expression for the vector field X, this equation can be written

$$\sum_i k_i x^i g_{ab,i} = \nu a g_{ab}. \tag{3.4.7}$$

Taking derivatives with respect to x^j this gives

$$\sum_i k_i x^i g_{ab,ij} + k_j g_{ab,j} = \nu a g_{ab,j}. \tag{3.4.8}$$

Thus, at the fixed point we have

$$(\nu a - k_j) g_{ab,j} = 0. \tag{3.4.9}$$

Going back to (3.4.8) and taking derivatives now with respect to the variable x^l we get

$$\sum_i k_i x^i g_{ab,ijl} + k_l g_{ab,lj} + k_j g_{ab,jl} = \nu a g_{ab,jl}. \tag{3.4.10}$$

Thus, at the fixed point we have

$$(\nu a - k_l - k_j) g_{ab,jl} = 0. \tag{3.4.11}$$

By induction we see that, at the fixed point, for all $m \geq 1$ and all choices of $i_1, ..., i_m$ in $\{1,2,3,4\}$ we have

$$(\nu a - \sum_{1 \leq p \leq m} k_{i_p}) g_{ab,i_1 i_2 \dots i_m} = 0. \tag{3.4.12}$$

Let us set

$$\Theta_{i_1,\dots,i_m} = \nu a - \sum_{1\le p\le m} k_{i_p}. \qquad (3.4.13)$$

From (3.4.12) we see that whenever $\Theta_{i_1\dots i_m} \neq 0$ we have $g_{ab,i_1\dots,i_m} = 0$ at the fixed point. So let us study the numbers $\Theta_{i_1\dots i_m}$. To do this let us denote by r_1 the number of terms $i_j = 1$ in the list $i_1,\dots,i_m$, by r_2 the number of terms $i_j = 2$ and by r_3 the number of terms $i_j = 3,4$ in the same list. Then we have

$$\Theta_{i_1\dots i_m} = \nu a - r_1(1-a) - r_2(1+a) - r_3,$$

that is

$$\Theta_{i_1\dots i_m} = (\nu + r_1 - r_2)a - (r_1 + r_2 + r_3). \qquad (3.4.14)$$

Notice that $r_1 + r_2 + r_3$ is exactly the order of the corresponding derivative. From this expression we see that $\nu + r_1 - r_2 = 0$ is not compatible with $\Theta_{i_1\dots i_m} = 0$, as we must have $r_1 + r_2 + r_3 > 0$.

Thus, we may assume that $\nu + r_1 - r_2 \neq 0$, in which case, $\Theta_{i_1\dots i_m} = 0$ if and only if we have

$$a = \frac{r_1 + r_2 + r_3}{\nu + r_1 - r_2}. \qquad (3.4.15)$$

This immediately shows that (as ν and the r_i are integers) if a is irrational then all the $\Theta_{i_1\dots i_m}$ are non-zero and so all derivatives vanish at the fixed point. Thus, we have by analytic continuation

THEOREM 3.4.2. *If a is irrational, $b = 0$ and (M, g) is analytic then (M, g) is flat.*

Now let us look for conditions on a for which $\Theta_{i_1..i_m} = 0$ when $m = 1, 2$. Such will be the values of a for wich the Riemann tensor of the space-time in question may not vanish at the fixed point.

In the case when we take $m = 1$ only one of the r_i is non-zero. If it is r_1, we have $r_1 = 1$ and so, from (3.4.16)

$$a = \frac{1}{\nu + 1},$$

and since $\nu = -2, -1, 0, 1$ or 2 we see that this is incompatible with the condition that $a > 1$. Similar considerations in the remaining cases show that for $m = 1$ all the $\Theta_{i_1..i_m}$ are non-zero.

In the case when $m = 2$ we have also several cases to consider. When $r_1 = 2$ (so that $r_2 = r_3 = 0$), we have

$$a = \frac{2}{\nu + 2},$$

and taking then $\nu = -1$ we see that $a = 2$ gives $\Theta_{11} = 0$ for $(a, b) = (2,3), (2,4)$.

Similar considerations in the remaining cases show that some of the second order derivatives of the g_{ab} may not vanish at the fixed point only when $a = 2\lambda (= 2)$. This proves T.3.2.2.(c).

The following examples illustrate some of the above results. As it will be proved, the Segre type of the Ricci tensor for these space-times is $\{(211)\}$ with zero eigenvalue (first example) or $\{(1,1),(11)\}$ (second example) where it does not vanish. The first of these space-times may represent either null-electromagnetic fields or pure radiation fields (see Ch.2, p.51).

EXAMPLES.I.-Segre type of the Ricci tensor: $\{(211)\}$ with zero eigenvalue.

Let M be an open connected subset of $\mathbf{R}^4$ and let (u, v, x, y) be (global) coordinates in M, with $0 \in M$ in these coordinates. Let $C : M \to \mathbf{R}$ be a differentiable function on M such that $1 + C$ never vanishes. depending only on the variables u, x. We define a metric g of Lorentz signature on M by

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (1+C)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A simple computation shows that the only component of the Ricci tensor of this metric that is not identically zero is

$$\mathbf{R}_{11} = \frac{-1}{(C+1)} \partial_u^2 C;$$

This together with the above form for g shows that the Ricci tensor of this metric has Segre type $\{(211)\}$ with zero eigenvalue at all points where the Ricci tensor does not vanish.

Let now a be a real number with $a > 1$. Consider the vector field X on M given in the above coordinates by

$$X_{(u,v,x,y)} = (1-a)u\partial_u + (1+a)v\partial_v + x\partial_x + y\partial_y.$$

Computing $\mathcal{L}_X g = 2h$ we get, in the above coordinates

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$F = (C+1)[(1-a)u\partial_u C + x\partial_x C + (C+1)].$$

Thus, X is a proper homothetic vector field of M if we have

$$(1-a)u\partial_u C + x\partial_x C = 0.$$

We see that for every integer value of a, choosing p integer and such that $p \geq 2$ and $q = (a-1)p$ then

$$C(u,x) = \pm u^p x^{(a-1)p},$$

is a solution of this equation defined and differentiable at 0. As a particular example take $a = 4$, $p = 2$ so that $q = 6$. Then, taking $C = -u^2x^6$, we have

$$\mathbf{R}_{11} = \frac{2x^6}{(1-u^2x^6)}.$$

This shows that, choosing M small enough, the energy conditions of Ch.2 are satisfied.

In this particular case the only components of the Riemann tensor which are not identically zero are given by

$$\mathbf{R}^2{}_{313} = 2x^6(1-u^2x^6);$$
$$\mathbf{R}^3{}_{113} = -\frac{2x^6}{(1-u^2x^6)},$$

and so we see that the Riemann tensor vanishes at the fixed point.

In the general case the only components of the Riemann tensor that may not be identically zero are

$$\mathbf{R}^2{}_{313} = -(1+C)\partial_u^2 C;$$
$$\mathbf{R}^3{}_{113} = \frac{1}{(1+C)}\partial_u^2 C.$$

Thus we see that these space-times can only be vacuum if they are locally flat.

The only components of the Weyl tensor that may not be zero are given by

$$\mathbf{C}_{1313} = -\frac{1}{2}(1+C)\partial_u^2 C;$$
$$\mathbf{C}_{1414} = \frac{1}{2(1+C)}\partial_u^2 C.$$

Let then $l = (l^1, l^2, l^3, l^4)$. The equation $C_{abcd}l^d = 0$, evaluated for $(a,b,c) = (3,1,3)$, gives $C_{3131}l^1 = 0$, hence l is a solution of the preceding equation only if $l^1 = 0$. If furthermore we ask that l be null, we have then $l^3 = l^4 = 0$. This shows that the only null solutions to the equation

$$C_{abcd}l^d = 0$$

are of the form $l = l^2\partial_v$. By Bel's criteria (cf. T.2.2.4.(a)) (M,g) is of Petrov type N at every point where the Weyl tensor does not vanish.

One can in fact prove that ∂_v is a covariantly constant null vector field, so that in fact (M,g) is a pp-wave space-time.

EXAMPLES.II- Segre type of the Ricci tensor: $\{(1,1)(11)\}$.

We take again M to be a connected open neighbourhood of $0 \in \mathbf{R}^4$ and we choose coordinates (u,v,x,y) centered at 0.

The metric given in these coordinates by

$$g = \begin{pmatrix} 0 & 1+u^4v^2 & 0 & 0 \\ 1+u^4v^2 & vu^5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

is of Petrov type D and its Ricci tensor is given by

$$\mathbf{R}_{12} = \frac{2vu^3}{(1+u^4v^2)^2},$$
$$\mathbf{R}_{22} = \frac{2v^2u^8}{(1+u^4v^2)^3}.$$

and so has Segre type $\{(1,1)(11)\}$ at all points where it is non-zero. The only component of the Riemann tensor that is not identically zero is

$$\mathbf{R}_{1212} = -\frac{2vu^3}{(1+v^2u^4)}.$$

Notice that (M,g) is decomposable in the sense of §.1.8, with two covariantly constant spacelike vector fields, ∂_x and ∂_y. It follows that (cf. TABLE.I, at the end of Ch.2) it's holonomy group is of type R_2 and consequentely it's Petrov type is D.

One proves easily that the vector field

$$X = -2u\partial_u + 4v\partial_v + x\partial_x + y\partial_y,$$

is a proper homothetic vector field of this space-time.

4.AFFINE VECTOR FIELDS

In Ch.1 we defined an affine vector field of a pseudo-riemannian manifold (M,g). The set of such vector fields, denoted $\mathcal{A}(M,g)$, is a finite-dimensional Lie algebra whose dimension is $\leq n(n+1)$, n being the dimension of M. All homothetic vector fields are affine vector fields and we say that an affine vector field is **proper affine** if it is not homothetic.

As we shall see, in general the Lie algebra $\mathcal{A}(M,g)$ coincides with the Lie algebra $\mathcal{H}(M,g)$ of homothetic vector fields. In such cases the study of affine vector fields reduces to the study of homothetic vector fields for which a large literature already exists (see, for instance [25], [26], [38], [39], [64]).

In this chapter we analyse the cases when (M,g) may admit proper affine vector fields. In all such cases a local characterization of affine vector fields is obtained, as well as an upper bound on the dimension of the Lie algebra $\mathcal{A}(M,g)$. The last section will be concerned with an extension result (of which a different proof has been given by Hall [32]) generalizing a theorem of Nomizu [65].

We recall that in all that follows the manifolds (M,g) considered are assumed connected. Furthermore, it is assumed that the Riemann tensor does not vanish on non-empty open subsets of M.

4.1.Generalities. The space $\mathfrak{h}(M,g)$.

Let (M,g) be a pseudo-riemannian manifold, X a vector field on M. As in the preceding chapter we define, by their local expressions, the following tensors

$$\overset{X}{h}_{ab} = \frac{1}{2}(X_{a;b} + X_{b;a}), \tag{4.1.1}$$

and

$$\overset{X}{f}_{ab} = \frac{1}{2}(X_{a;b} - X_{b;a}), \tag{4.1.2}$$

the superscript X being dropped when no confusion is possible. One has then the following well known result [40], [81], [55]

THEOREM 4.1.1. *Let (M,g) be a pseudo-riemannian manifold. If X is an affine vector field of M, then the tensor fields $\overset{X}{f}$ and $\overset{X}{h}$ satisfy*

$$\overset{X}{h}_{ab;c} = 0,$$
$$\overset{X}{f}_{ab;c} = X^d \mathbf{R}_{abcd}.$$

This theorem leads to the consideration of the set $\mathsf{h}(M,g)$ of covariantly constant symmetric tensor fields of type (0,2) of the pseudo-riemannian manifold (M,g). $\mathsf{h}(M,g)$ is obviously a real vector space and, since $g \in \mathsf{h}(M,g)$, its dimension is at least 1. If its dimension is 1, then it is spanned by g and so all affine vector fields of M are homothetic. Thus, we see that (M,g) can admit proper affine vector fields only if $\dim \mathsf{h}(M,g) \geq 2$.

It is known [40], [43] that if (M,g) is a space-time such that $\dim \mathsf{h}(M,g) \geq 2$, then (M,g) is reducible. Thus, we are lead to the consideration of the holonomy group of (M,g).

Let us recall (cf. §.1.8) that (M,g) is said to be non degenerately reducible if a non-trivial, non-null, subspace of $T_p(M)$ is invariant under holonomy. In such cases, as stated in §.1.8, (M,g) is locally decomposable and we have the following possibilities [40], [44], [33] where we use the labelling of the holonomy algebra introduced in Ch.2

I. The holonomy algebra is of type R_6, R_{10} or R_{13}.

In these cases (M,g) admits a (unique up to constant multiples) nowhere zero, global, covariantly constant non-null vector field u. u is spacelike in the R_6 and R_{10} cases and timelike in the R_{13} case.

The vector space $\mathsf{h}(M,g)$ is in this case 2-dimensional and spanned by the tensor fields g and $u_a u_b$.

In these cases, denote by Δ the distribution spanned by u and by $\Delta^{\perp}$ the distribution defined at each point p of M by the orthogonal complement of $\Delta(p)$. Then Δ and $\Delta^{\perp}$ are both integrable distributions. Given a point p in M we can find then an open neighbourhood U of p such that if I (resp. J) denotes the maximal integral submanifold of $\Delta_{|U}$ (resp. $\Delta^{\perp}_{|U}$), then $(U, g_{|U})$ is isometric to $(I \times J, i \oplus j)$, where i (resp. j) is the restriction of g to I (resp. J).

Space-times in this class are called "1+3"-(locally) decomposable.

It should be noticed that in these cases the manifold (J,j) is not non degenerately reducible. However in the case when the holonomy algebra is R_6, J (and so M as well) admits a recurrent null vector field, and so it is reducible. This null vector, however, does not contract the Riemann tensor to zero and so [44] it cannot be scaled to a covariantly constant vector field.

It should also be noticed that when the holonomy algebra is R_3, (M, g) admits a spacelike covariantly constant vector field, and so can be considered as non-degenerately reducible. In this case, however, (M, g) also admits a covariantly constant null vector field. Because of this, we exclude it from the above class, and we shall treat it separately in §.4.3.

II. The holonomy algebra is of type R_7.

In this case (M, g) does not admit covariantly constant vector fields. However, the holonomy determines a pair of mutually orthogonal 2-dimensional distributions Δ and $\Delta^{\perp}$, one timelike, the other spacelike.

In this case $\mathfrak{h}(M, g)$ is two dimensional and spanned by the tensor fields one gets by restricting g to Δ and $\Delta^{\perp}$.

Given a point p in M we can find then an open neighbourhood U of p such that if I (resp. J) denotes the maximal integral submanifold of $\Delta_{|U}$ (resp. $\Delta^{\perp}_{|U}$), then $(U, g_{|U})$ is isometric to $(I \times J, i \oplus j)$, where i (resp. j) is the restriction of g to I (resp. J).

In this case U can be chosen sufficiently small for a null tetrad (l, n, x, y) to exist on U with the property that $\Delta(p)$ and $\Delta^{\perp}(p)$ are the blades of the bivectors $2x_{[a}y_{b]}$ and $2l_{[a}n_{b]}$ at $p \in U$, respectively. These bivectors are then covariantly constant and the tensor fields i and j described above span $\mathfrak{h}(M, g)$ (in U) and are given by

$$i_{ab} = 2l_{(a}n_{b)},$$
$$j_{ab} = x_a x_b + y_a y_b.$$

Space-times in this class are called "2+2"-(locally) decomposable.

III. The holonomy algebra is of type R_2 or R_4.

In these cases (M, g) admits a (unique up to constant linear combinations) pair u, v of nowhere zero, global covariantly constant vector fields such that the distribution $span(u, v)$ is non-null. $span(u, v)$ is spacelike in the R_2 case and timelike in the R_4 case.

In this case, the vector space $\mathfrak{h}(M, g)$ is 4-dimensional and spanned by the tensor fields g_{ab}, $u_a u_b$, $2u_{(a}v_{b)}$ and $v_a v_b$.

In these cases, denote by Δ the distribution $span(u, v)$ and by $\Delta^{\perp}$ the distribution defined at each point p of M by the orthogonal complement of $\Delta(p)$. Then

Δ and $\Delta^{\perp}$ are both integrable distributions. Given a point p in M we can find then an open neighbourhood U of p such that if I (resp. J) denotes the maximal integral submanifold of $\Delta_{|U}$ (resp. $\Delta^{\perp}_{|U}$), then $(U, g_{|U})$ is isometric to $(I \times J, i \oplus j)$, where i (resp. j) is the restriction of g to I (resp. J).

Space-times in this class are called "1+1+2"-(locally) decomposable.

As for the Petrov and Segre types they can be read from TABLE.I at the end of Ch.2 (notice however that in this table the holonomy group considered is the infinitesimal holonomy group and not the holonomy group).

When (M, g) is not non-degenerately reducible, the space $\mathsf{h}(M, g)$ can have dimension 2 or more only in the following case

IV. The holonomy algebra is R_8 or R_{11}.

In this case, if one assumes, furthermore, that M is simply-connected, then (M, g) admits a (unique up to constant multiples) nowhere zero, global covariantly constant null vector field l.

$\mathsf{h}(M, g)$ is, in this case, a 2-dimensional vector space, being spanned by g and $l_a l_b$.

Finally, we consider the case

V. The holonomy algebra has type R_3.

In this case, as mentioned above, if one requires furthermore that M be simply-connected, then (M, g) admits a (unique up to constant linear combinations) pair u, v of nowhere zero global covariantly constant vector fields which span a 2-dimensional null distribution.

In this case $\mathsf{h}(M, g)$ is 4-dimensional and spanned by g_{ab}, $u_a u_b$, $2u_{(a}v_{b)}$ and $v_a v_b$.

Again the possible Petrov and Segre types for these twos cases can be read from TABLE.I given in Ch.2 (cf. remark above).

NOTE.4.1.1. These results are due to Hall [40]. A more detailed solution can be found in Hall's paper [33], where the problem is treated more in the way needed here.

NOTE.4.1.2. Consider the map $\sigma : \mathcal{A}(M,g) \to \mathsf{h}(M,g)$, given for $X \in \mathcal{A}(M,g)$ by

$$\sigma(X) = h^{X}.$$

σ is a linear map and so $K = \sigma(\mathcal{A}(M,g))$ is a vector subspace of $\mathsf{h}(M,g)$. Let $r = \dim \mathcal{A}(M,g)$ and $m = \dim K$ and assume that $g \notin K$. Let then $h_1, .., h_m$ be a basis of K, so that there exist $X_1, ..., X_m \in \mathcal{A}(M,g)$ such that $\sigma(X_i) = h_i$. The X_i are obviously proper affine vector fields. Complete $(X_1, ..., X_m)$ into a basis $(X_1, .., X_m, Y_{m+1}, .., Y_r)$ of $\mathcal{A}(M,g)$. Then for each Y_j, as $\sigma(Y_j) \in K$, there exist real constants C^i_j, $1 \leq i \leq m$ such that $\sigma(Y_j) = C^i_j h_i$. Thus $Y_j - C^i_j X_i$ is a Killing vector field. Setting then $X_j = Y_j - C^i_j X_i$, the family $(X_1, ..., X_m)$ is a basis of $\mathcal{A}(M,g)$ whose first m elements are proper affine vector fields and whose last $r - m$ elements are Killing vector fields.

In the case when $g \in K$ we can choose the h_i in such away that $h_1 = g$. In this case the basis $(X_1, ..., X_m)$ above is such that X_1 is a proper homothetic vector field, $X_2, ..., X_m$ are proper affine and $X_{m+1}, ..., X_r$ are Killing vector fields.

The number m (or $m - 1$) represents thus the maximum number of independent proper affine vector fields (M,g) can admit (in the sense that every other proper affine vector field will then be a combination of those already described and homothetic or Killing vector fields). (These considerations where suggested to me by my supervisor Dr. Graham Hall).

NOTE.4.1.3. The remaining types for the holonomy algebra of (M,g) are R_9, R_{12}, R_{14} and R_{15} (R_5 is impossible, see Ch.2). In the first three cases there exists a recurrent null vector field but no covariantly constant vector fields [33]. Thus, in theses cases (M,g) is still reducible but $\dim \mathsf{h}(M,g) = 1$ and so no proper affine vector fields are admitted. R_{15} is the irreducible case.

The above results provide us with a preliminary classification of those space-times that may admit proper affine vector fields. In the next sections we shall analyse the classes described above. The types described in I, II and III will be analysed in the next section, types IV and V in the following one.

4.2.The types I, II and III.

Let then (M, g) be a space time of one of the types described in I, II or III. Given $p \in M$ we can then find an open neighbourhood U of p in M and manifolds (I, i) and (J, j) such that $(U, g_{|U})$ is isometric to $(I \times J, i \oplus j)$. We identify $(U, g_{|U})$ and $(I \times J, i \oplus j)$ and, using the notions introduced in §.1.3, we do not distinguish a vector field on I (or J) from its extension to U.

A neighbourhood such as U above will be called a **decomposable neighbourhood** of p and the manifolds I, J the **factors** of the decomposition. We make the convention that for "1+3"-decomposable space-times (I, i) stands for the 1-dimensional factor; for "2+2"-decomposable space-times (I, i) stands for the timelike factor in the decomposition and, in the case of "1+1+2"-decomposable space-times, (I, i) stands for the flat 2-dimensional manifold - the integral submanifold of $span(u, v)$ through some point of U - of the decomposition.

In all that follows U denotes a contractible decomposable neighbourhood of some point in M.

With these notations the following lemma is evident

Lemma 4.2.1. *Let V (resp. W) be a vector field on I (resp. J)and denote also by V, W their extensions to U (in the sense of §.1.3). Then, for every vector field S (resp. T) everywhere orthogonal to I (resp. J)*

$$T^b W_{a;b} = S^b V_{a;b} = 0.$$

We also have

Lemma 4.2.2 *Let X be an affine vector field of (M, g) and let f be its bivector. Then for all vector fields V on I and W on J*

$$f_{ab} V^a W^b = 0.$$

Proof. We analyse each case separately.

The "2+2"-decomposable case.

In this case by restricting U if necessary we can assume the existence in U of a null tetrad (l, n, x, y) of vector fields such that l, n are vector fields on I and x, y

are vector fields on J. A simple computation shows then that there exist 1-forms ρ on I and μ on J such that (see [44])

$$\begin{aligned} l_{a;b} &= \rho_b l_a, \\ n_{a;b} &= -\rho_b n_a, \\ x_{a;b} &= \mu_b y_a, \\ y_{a;b} &= -\mu_b x_a. \end{aligned} \tag{4.2.1}$$

As was proved by Hall and Kay [44], in this case there exist functions Ω^1 and Ω^2 on U such that, setting

$$F_{ab} = 2l_{[a}n_{b]},$$

the Riemann tensor of (M, g) takes the form

$$\mathbf{R}_{abcd} = \Omega^1 F_{ab} F_{cd} + \Omega^2 F^*_{ab} F^*_{cd}. \tag{4.2.2}$$

Using this expression we have then

$$f_{ab;c} V^a W^b = 0, \tag{4.2.3}$$

since V^a contracts the second term of the Riemann tensor to zero and W^b contracts the first term to zero. Defining then $\zeta = f_{ab} l^a x^b$ and $\eta = f_{ab} l^a y^b$, and using (4.2.3) and (4.2.2) it is easy to prove that

$$\begin{aligned} \zeta_{;c} - \zeta\rho_c - \eta\mu_c &= 0, \\ \eta_{;c} - \eta\rho_c + \zeta\mu_c &= 0. \end{aligned}$$

Setting then $\phi = \zeta^2 + \eta^2$, the above relations give

$$\phi\phi_{;c} - \phi^2\rho_c = 0.$$

If at some point of U ϕ does not vanish then in some open neighbourhood of that point ϕ satisfies $\phi_{;c} - \phi\rho_c = 0$. It follows then that in that neighbourhood the vector field ϕn is covariantly constant, thus contradicting the fact that (M, g) is "2+2"-decomposable. The same method can be applied to $f_{ab} n^a x^b$ and $f_{ab} n^a y^b$, proving that these quantities also vanish.

The "1+1+2"-decomposable case.

In this case we can use the same method as above, since it can be considered as a particular case of the above class (with either $\rho = \Omega^1 = 0$ or $\mu = \Omega^2 = 0$). One obtains the same result.

In particular it follows from (4.2.3) that, in both these cases there exist functions A^X, B^X on U such that

$$f_{ab} = A^X F_{ab} + B^X F^{*}_{ab}. \tag{4.2.4}$$

The "1+3"-decomposable case.

In this case, as v is covariantly constant, we have by T.4.1.1

$$(f_{ab}v^a)_{;c} = f_{ab;c}v^a = X^d v^a \mathbf{R}_{abcd} = 0,$$

since as we have $v_{a;b} = 0$, we have, by the Ricci identity

$$\mathbf{R}_{abcd}v^d = 0.$$

As v is the unique (up to constant multiples) covariantly constant vector field on U we deduce that $f_{ab}v^b = 0$, so that f is simple and its blade is orthogonal to v. This gives immediately the result of the lemma ■.

This result leads to the following theorem [43]

THEOREM 4.2.3 *Let (M, g) be a space-time whose holonomy algebra belogs to one of the types in I, II or III and let X be an affine vector field of (M, g). Given any point p in M and a decomposable neighbourhood U of p in M*
(a). The vector field X is projectable over the decomposition $(I, i), (J, j)$ of $(U, g_{|U})$;
(b). If Y (resp. Z) is the projection of X over I (resp. J) then Y (resp. Z) is an affine vector field of (I, i) (resp. (J, j)).

Proof. The vector fields Y and Z are given locally by, respectively

$$Y^a = i^a{}_b X^b, \tag{4.2.5}$$

and

$$Z^a = j^a{}_b X^b. \tag{4.2.6}$$

Let us consider first "1+3" and "2+2"-decomposable space-times. In this cases i, j span $\mathsf{h}(M, g)$ and so there exist real constants α, β such that

$$\overset{X}{h}_{ab} = \alpha i_{ab} + \beta j_{ab}. \tag{4.2.7}$$

This gives then,

$$Y_{a;b} = i^a{}_c X^c{}_{;b},$$

since i is covariantly constant. Hence, we have

$$Y_{a;b} = \alpha i_{ab} + i_a{}^c f_{cb}. \tag{4.2.8}$$

Let then W be a vector field on J and denote also by W its extension to U (in the sense of §.1.3.).

We have then

$$[W, Y]_a = W^b Y_{a;b} - Y^b W_{a;b},$$

thus, using the above expression (4.2.8) for $Y_{a;b}$, we have

$$[W, Y]_a = i_a{}^c f_{cb} W^b - Y^b W_{a;b} = 0,$$

since on the right hand side of the first equality, the second term vanishes by L.4.2.1 and the second term vanishes by L.4.2.2.

The same method can be applied to prove that if V is a vector field on I then $[V, Z] = 0$.

This proves (a), as follows from T.1.3.3.

Notice then that, by (4.2.8), Y is a homothetic vector field of (I, i). Similarly Z is a homothetic vector field of (J, j), and this establishes (b) for "1+3" and "2+2"-decomposable space-times.

In the case of "1+1+2"-decomposable space-times, whilst for Z one still gets

$$Z_{a;b} = \beta j_{ab} + j_a{}^c f_{cb},$$

the same does not happen for Y (recall that in this case we have made the convention that (I, i) is the flat factor in the decomposition).

However, in this case, as follows from the observations in case III, there exist real constants α, γ, δ and β such that

$$\overset{\times}{h}_{ab} = \beta j_{ab} + \alpha u_a u_b + 2\gamma u_{(a} v_{b)} + \delta v_a v_b.$$

A simple computation shows then that

$$Y_{a;b} = \alpha u_a u_b + 2\gamma u_{(a} v_{b)} + \delta v_a v_b + i_a{}^c f_{cb}.$$

The proof is then virtually the same as in the preceding cases ■

It follows from the above theorem that [43]

COROLLARY 4.2.4. *If the type of the holonomy algebra of (M,g) is R_7, then the vector field Y is a homothetic vector field of (I,i). In all cases, Z is a homothetic vector field of (J,j).*

Another consequence of T.4.2.3 is that [43]

THEOREM 4.2.5.*Keeping the notations of T.4.2.3, the Lie algebras $\mathcal{A}(U,g_{|U})$ and $\mathcal{A}(I,i)\times\mathcal{H}(J,j)$ are isomorphic.*

Proof. By T.4.2.3., the mapping $\Theta : X \mapsto (Y,Z)$ is well defined. It clearly is one-to-one, onto and linear. Now, for all $X_1,X_2 \in \mathcal{A}(U,g_{|U})$ we have

$$[X_1,X_2]=[Y_1,Y_2]+[Z_1,Z_2],$$

since, as both X_1,X_2 are projectable

$$[Y_1,Z_2]=[Z_1,Y_2]=0.$$

This proves that Θ is a Lie algebra endomorphism ■

It follows that [43]

COROLLARY 4.2.6.*Keeping the notation of the above theorems*
(a).If the holonomy algebra of (M,g) is R_6, R_{10} or R_{13} then $2\leq \dim\mathcal{A}(U,g_{|U})\leq 8$;
(b).If the holonomy algebra of (M,g) is R_7 then $0\leq \dim\mathcal{A}(U,g_{|U})\leq 6$;
(b).If the holonomy algebra of (M,g) is R_2 or R_4 then $6\leq \dim\mathcal{A}(U,g_{|U})\leq 9$.

Proof. (a). In this case (J,j) is 3-dimensional and non-flat, so $\dim\mathcal{H}(J,j)\leq 6$, as follows from T.1.9.5.

Let us consider then affine vector fields of $(U,g_{|U})$ which are parallel to the vector field u.

Given one-such vector field X, there exists a function λ on U such that $X = \lambda u$. This gives then

$$X_{a;b}=\lambda_{;b}u_a,$$

since u is covariantly constant. It follows then from the fact that

$$\overset{X}{h}_{ab}=\alpha g_{ab}+\beta u_a u_b,$$

for some real constants α,β, that we must have

$$\lambda_{;b}=cu_b,$$

for some real constant c. Since U is contractible and u is covariantly constant, there exists a differentiable function ϕ on U such that $\phi_{;b} = u_b$. The above result shows that $= c\phi + d$, where c, d are real constants. Thus, $\dim \mathcal{A}(I,i) = 2$ and this proves (a).

(b). In this case, as mentioned in C.4.2.4, if X is an affine vector field of (M,g) then Y is a homothetic vector field of (I,i). Thus, in this case since we have $\dim \mathcal{H}(I,i)$, $\dim \mathcal{H}(J,j) \leq 3$ (since both these manifolds are non-flat, see T.1.9.5), it follows that

$$\dim \mathcal{A}(U, g_{|U}) \leq 6.$$

In this case the lower bound is obviously 0, as both (I,i) and (J,j) may not admit homothetic vector fields.

(c). As for cases (b), we have in this case $\dim \mathcal{H}(J,j) \leq 3$, since (J,j) is non-flat. Since in this case (I,i) is a 2-dimensional flat manifold, we have $\dim \mathcal{A}(I,i) = 6$, as follows from T.1.9.2 ■

In general, these results hold only locally, as (M,g) may not be (globally) decomposable. If such is the case, the isomorphism of the local Lie algebras given by T.4.2.5 does not lead to a similar isomorphism for the global Lie algebra. However, given a decomposable open subset U of M, the restriction map, $X \mapsto X_{|U}$, defines a one-to-one homomorphism of the Lie algebra $\mathcal{A}(M,g)$ into the Lie algebra $\mathcal{A}(U, g_{|U})$ (see §.4.5), and so C.4.2.6 still provides us with upper bounds on the dimension of $\mathcal{A}(M,g)$.

Let now (M,g) be a connected, simply connected and geodesically complete space-time. In such case all affine vector fields of (M,g) are complete vector fields (T.1.9.2.(b)) and $\mathcal{A}(M,g)$ is then the Lie algebra of the Lie group $\mathbf{A}(M,g)$ of affine transformations of M, as follows from Palais' theorem T.1.5.6.

If furthermore (M,g) is assumed non-degenerately reducible then it follows from Wu's theorem T.1.8.9 that (M,g) is decomposable. Combining these observations with the previous local results we get the following theorem [43], where we have denoted by $\mathcal{A}(n)$ the $n(n+1)$-dimensional Lie group of affine transformations of $\mathbf{R}^n$

THEOREM 4.2.7. *Let (M,g) be a non-degenerately reducible, connected, simply connected and geodesically complete space-time. Then*

(a).If the holonomy algebra of (M,g) is of type R_6, R_{10} or R_{13}, then the Lie group $\mathbf{A}(M,g)$ is isomorphic to the Lie group $\mathcal{A}(1) \times \mathbf{H}(J,j)$ in the case when u is spacelike, and isomorphic to the group $\mathcal{A}(1) \times \mathbf{I}(J,j)$ in the case when u is timelike. In particular its dimension is at most 6;

(b).If the holonomy algebra of (M,g) is of type R_7, then the Lie group $\mathbf{A}(M,g)$ is isomorphic to the Lie group $\mathbf{H}(I,i) \times \mathbf{I}(J,j)$. In particular its dimension is at most 6;

(c).If the holonomy algebra of (M,g) is of type R_2 or R_4, then the Lie group $\mathbf{A}(M,g)$ is isomorphic to the Lie group $\mathcal{A}(2) \times \mathbf{H}(J,j)$ in the case when $span(u,v)$ is spacelike, and isomorphic to the group $\mathcal{A}(2) \times \mathbf{I}(J,j)$ in the case when $span(u,v)$ is timelike. In particular its dimension is at most 9.

The appearance of the isometry group of (J,j) in some parts of the above theorem is due to the following fact. Since (M,g) is complete and (J,j) is totally geodesic (cf. T.1.8.5), it follows that (J,j) is also geodesically complete. Now when in case (a) u (or $span(u,v)$ in case (c)) is timelike, (J,j) is a riemannian manifold. It is well known that complete riemannian manifolds can admit proper homothetic vector fields only if they are flat [55]. As (J,j) is assumed non-flat, we see that in these cases we can replace the homothety group by the isometry group in the formulation of the theorem.

4.3.The types IV and V.

As follows from §.4.1, in these cases (M,g) admits a globally defined null covariantly constant vector field l. If, furthermore, (M,g) is of type IV then l is the unique (up to constant multiples) covariantly constant vector field on M. If (M,g) is of type V then, besides l, (M,g) admits a spacelike covariantly constant vector field v.

In these cases as the spaces spanned by u or $span(u,v)$ are null, no well defined projection operator exists to play the rôle of i, j in the previous section. As a consequence, for these types, splitting theorems as T.4.2.3 and T.4.2.7 are not to be expected.

We analyse the two possible cases described above, separately.

Type IV.

In such case, if $X \in \mathcal{A}(M,g)$, then there exist real constants α, β such that

$$\overset{X}{h}_{ab} = \alpha g_{ab} + \beta l_a l_b.$$

Let then $p_o \in M$ and let U be a contractible neighbourhood of p in M. As l is covariantly constant it follows from the Poincaré Lemma that there exists a differentiable function λ on U such that in U

$$\lambda_{;a} = l_a.$$

Furthermore, λ is unique if we ask that $\lambda(p_o) = 0$.

Consider then the vector field on U given by $L = \beta\lambda l$. Then we have

$$L_{a;b} = \beta l_a l_b,$$

and so $X - L$ is a homothetic vector field of $(U, g_{|U})$.

Thus, we have [43]

THEOREM 4.3.1. *Assume the holonomy algebra of (M,g) is of type R_8 or R_{11} and let X be an affine vector field of (M,g). Then, given any point p_o in M and any contractible neighbourhood U of p_o there exists essentially one proper affine vector field L on U, in the sense that there exists a real constant β such that $X - \beta L$ is a homothetic vector field of $(U, g_{|U})$. In particular, the Lie algebra $\mathcal{A}(U, g_{|U})$ is at most of dimension 9.*

Proof. The first part has been proved above. Consider then the map $\pi : \mathcal{A}(U, g_{|U}) \to \mathcal{H}(U, g_{|U})$ given by $\pi(X) = X - \beta L$. π is obviously linear and onto, since for every homothetic vector field Y on U, $Y + L$ is a proper affine vector field. Since the kernel of π is spanned by L and π is onto , we see that

$$\dim \mathcal{A}(U, g_{|U}) = 1 + \dim \mathcal{H}(U, g_{|U}).$$

Now, from T.1.9.5, $\dim \mathcal{H}(U, g_{|U}) \leq 11$, the equality being reached only if $(U, g_{|U})$ is flat. This case has been excluded from our considerations. The case $\dim \mathcal{H}(U, g_{|U}) = 10$ is impossible for it implies the existence of a 9 or 10 dimensional group of isometries; the first case is excluded by Fubini's theorem [57]; the second case implies (by T.1.9.4) that $(U, g_{|U})$ is a manifold of constant curvature, so that we have

$$\mathbf{R}_{abcd} = K g_{a[c} g_{d]b}.$$

Contracting this relation with l^d we get, using the Ricci identity (T.1.6.4), since l is covariantly constant

$$K(g_{ac} l_b - l_a g_{bc}) = 0,$$

and contracting now with l^a we get $K l_c l_b = 0$, thus proving that $(U, g_{|U})$ is flat.

If $\dim \mathcal{H}(U, g_{|U}) = 9$, then, again by Fubini's theorem, $\mathcal{H}(U, g_{|U})$ must contain at least a proper homothetic vector field, and so, in this case $\dim \mathcal{I}(U, g_{|U}) = 8$. It is well known [57] that this is impossible.

Thus $\dim \mathcal{H}(U, g_{|U}) \leq 8$ and this proves the theorem ■

NOTE.4.3.1. In fact the dimension 9 can be attained, because plane waves with an 8-dimensional Lie algebra of homothetic vector fields are known (and have a covariantly constant null vector field. See, for instance [48]).

Type V.

In this case the vector fields u, v can be chosen to be such that $u = l$ is null and v is spacelike with $v^a v_a = 1.$. Take then again a point p_o in M and a contractible neighbourhood U of p_o in M, so that there exist differentiable functions λ, ν on U such that

$$\lambda_{;a} = l_a,$$

$$\nu_{;a} = v_a.$$

Let then X be an affine vector field of (M, g). Then there exist real constants $\alpha, \beta, \gamma, \delta$ such that

$$h_{ab} = \alpha g_{ab} + \beta l_a l_b + 2\gamma l_{(a} v_{b)} + \delta v_a v_b.$$

Consider then the vector field defined on U by

$$V = (\beta\lambda + \gamma\nu) l + (\gamma\lambda + \delta\nu) v.$$

We have then

$$V_{a;b} = (\beta l_b + \gamma v_b) l_a + (\gamma l_b + \delta v_b) v_a,$$

so that $X - V$ is a homothetic vector field of (M, g).

Thus, in a similar fashion to the previous case we have [43]

THEOREM 4.3.2.*Assume the holonomy algebra of (M, g) is of type R_3 and let X be an affine vector field of (M, g). Then, given any point p_o in M and any contractible neighbourhood U of p_o there exist essentially three proper affine vector fields L_1, L_2, L_3 on U, in the sense that there exist real numbers $\mathsf{a}, \mathsf{e}, \mathsf{c}$ such that $X - \mathsf{a}\,L_1 - \mathsf{e}\,L_2 - \mathsf{c}\,L_3$ is a homothetic vector field of $(U, g_{|U})$. In particular, the Lie algebra $\mathcal{A}(U, g_{|U})$ is at most of dimension 10.*

Proof. As in the previous theorem, the map $\pi : \mathcal{A}(U, g_{|U}) \rightarrow \mathcal{H}(U, g_{|U})$given by $\pi(X) = X - V$ is an onto Lie algebra homomorphism, whose kernel is spanned by the vector fields V. Since the space spanned by these vector fields has dimension 3 (cf. C.4.1.6.(b)), it follows that

$$\dim \mathcal{A}(U, g_{|U}) = 3 + \dim \mathcal{H}(U, g_{|U}).$$

The proof of the preceding theorem shows that $\dim \mathcal{H}(U, g_{|U}) \leq 8$. Assume that $\dim \mathcal{H}(U, g_{|U}) = 8$. In such case, it is well known [57] that $(U, g_{|U})$ is conformally flat. On the other hand, in this case, the Riemann tensor has rank 1 with null spanning bivector, thus corresponding to the second line in TABLE.1 of Ch.2, so that its Petrov type must be N!

It follows that $\dim \mathcal{H}(U, g_{|U}) \leq 7$ and this proves the theorem ■

NOTE.4.3.2. In [43], erroneously, it was affirmed that the maximum dimension of the local Lie algebra of affine vector fields was 9. The error was due to the fact that it was overlooked that in this last class the Lie algebra of homothetic vector fields can be 7-dimensional, as was pointed out to the authors by J. Steele, of the University of Aberdeen. Steele [73] (see also [48]), in fact, has found all space-times which fall in this class. The family in question is given in local coordinates (u, v, x, y) by

$$g = dx^2 + dy^2 + 2dudv + 2A(u)(x + cy)^2 du^2,$$

where c is a real number and the function A is either constant or of the form d/u^2, where $d \in \mathbf{R}$. The author would like to thank J.Steele for comunicating him these facts.

4.4. Local and global affine vector fields.

Let (M, g) be a pseudo-riemannian manifold, U an open subset of M. Under what conditions on M can we ensure that an affine vector field on U is in fact the restriction to U of an affine vector field of (M, g)?

In this section we give an answer to this question, which generalizes to affine vector fields a result of Nomizu [65]. The proof of the result in question that we give here is different from that of Hall [32], to whom this result is due.

As always, we assume the manifolds under consideration are connected.

Let then (M, g) be a pseudo-riemannian manifold and, for $p \in M$, let $\mathsf{h}(p)$ denote the subspace of the space of (0,2)-symmetric tensors at p spanned by the evaluation at p of all elements of $\mathsf{h}(M, g)$. Denote also by $A^2(p)$ the space of bivectors at p and consider

$$\mathsf{a}(p) = T_p(M) \oplus \mathsf{h}(p) \oplus A^2(p).$$

If $m = \dim \mathsf{h}(M, g)$, then $\dim \mathsf{a}(p) = \frac{n(n+1)}{2} + m$, where n is the dimension of M.

Given an open subset U of M consider, for every $p \in U$, the map $\sigma(U, p) : \mathcal{A}(U, g_{|U}) \to \mathsf{a}(p)$, given by $\sigma(U, p)(X) = (X_p, h_p^X, f_p^X)$. $\sigma(U, p)$ is a linear map. We have then (see [32])

Lemma 4.4.1. *The map $\sigma(M, p)$ is one-to-one for all $p \in M$.*

Proof. Let $q \in M$ and let $\gamma : [0, 1] \to M$ be a curve from p to q. Without any loss of generality we may assume that γ is differentiable and that $\gamma([0, 1])$ is contained in the domain U of a coordinate chart of M. Denoting by $(x^1, ..., x^n)$ these coordinates we have then in U, as X is an affine vector field

$$X_{i;j} = h_{ij} + f_{ij},$$
$$h_{ij;k} = 0,$$
$$f_{ij;k} = X^m \mathbf{R}_{ijkm}.$$

Using then the local expressions for these covariant derivatives in terms of the Christoffel symbols, we get, setting

$$V^i = \frac{d\gamma^i}{dt},$$
$$B^m{}_i = V^k \Gamma^m{}_{ik},$$
$$C^m{}_{ij} = V^k \mathbf{R}^m{}_{kij},$$

the following differential system,

$$\begin{aligned}\frac{dX^i}{dt} &= V^j h_{ij} + V^j f_{ij} + B^k{}_i X_k,\\ \frac{dh_{ij}}{dt} &= B^m{}_i h_{mj} + B^m{}_j h_{im},\\ \frac{df_{ij}}{dt} &= B^m{}_i f_{mj} + B^m{}_j f_{im} - C^m{}_{ij} X_m.\end{aligned} \tag{S}$$

Since all functions on the right hand side of this system are differentiable we see that, as the system is linear, it has, for a given set of initial conditions, a unique solution which is defined for all values of $t \in [0,1]$. This proves the lemma ■

Thus, if U is any open connected subset of M every affine vector field on U is uniquely determined in U by the value of $\sigma(U,p)(X)$, p being any point in U.

Let $p \in M$. Given vector fields X, Y defined in some open neighbourhood of p we say that X and Y have the same **germ at** p if there exists an open neighbourhood U of p in M such that X and Y coincide in U. Obviously, the relation, "X and Y have the same germ at p", is an equivalence relation in the set of vector fields which are defined at p.

Consider then the set

$$A(U) = \bigcup \mathcal{A}(U, g_{|U}),$$

where U runs over all connected open neighbourhoods of p in M. Then the above relation is an equivalence relation in $A(U)$. The quotient of $A(U)$ by this relation is denoted by $\mathsf{a}^*(p)$. We have then [32] [65]

Lemma 4.4.2. *Given any $p \in M$, $\mathsf{a}^*(p)$ is a real vector space. Moreover, there exists an open connected neighbourhood U of p in M such that the map,*

$$\Psi : \mathcal{A}(U, g_{|U}) \to \mathsf{a}^*(p),$$

given by $\Psi(X) =$ "germ of X at p", is vector space isomorphism.

Proof. The first assertion is obvious. To prove the second, consider a sequence of open conneceted neighbourhoods (U_m) of p such that $U_{m+1} \subset U_m$ and $\cap U_m = \{p\}$. Then, if $X \in \mathcal{A}(U_m, g_{|U_m})$, the restriction of X to U_{m+1} is an affine vector field on U_{m+1}. This restiction map is clearly linear and, by the preceding lemma, it is one-to-one. Consequentely, the sequence $\dim \mathcal{A}(U_m, g_{|U_m})$ is an increasing sequence.

Since it is bounded by $n(n+1)$, where $n = \dim M$, we deduce the existence of a m_o such that for $m \geq m_o$, the algebras $\mathcal{A}(U_m, g_{|U_m})$ coincide. Consider then $\mathsf{a}^*(p)$ and consider the map $\Psi_m : \mathcal{A}(U_m, g_{|U_m}) \to \mathsf{a}^*(p)$ which associates to every element of $\mathcal{A}(U_m, g_{|U_m})$ its germ at p. These maps are all linear and, for $m \geq m_o$, they are onto. To see this take an element v of $\mathsf{a}^*(p)$, so that there exists an open neighbourhood V of p in M and an affine vector field X on V whose germ at p is v. The conditions on the sequence (U_m) garantee the existence of an integer k such that, for $m \geq k$, $U_m \subset V$, so that v is the germ at p of some affine vector field on U_m for $m \geq k$.

It follows that, for $m \geq m_o$ the maps Ψ_m are isomorphisms.

This proves the lemma ■

Given a point p in M we say (following Nomizu, [65]) that an open neighbourhood U of p in M is **special** if the map $\Psi : \mathcal{A}(U, g_{|U}) \to \mathsf{a}^*(p)$ is a vector space isomorphism. Clearly if U is a special neighbourhood of p and W is a neighbourhood of p such that $W \subset U$, then W is also a special neighbourhood of p. The preceding lemma states that every point of M has a special neighbourhood.

We have then

Lemma 4.4.3. *Let $p \in M$ and let U be a special neighbourhood of p. Then for every point $q \in U$ the map*

$$\Psi_q : \mathcal{A}(U, g_{|U}) \to \mathsf{a}^*(q),$$

given by $\Psi_q(X)$ = "germ of X at q", is one-to-one and linear. In particular $\dim \mathsf{a}^*(p) \leq \dim \mathsf{a}^*(q)$.

Proof. This follows immediately from the definition of special neighbourhood ■

A point p of M is said to be **a^*-special** if it admits a special neighbourhood U such that $q \mapsto \dim \mathsf{a}^*(q)$ is constant in U. The set of a^*-special points of M is obviously open in M. We have then

Lemma 4.4.4. *The function $p \mapsto \dim \mathsf{a}^*(p)$ is constant in M if and only if all points of M are a^*-special.*

Proof. It is obvious that if $p \mapsto \dim \mathsf{a}^*(p)$ is constant then every point of M is a^*-special. Conversely, suppose every point of M is a^*-special. Let then $p_o \in M$ and let V be the subset of M of all those points p such that $\dim \mathsf{a}^*(p) = \dim \mathsf{a}^*(p_o)$.

Let $q \in V$. Since q is a^*-special, there exists an open neighbourhood W of q in M such that $\dim \mathsf{a}^*$ is constant in W. Thus, $W \subset V$ and so V is open in M. Let now q belong to the closure of V. Again because q is a^*-special we can find an open neighbourhood W of q such that $\dim \mathsf{a}^*$ is constant in W. As $W \cap V$ is non-empty, we deduce that $q \in V$ and so V is closed. As V is non-empty and M is connected we deduce that $V = M$ ■

If p is a^*-special and U is a special neighbourhood of p where $\dim \mathsf{a}^*$ is constant, then for every point q in U and every $v \in \mathsf{a}^*(q)$ there exists a unique $Y \in \mathcal{A}(U, g_{|U})$ such that the germ of Y at q is exactly v. This follows from the fact that by L.4.4.3, the map Ψ_q is one-to-one and $\dim \mathsf{a}^*$ is constant in U.

Let now p be a^*-special and let $\gamma : [0,1] \to M$ be a curve on M from p to some point q. For $t \in [0,1]$ let X_t be an element of $\mathsf{a}^*(\gamma(t))$, U_t the open neighbourhood of $\gamma(t)$ where X_t is defined (by L.4.4.2). Let $X_o \in \mathsf{a}^*(\gamma(0))$. A **prolongation** of X_o along γ from p to q is a family (X_t) as the one above, such that the family

$$(X_t, h^{X_t}, f^{X_t}),$$

for $t \in [0,1]$, satisfies the differential system (S) of L.4.4.1 for the initial conditions (X_o, h^{X_o}, f^{X_o}).

It is clear that if a prolongation exists then it is unique. Moreover

Lemma 4.4.5. *A prolongation exists along every piecewise differentiable curve consisting of a^*-special points.*

Proof. Whithout any loss of generality we may assume that γ is differentiable, and it is clear that a prolongation of X_o along $\gamma([0,1]) \cap U_o$ exists. Let then I denote the subset of $[0,1]$ of all those t such that a prolongation exists along γ from p to $\gamma(t)$. The above argument shows that I is open in $[0,1]$ and non-empty. Let t_o be its supremum and assume that $t_o < 1$. Since t_o belongs to the closure of I, given any open neighbourhood W of $\gamma(t_o)$ in M there exists $\epsilon > 0$ such that for $|t - t_o| < \epsilon$, $\gamma(t) \in W$.

As $\gamma(t_o)$ is special, choose the neighbourhood W to be a special neighbourhood of $\gamma(t_o)$ such that $\dim \mathsf{a}^*$ is constant in W.

Let then t_1 be such that $t_o - \epsilon < t_1 < t_o$. Since $t_1 \in I$, there exists $X_{t_1} \in \mathsf{a}^*(\gamma(t_1))$ which is a prolongation of X_o. Since $\gamma(t_1) \in W$ it follows from L.4.4.4 that there exists $X_1 \in \mathcal{A}(W, g_{|W})$ such that $\Psi_{\gamma(t_1)}(X_1) = X_{t_1}$.

As X_1 is an affine vector field of W it is clear that the family $(X_{1t}, h_t^{X_1}, f_t^{X_1})$ satisfies the differential system (S).

Thus, it remains to be proved that for every t such that $t_o - \epsilon < t < t_o + \epsilon$ we have $X_{1\gamma(t)} = X_{t_1|\gamma(t)}$.

But, to see that this holds, notice that both families satisfy (S) and that at t_1 they coincide (cf. L.4.4.1). It follows that X_1 defines a prolongation of X_o to $[t_o, t_o + \epsilon[$, and this contradicts the definition of t_o, unless if $t_o = 1$ ∎

This leads to the following [32]

THEOREM 4.4.6. *Let (M, g) be a connected and simply connected pseudo-riemannian manifold and assume all points of M are a^*-special. Then for all $p \in M$ and all $v \in \mathsf{a}^*(p)$ there exists a unique $X \in \mathcal{A}(M, g)$ such that X is a prolongation of v.*

Proof. If such a prolongation exists it is unique as follows from L.4.4.1.

To prove the existence, let q be a point in M and let $\gamma : [0,1] \to M$ be a curve in M from p to q. As all points in M are a^*-special, it follows from the preceding lemma that a prolongation along γ from p to q exists. If this prolongation gives a value at q for X which does not depend on the curve γ chosen then the theorem is proved.

Let then q be a point in M and let $\gamma, \tau : [0,1] \to M$ be curves on M from p to q. Without any loss of generality, we may assume that both curves are differentiable. As M is simply connected, there exists a differentiable [16] homothopy

$$\phi : [0,1] \times [0,1] \to M$$

such that, for all $s \in [0,1]$, $t \mapsto \phi(s,t)$ is piecewise differentiable, $\phi(0,t) = \gamma(t)$ and $\phi(1,t) = \tau(t)$.

Let us set $K = [0,1] \times [0,1]$ and $K' = \phi(K)$, so that K' is a compact subset of M. For each fixed $s \in [0,1]$ let us denote by L_s the line $\{s\} \times [0,1]$ of K and by γ_s the curve $t \mapsto \phi(s,t)$, so that γ_s is a differentiable curve from p to q. For every $(s,t) \in K$ let $U_{(s,t)}$ be a special neighbourhood of $\phi(s,t)$ in M and let $V_{(s,t)} = \phi^{-1}(U_{(s,t)})$.

As each $V_{(s,t)}$ is open and contains the point (s,t), we can find a real number $\epsilon(s,t) > 0$ such that the square with center at (s,t), sides parallel to the sides of

K and width $2\epsilon(s,t)$ is contained in $V_{(s,t)}$. Let us denote this square by $Q_{(s,t)}$ and let us consider the family af all such squares for a fixed value s_o of s. As they cover L_{s_o} and L_{s_o} is compact, we can find a finite family $t_1, .., t_r$ in $[0,1]$ such that the squares $Q_{(s_o,t_i)}$ also cover L_{s_o}. Take then $\epsilon(s_o)$ to be the smallest of the real numbers $\epsilon(s_o, t_i)$. Consider then the band B_{s_o} with sides parallel to L_{s_o} and of width $\epsilon(s_o)$. This band is contained in the union of the squares $Q_{(s_o,t_i)}$ and so it is also contained in the union of the $V_{(s_o,t)}$, $0 \leq t \leq 1$. Now consider the bands B_s, $0 \leq s \leq 1$. These bands cover K. Since K is compact, we can find a finite family $B_1 = B_{s_1}, ..., B_n = B_{s_n}$ of these bands which still cover K.

Obviously, the bands B_i can be chosen of shorter width if necessary, to guarantee that, ordering the s_i in such a way that $s_1 < ... < s_n$, then the band B_{s_i} interesects only the bands $B_{s_{(i-1)}}$ and $B_{s_{(i+1)}}$ for $2 \leq i \leq n-1$ and $L_o \subset B_1$. $L_1 \subset B_n$.

Now we choose $z_o = 0$, z_1 to be such that the line L_{z_1} lies in the intersection of B_1 and B_2, z_2 such that the line L_{z_2} lies in the intersection of B_2 and B_3, and we continue the process, until we have chosen z_{n-1}, which lies in the intersection of B_{n-1} with B_n. We set then $z_n = 1$.

Consider now the curves γ_{z_i}. These are differentiable curves from p to q and so to prove the theorem we just have to prove that prolongation along γ_{z_i} and $\gamma_{z_{i+1}}$ for $0 \leq i \leq n-1$ yields the same result at q. So consider the lines L_{z_i} and $L_{z_{i+1}}$, $0 \leq i \leq n-1$. These lines are both entirely contained in some band B_{s_o}. On the other hand the band B_{s_o} is contained in the union of squares $Q_{(s_o,t_i)}$.

Denote by a_j (resp. b_j), $1 \leq j \leq m$, the points of intersection of the line L_{z_i} (resp. $L_{z_{(i+1)}}$) with all the squares $Q_{(s_o,t_i)}$, and let us assume that the a_j, b_j are ordered by $a_1 < ... < a_m$, $b_1 < ... < b_m$. Suppose then that the prolongations v_i and v_{i+1} of v along γ_{z_i} and $\gamma_{z_{i+1}}$ do not coincide at q.

Then the prolongation of v along the curve γ_{z_i} from $\phi(a_0)$ to $\phi(a_m)$ does not coincide with the prolongation along the curve which is the image by ϕ of the segments joining a_0 to b_0, b_0 to b_m and b_m to a_m.

In fact if they coincided, then, because a_m lies in a special neighbourhood of q, the prolongations would coincide at q!

Obviously, using the same process, we prove that the prolongation is then not uniquely defined at all points $a_{m-1}, a_{m-2}, \ldots, a_1$. Since a_1 lies in a special neighbourhood of p this is a contradiction and the theorem is proved ■

5.CURVATURE COLLINEATIONS.

In this chapter we study the Lie algebra of curvature collineations of a pseudo-riemannian manifold . As it has been pointed out in Ch.1, this Lie algebra may be infinite dimensional; one of our aims is precisely the characterization of pseudo-riemannian manifolds for which this happens.

As in the preceding chapters, all manifolds considered here are assumed connected.

5.1.Generalities.

Let (M,g) be a pseudo-riemannian manifold of dimension n; we recall that (see §.1.9) a vector field X on M is said to be a curvature collineation if it satisfies:

$$\mathcal{L}_X Riemann = 0. \tag{5.1.1}$$

Locally, this equation translates into:

$$X^m \mathbf{R}^k{}_{jhi;m} + X^m{}_{;j}\mathbf{R}^k{}_{mhi} + X^m{}_{;h}\mathbf{R}^k{}_{jmi} + X^m{}_{;i}\mathbf{R}^k{}_{jhm} - X^k{}_{;m}\mathbf{R}^m{}_{jhi} = 0. \tag{5.1.2}$$

As it has been shown in Ch.1, the set of curvature collineations of (M,g) is a Lie algebra over the reals; we denote this Lie algebra by $\mathcal{K}(M,g)$. It is also well known that every affine vector field (cf. Ch.1) is a curvature collineation , so that $\mathcal{A}(M,g)$ is a subalgebra of $\mathcal{K}(M,g)$. A curvature collineation is said to be **proper** if it is not an affine vector field.

Given any vector field X on M we set, as in previous chapters:

$$\overset{x}{h}_{ab} = \frac{1}{2}(X_{a;b} + X_{b;a}); \tag{5.1.3}$$

and

$$\overset{x}{f}_{ab} = \frac{1}{2}(X_{a;b} - X_{b;a}); \tag{5.1.4}$$

the symbol X in the above expressions being dropped when no confusion arises. One has then the following [52]:

THEOREM 5.1.1. *Let (M,g) be a pseudo-riemannian manifold and X be a vector field on M. Then:*
(a).If $X \in \mathcal{K}(M,g)$ then $h_{i(a}\mathbf{R}^{i}{}_{b)cd} = 0$;
(b).$X \in \mathcal{K}(M,g)$ if and only if

$$(h_{im;j} + h_{mj;i} - h_{ij;m})_{;h} - (h_{hm;j} + h_{mj;h} - h_{hj;m})_{;i} = 0.$$

From now on we assume that (M,g) is a space-time in the sense of Ch.2.

The equation in (a) of the above theorem has been mentioned in Ch.2, where its solutions where displayed, as well as the relationship between the rank of the Riemann tensor and the nature of the solutions. In particular it was stated in Ch.2 that whenever the rank of the Riemann tensor is ≥ 4 the only solutions to that equation are multiples of the metric g. If X is a curvature collineation , and if

$$\mathcal{L}_X g = 2h^{X},$$

has rank 4 at every point of M then we can consider h^{X} as a new metric on M with the same Riemann tensor as g. In such circumstances Hall [35] has proved that h^{X} is a constant multiple of g. Thus we have the following result due to Hall [35]

THEOREM 5.1.2. *If the Riemann tensor of a space-time (M,g) has rank ≥ 4 in an open dense subset of M, every curvature collineation of M is a homothetic vector field, that is $\mathcal{K}(M,g) = \mathcal{H}(M,g)$.*

On the other hand it is obvious that if a given space-time has no proper curvature collineations then one has $\mathcal{K}(M,g) = \mathcal{A}(M,g)$ and so the question of finite dimensionality of $\mathcal{K}(M,g)$ is of no interest (see [42]). Thus, we consider from now on space-times which can admit proper curvature collineations.

These space-times are those for which the equation in (a) of T.5.1.1. has solutions which are not constant multiples of the metric. We use then the results of Hall and McIntosh [45], mentioned in Ch.2, to split our analysis into the separate study of the classes described in T.2.3.1.

We shall take here the following order in the analysis of these classes:

In §.5.2 we shall study the case of 2+2-decomposable space-times; they correspond to (c) of T.2.3.1; space-times in this class will be called "class (I) space-times".

In §.5.3 we shall study the class that includes the 1+3-decomposable space-times; it corresponds to (b) of T.2.3.1. For reasons of convenience we shall call these space-times "class (II)(a) space-times". In contrast with the case for affine vector fields, the existence of a proper curvature collineation in a space-time in this class does not imply that the vector field u, whose existence is guaranteed by T.2.3.1.(b), is covariantly constant.

The case when this vector field is null will be analysed in §.5.5; space-times in this class will be called "class (II)(b) space-times".

Finally, in §.5.4 we will study "class (III) space-times", that is, space-times corresponding to (d) of T.2.3.1. Here we will follow a scheme similar to that of class (II).

Let (M, g) be an n-dimensional pseudo-riemannian manifold and let $p \in M$. For every open connected neighbourhood W of p in M we define:

$$\mathcal{K}_p(W) = \{v \in T_p(M) : \exists X \in \mathcal{K}(W, g_{|W}) : X_p = v\}.$$

An element of $\mathcal{K}_p(W)$ is the evaluations at p of a curvature collineation defined in W.

Obviously, $\mathcal{K}_p(W)$ is a vector subspace of $T_p(M)$. We define then:

$$\mathcal{K}_p = \bigcup_W \mathcal{K}_p(W),$$

where W runs over all connected open neighbourhoods of p in M. $\mathcal{K}_p$ is a vector subspace of $T_p(M)$. To see this, let $v, w \in \mathcal{K}_p$; then, by definition, there exist connected open neighbourhoods V, W of p in M such that $v \in \mathcal{K}_p(V)$ and $w \in \mathcal{K}_p(W)$. Let then U be the connected component of p in $V \cap W$. As $v \in \mathcal{K}_p(V)$, there exists $X \in \mathcal{K}(V, g_{|V})$ such that $v = X_p$; similarly, there exists $Y \in \mathcal{K}(W, g_{|W})$ such that $Y_p = w$. As $X_{|U}$ and $Y_{|U}$ both belong to $\mathcal{K}(U, g_{|U})$, one sees that for all reals a, b $av + bw \in \mathcal{K}_p(U) \subset \mathcal{K}_p$.

Notice now that if V, W are connected open neighbourhoods of p with $V \subset W$ then for every $X \in \mathcal{K}(W, g_{|W})$, $X_{|V}$ belongs to $\mathcal{K}(V, g_{|V})$; this shows that $\dim \mathcal{K}_p(W) \leq \dim \mathcal{K}_p(V)$. Consider then the map $W \mapsto \dim \mathcal{K}_p(W)$, where W runs over all connected open neighbourhoods of p in M. This is a map which takes its values in $\mathbf{N}$ and is bounded (by the dimension of M). Consequently, there exists an open connected neighbourhood U of p such that, for every connected open neighbourhood W of p:

$$\dim \mathcal{K}_p(U) \geq \dim \mathcal{K}_p(W).$$

Since $\mathcal{K}_p(U) \subset \mathcal{K}_p$, one has $\dim \mathcal{K}_p(U) \leq \dim \mathcal{K}_p$. Assume this is a strict inequality; then we can find $v \in \mathcal{K}_p$ such that $v \notin \mathcal{K}_p(U)$. Now, there exists an open connected neighbourhood W of p such that $v = X_p$ for some curvature collineation X in W. Let V be the connected component of p in $W \cap U$. Then $v \in \mathcal{K}_p(V)$ and $\mathcal{K}_p(U) \subset \mathcal{K}_p(V)$. It follows that $\mathcal{K}_p(U)$ is strictly contained in $\mathcal{K}_p(V)$ and so we have $\dim \mathcal{K}_p(V) > \dim \mathcal{K}_p(U)$, and this contradicts the definition of U. Thus, given every point p in M there exists an open connected neighbourhood U of p in M such that $\mathcal{K}_p = \mathcal{K}_p(U)$. An open connected neighbourhood U of p with the property that $\mathcal{K}_p = \mathcal{K}_p(U)$ is called a **regular neighbourhood** of p.

An open subset U of M will be called **regular** if it is a regular neighbourhood of each of its points. Finaly, define the map $\mathcal{R} : M \to \mathbf{N}$ by $p \mapsto \mathcal{R}(p) = \dim \mathcal{K}_p$. $\mathcal{R}$ is upper semicontinuous, that is, for every $k \in \mathbf{N}$ the set $\{p \in M : \mathcal{R}(p) \geq k\}$ is open in M.

We say that $\mathcal{R}$ is **locally constant** if for every point $p \in M$ there exists a neighbourhood W of p such that $\mathcal{R}$ is constant in W. (M, g) will be said to be **$\mathcal{K}$-regular** if $\mathcal{R}$ is locally constant. In such case, as M is assumed connected, $\mathcal{R}$ is constant.

Consider now a space -time (M, g) such that there exists an open subset U of M with the property that $(U, g_{|U})$ is flat. In such case, as the Riemann tensor of $(U, g_{|U})$ is identically zero every vector field on U is a curvatur collineation, that is, we have $\mathcal{K}(U, g_{|U}) = \mathcal{D}_1(U)$. Let then K be any compact subset of U and let $f \in \mathcal{F}(M)$ be such that $f_{|K} = 1$ and $f_{|(M \setminus U)} = 0$ (the existence of such a function is guaranteed by the theorem of partitions of unity). Given then any vector field X on U we can define a vector field X' on M by setting $X'_{|(M \setminus U)} = 0$ and $X'_{|U} = fX$. X' is then a curvature collineation of (M, g). Thus, we see that whenever M contains flat regions with non-empty interior, its Lie algebra of curvature collineations is infinite dimensional [42]. For this reason we shall assume from now on that the space-times under consideration do not contain such regions or, equivalently, that their Riemann tensors do not vanish on open sets.

5.2. Space-times of class (I) (The 2+2 case).

In this section we mean by class (I) space-time a space-time (M, g) such that for every point $p \in M$ there exists an open neighbourhood W of $p \in M$ such that $(W, g_{|W})$ is strictly 2+2-decomposable (see Ch.4). As mentioned in Ch.4, Hall and Kay proved that if (M, g) is a space-time in this class, then for every point $m \in M$ there exists an open neighbourhood W of m in M and a pair of 2-dimensional pseudo-riemannian manifolds, (S, i) and (T, j), the first of which is riemannian, such that $(W, g_{|W})$ is isometric to the product manifold $(T \times S, j \oplus i)$. Our assumptions above imply then that neither (T, j) nor (S, i) are such that they contain open subsets where their respective Riemann tensors vanish.

In local coordinates this can be translated as follows; for any $m_0 \in M$ one can find an open neighbourhood W of m_0 in M and a null tetrad (l, n, x, y) on W such that the distributions defined on W by $m \mapsto span(x_m, y_m)$ and $m \mapsto span(l_m, n_m)$ are integrable; one can then take for S and T their respective integral submanifolds through m_0, i and j being then the metrics induced by g on S and T.

One has then:

$$i_{ab} = 2l_{(a}n_{b)};$$

and

$$j_{ab} = x_a x_b + y_a y_b.$$

One can then (by restricting W if necessary) choose coordinates (s, t, u, v) in W adapted to the above decomposition (i.e. with (s, t) coordinates in T, and (u, v) coordinates in S). In these coordinates the metric then has a matrix of the form:

$$\begin{pmatrix} i_{11}(s,t) & i_{12}(s,t) & 0 & 0 \\ i_{21}(s,t) & i_{22}(s,t) & 0 & 0 \\ 0 & 0 & j_{33}(u,v) & j_{34}(u,v) \\ 0 & 0 & j_{43}(u,v) & j_{44}(u,v) \end{pmatrix}$$

If then one denotes by F_{ab} anf G_{ab} the bivectors $2l_{[a}n_{b]}$ and $2x_{[a}y_{b]}$, a simple calculation shows the existence of functions α and β on W such that [44]:

$$\mathbf{R}_{abcd} = \alpha F_{ab}F_{cd} + \beta G_{ab}G_{cd}, \tag{5.2.1}$$

where the functions α and β do not vanish on open subsets of W. Furthermore [44], α does not depend on u, v and β does not depend on s, t. This can be translated into:

$$x^b\alpha_{;b} = y^b\alpha_{;b} = l^b\beta_{;b} = n^b\beta_{;b} = 0. \tag{5.2.2}$$

The vector fields l, n are recurrent; we denote the recurrent vector by q, so that [44]:

$$\begin{aligned} l_{a;b} &= q_b l_a \\ n_{a;b} &= -q_b n_a. \end{aligned} \tag{5.2.3}$$

As for x and y, one can prove the existence of a 1-form p on W such that [44]:

$$\begin{aligned} x_{a;b} &= p_b y_a \\ y_{a;b} &= -p_b x_a. \end{aligned} \tag{5.2.4}$$

NOTE.5.2.1. The vector fields p, q satisfy $l^b p_b = n^b p_b = x^b q_b = y^b q_b = 0$ [44].

Using these relations on proves easily that F and G are both covariantly constant bivectors; the tensor fields i and j are also covariantly constant. One has therefore:

$$\mathbf{R}_{abcd;e} = \alpha_{;e}F_{ab}F_{cd} + \beta_{;e}G_{ab}G_{cd}. \tag{5.2.5}$$

Let us assume then that (M, g) admits a proper curvature collineation X so that relation (5.1.2) holds on M. In the open neighbourhood W above, a contraction of (5.1.2) with $l^i n^h$ gives:

$$X^m\alpha_{;m}F^k{}_j + \alpha X^m{}_{;j}F^k{}_m + \alpha X^m{}_{;h}n^h l_m F^k{}_j + \alpha X^m{}_{;i}l^i n_m F^k{}_j - \alpha X^k{}_{;m}F^m{}_j = 0. \tag{5.2.6}$$

As $F^k{}_j x^j = 0$, contracting this relation with x^j we get:

$$\alpha X^m{}_{;j}x^j F^k{}_m = 0. \tag{5.2.7}$$

Similarly, contracting (5.2.4) with y^j gives:

$$\alpha X^m{}_{;j}y^j F^k{}_m = 0. \tag{5.2.8}$$

Contracting these last two relations with l_k and n_k we get:

$$X^m{}_{;j}x^j l_m = X^m{}_{;j}x^j n_m = X^m{}_{;j}y^j l_m = X^m{}_{;j}y^j n_m = 0, \tag{5.2.9}$$

since, by assumption, α does not vanish on open subsets of W (otherwise $(W, g_{|W})$ is not strictly 2+2-decomposable). Using then (5.2.3) and (5.2.4) one gets:

$$x^j(l_m X^m)_{;j} = y^j(l_m X^m)_{;j} = x^j(n_m X^m)_{;j} = y^j(n_m X^m)_{;j} = 0. \tag{5.2.10}$$

Starting again from (5.1.2) but contracting now with $x^i y^h$ and then with l^j (n^j) we easily obtain, by the same method (and since β does not vanish on open subsets of W):

$$l^j(x_m X^m)_{;j} = n^j(x_m X^m)_{;j} = l^j(y_m X^m)_{;j} = n^j(y_m X^m)_{;j} = 0. \qquad (5.2.11)$$

These relations mean that if one sets $A = n_m X^m$, $B = l_m X^m$, $C = x_m X^m$ and $D = y_m X^m$, then, in the above described coordinate system, the functions A, B do not depend on u, v and the functions C, D do not depend on s, t. In other words (cf. §1.3) X is projectable with respect to the decomposition $\{(T, j), (S, i)\}$.

Define then $Y^a = j^a{}_b X^b$ and $Z^a = i^a{}_b X^b$ (so that $Y = Cx + Dy$ and $Z = Al + Bn$). Then $Y \in \mathcal{D}_1(S)$ and $Z \in \mathcal{D}_1(T)$. Furthermore, let :

$${}^1\mathbf{R}_{abcd} = \alpha F_{ab} F_{cd}, \qquad (5.2.12)$$

and

$${}^2\mathbf{R}_{abcd} = \beta G_{ab} G_{cd}. \qquad (5.2.13)$$

${}^1\mathbf{R}$ (resp ${}^2\mathbf{R}$) can be identified to the Riemann tensor of (T, j) (resp. (S, i)).

As $X = Z + Y$, we have, using (5.2.5):

$$X^m \mathbf{R}^k{}_{jhi;m} = {}^2\mathbf{R}^k{}_{jhi;m} Y^m + {}^1\mathbf{R}^k{}_{jhi;m} Z^m,$$

since we have :

$${}^1\mathbf{R}^k{}_{jhi;m} Y^m = {}^2\mathbf{R}^k{}_{jhi;m} Z^m = 0.$$

Similar calculations show that (5.1.2) reduces to an expression of the form:

$$A^k{}_{jhi} + B^k{}_{jhi} = 0,$$

where $A^k{}_{jhi}$ (resp. $B^k{}_{jhi}$) is relation (5.1.2) with Y (resp. Z) in the place of X and ${}^2\mathbf{R}$ (resp. ${}^1\mathbf{R}$) in the place of $\mathbf{R}$. As $A^k{}_{jhi}$ (resp. $B^k{}_{jhi}$) only contains terms in x and y (resp. l and n), one concludes that:

$$A^k{}_{jhi} = B^k{}_{jhi} = 0.$$

Thus, we have:

THEOREM 5.2.1. *Let (M, g) be a space-time in class (I), $m \in M$, and let $L = \{(T, j), (S, i)\}$ be its local decomposition around m. Then for every $X \in \mathcal{K}(M, g)$:*
(a). $X_{|W}$ is projectable;

(b).The projection Y (resp. Z) of X over S (resp. T) is a curvature collineation of (S,i) (resp. (T,j)).
(c).The Lie algebra $\mathcal{K}(W,g_{|W})$ is isomorphic to the product of the Lie algebras $\mathcal{K}(T,j)$ and $\mathcal{K}(S,i)$.

The importance of this theorem lies in the fact that it reduces the study of curvature collineations in class (I) space-times to the study of curvature collineations of 2-dimensional pseudo-riemannian manifolds.

Curvature collineations in 2-dimensional pseudo-riemannian manifolds

The main result for this type of manifolds is the following, due to Katzin et al. [52]:

THEOREM 5.2.2. *Let (V,j) be a 2-dimensional pseudo-riemannian manifold whose Riemann tensor does not vanish on open subsets of V and let X be a vector field on V. Then X is a curvature collineation if and only if :*
(a). It is a conformal vector field of (V,j) and,
(b). ϕ being its conformal function, it satisfies the equation $\mathcal{L}_X\mathbf{R}+2\phi\mathbf{R}=0$.

Proof. We analyse first the riemannian case. Let $p \in V$ and let U be an open neighbourhood of p in V sufficiently small for an orthonormal basis (x,y) of vector fields to exist on it. A simple calculation shows then that in U:

$$\mathbf{R}_{abcd}=\frac{1}{2}\mathbf{R}(j_{ac}j_{bd}-j_{ad}j_{bc}). \tag{5.2.14}$$

As x_ax_b, $2x_{(a}y_{b)}$ and y_ay_b span the space of $(0,2)$ symmetric tensor fields on U, one has (cf. (5.1.3)):

$$h_{ab}=\lambda x_ax_b+2\mu x_{(a}y_{b)}+\delta y_ay_b, \tag{5.2.15}$$

where λ,μ,δ are differentiable functions on U.

The equation in T.5.1.1.(a), when contracted with x^cy^d reduces to:

$$2(\lambda-\delta)x_{(a}y_{b)}+2\mu(y_ay_b-x_ax_b)=0,$$

and this shows that $\mu=\lambda-\delta=0$ in U, thus that X is conformal. There exists, therefore a differentiable function ϕ on M such that:

$$\mathcal{L}_Xj=2\phi j.$$

Now, as the Lie derivative commutes with contractions, one has:

$$\mathcal{L}_X Ricci = 0, \tag{5.2.16}$$

so, as $\mathbf{R} = j^{ab}\mathbf{R}_{ab}$, we have:

$$(\mathcal{L}_X \mathbf{R}) = \mathbf{R}_{ab}(\mathcal{L}_X j')^{ab},$$

where j' is the tensor with components j^{ab}. Thus:

$$\mathcal{L}_X \mathbf{R} + 2\phi \mathbf{R} = 0.$$

Reciprocally, suppose X is conformal and that it satisfies the above equation. Using then (5.2.14), on finds that:

$$2(\mathcal{L}_X \mathbf{R})^a{}_{bcd} = (\mathcal{L}_X \mathbf{R})(j^a{}_c j_{bd} - j^a{}_d j_{bc}) + \mathbf{R}(\delta^a{}_c(\mathcal{L}_X j)_{bd} - \delta^a{}_d(\mathcal{L}_X j)_{bc} = 0.$$

This establishes the theorem for the riemannian case. The proof is virtually the same for the lorentzian case ■

NOTE.5.2.2. A different proof of the conformallity of X can be given by adapting the method described by Hall in [35]. This immediately gives the:

COROLLARY 5.2.3.*If* (V, j) *is such that* $\mathbf{R}$ *is constant and non-zero, every curvature collineation of* (V, j) *is a Killing vector field,*

Using now the notion of $\mathcal{K}$-regularity of the preceding section, we have:

THEOREM 5.2.4.*Let* (V, j) *be a* $\mathcal{K}$*-regular 2-dimensional pseudo-riemannian manifold whose Riemann tensor never vanishes. Then for every* $p \in V$ *there exists an open connected neighbourhood* W *of* p *in* V *such that* $\mathcal{K}(W, j_{|W})$ *is finite dimensional, its dimension being at most 3.*

NOTE.5.2.3. As mentioned in §.1.9, the Lie algebra of conformal vector fields of a 2-dimensional pseudo-riemannian manifold is infinite dimensional.

Proof. Let $p_o \in V$. If $\mathcal{R}(p_o) = 0$ then, due to the $\mathcal{K}$-regularity of V, $\mathcal{R}$ is identically zero in some open neighbourhood of p_o and there is nothing to prove.

If $\mathcal{R}(p_o) = 1$, we can choose an open connected neighbourhood U of p_o in V such that $\mathcal{R} = 1$ in U. Let then U' be a regular neighbourhood of p_o (cf. §.5.1);

as $\mathcal{R}(p_o) = 1$ there exists then $X \in \mathcal{K}(U', j_{|U'})$ such that $X_{p_o} \neq 0$. Let then W be the connected component of p_o in $U \cap U'$. Restricting W if necessary, we can assume, due to the continuity of X, that X does not vanish in W.

If Y is any other curvature collineation , it follows from our assumption on $\mathcal{R}$ that there exists a differentiable function λ on W such that $Y = \lambda X$. This gives:

$$Y^a{}_{;b} = \lambda_{;b} X^a + \lambda X^a{}_{;b}. \qquad (*)$$

From T.5.2.2. and our assumptions on the Riemann tensor of (V, j),we know that both X and Y are conformal; thus, there exist differentiable functions ϕ and ψ on W such that $h^X = \phi j$ and $h^Y = \psi j$. $(*)$ gives, therefore:

$$\lambda_a X_b + \lambda_b X_a = 2(\psi - \lambda\phi) j_{ab}. \qquad (**)$$

In the riemannian case, as $X^a X_a \neq 0$, we can find a vector field Z on W (restricting again W if necessary) which is orthogonal to X and linearly independent of X. Contracting then $(**)$ with Z^a we get $Z^a \lambda_a X_b = 2(\psi - \lambda\phi) Z_b$, and this shows that $Z^a \lambda_a = \psi - \lambda\phi = 0$. Thus, as (X, Z) span the tangent space at each point of W, one deduces that λ_a is parallel to X_a. $(**)$ shows then that $\lambda_a = 0$; since W is connected, this shows that λ is constant.

In the lorentzian case the same argument holds if X is non-null. So, let us assume that $X = l$ is null and let n be another null vector field on W such that $l^a n_a = 1$. Using $(*)$ and the fact that both X and Y are curvature collineations (5.1.2) gives

$$\mathcal{L}_Y Riemann - \lambda \mathcal{L}_X Riemann = 0,$$

so

$$\lambda_b l^m \mathbf{R}^a{}_{mcd} + \lambda_c l^m \mathbf{R}^a{}_{bmd} + \lambda_d l^m \mathbf{R}^a{}_{bcm} - \lambda_m l^a \mathbf{R}^m{}_{bcd} = 0.$$

Since in this case we have $\mathbf{R}^a{}_{bcd} = \alpha(l^a n_b - l_b n^a)(l_c n_d - l_d n_c)$, for some function α on W (which does not vanish on open subsets of W), we find that, contracting with $l_a n^b$:

$$\lambda_c l_d - \lambda_d l_c = 0,$$

and this shows that λ_c is parallel to l. Replacing in $(**)$, and considering the rank of each side, we see that $\lambda_c = 0$; since W is connected, this shows that λ is constant.

Consider now the case when $\mathcal{R}(p_o) = 2$.

In this case we can find a regular neighbourhood W of p_o in V such that $\mathcal{R} = 2$ in W.

We analyse separately the two possible cases.

The riemannian case

Restricting W if necessary, there exists then $X \in \mathcal{K}(W, j_{|W})$ such that it never vanishes in W. Restricting again W, if necessary, we can then choose coordinates (u, v) in W such that $X = \partial_u$. As X is conformal, one has $\mathcal{L}_X j = 2\phi j$ for some differentiable function ϕ on W. This gives, if j_{ab} are the components of j with respect to the above coordinates, $\partial_u j_{ab} = 2\phi j_{ab}$. Let then Φ be a differentiable function on W such that $\partial_u \Phi = \phi$. Then the above equation shows that one has

$$j_{ab} = e^{2\Phi} \rho_{ab},$$

where the functions ρ_{ab} depend only on the variable v. Thus, in the above coordinates j has the form:

$$j = e^{2\Phi(u,v)} \begin{pmatrix} a(v) & b(v) \\ b(v) & c(v) \end{pmatrix}.$$

Since j is positive definite, the functions a, c never vanish on W. This allows us to define $B = a/\sqrt{ac - b^2}$ and $C = -bB/a$. The vector field $Z = C\partial_u + B\partial_v$ is then orthogonal to X and satisfies $j(Z, Z) = a$ and $[X, Z] = 0$. This last relation shows that there exists a coordinate system (u, w) on W such that $X = \partial_u$ and $Z = \partial_w$. Thus, we may assume from the start that $b = 0$ and $c = a$. j becomes then:

$$j = e^{2(\Phi + A)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $2A = \operatorname{Log} a$ (notice that as we are in the riemannian case, $a > 0$). A simple calculation shows that one has then:

$$\mathbf{R} = -2e^{-2(\Phi + A)}(\partial_v^2 A + \Delta\Phi), \tag{5.2.17}$$

where:

$$\Delta\Phi = \partial_u^2 \Phi + \partial_v^2 \Phi. \tag{5.2.18}$$

As X is a curvature collineation , we have $\mathcal{L}_X\mathbf{R} + 2\phi\mathbf{R} = 0$. But, since $X = \partial_u$, we have, from (5.2.17), as $\partial_u A = 0$:

$$\mathcal{L}_X\mathbf{R} = \partial_u\mathbf{R} = (-2\partial_u\Phi)\mathbf{R} - 2e^{-2(A+\Phi)}\partial_u\Delta\Phi,$$

so that, as $\partial_u\Phi = \phi$:

$$\mathcal{L}_X\mathbf{R} + 2\phi\mathbf{R} = -2e^{-2(A+\Phi)}\partial_u\Delta\Phi.$$

Thus, we have:

$$\partial_u\Delta\Phi = 0. \tag{5.2.19}$$

Let now $Y = \alpha\partial_u + \beta\partial_v$ be another curvature collineation . Since Y is conformal, we have in W, $\mathcal{L}_Y j = 2\psi j$, for some differentiable function ψ on W. This gives, after some simple calculations:

$$\begin{aligned}\mathcal{L}_Y\Phi + \beta\partial_v A + \partial_u\alpha &= \psi \\ \partial_u\beta + \partial_v\alpha &= 0\ . \\ \partial_u\alpha - \partial_v\beta &= 0\end{aligned} \tag{5.2.20}$$

Since Y is a curvature collineation , we have (cf. T.5.2.2) $\mathcal{L}_Y\mathbf{R} + 2\psi\mathbf{R} = 0$. Let us set:

$$\Lambda = -2e^{-2A}(\partial_v^2 A + \Delta\Phi), \tag{5.2.21}$$

so that $\mathbf{R} = \Lambda e^{-2\Phi}$. The equation $\mathcal{L}_Y\mathbf{R} + 2\psi\mathbf{R} = 0$ gives then:

$$\mathcal{L}_Y\Lambda - 2\Lambda\mathcal{L}_Y\Phi + 2\psi\Lambda = 0. \tag{5.2.22}$$

Combining this with the first equation in (5.2.20) we get:

$$\mathcal{L}_Y\Lambda + 2\Lambda\beta\partial_v A + 2\Lambda\partial_u\alpha = 0,$$

and since $\partial_u\Lambda = 0$ (from (5.2.19) and (5.2.21)), this can be written as:

$$\beta\partial_v\Lambda + 2\Lambda\beta\partial_v A + 2\Lambda\partial_u\alpha = 0. \tag{5.2.23}$$

Using then the third relation in (5.2.20) one gets

$$\beta(\partial_v\Lambda + 2\Lambda\partial_v A) + 2\Lambda\partial_v\beta = 0. \tag{5.2.24}$$

Let then $\gamma = \beta^2\Lambda e^{2A}$. Using the above relation, one easily proves that $\partial_v\gamma = 0$. This result and the definition of γ show then that β has the form $\beta(u,v) =$

$C(u)D(v)$. This gives then, since the second and third equations in (5.2.20) show that β is harmonic:

$$\frac{d^2C}{du^2}D + \frac{d^2D}{dv^2}C = 0.$$

As β does not vanish and the functions C and D do not depend on the same variables, we must have:

$$\frac{d^2C}{du^2} = KC; \tag{5.2.25}$$

and

$$\frac{d^2D}{dv^2} = -KD; \tag{5.2.26}$$

for some real constant K.

Notice that D is not an arbitrary function, it depends on the metric j only, and so it determines a unique value for the real number K. In fact (5.2.24) gives

$$C(D(\partial_v\Lambda + 2\Lambda\partial_v A) + 2\Lambda\partial_v D) = 0,$$

and this shows that D is uniquely determined (up to a multiplicative constant) by j

K being fixed, consider then the equation on C. This is a second order ordinary linear differential equation; consequently its set of solutions (S) is a 2-dimensional real vector space.

Notice that, by (5.2.20), α is, up to an additive constant, uniquely determined by β. However, this last constant accounts for the curvature collineation X we started with.

This shows that in the riemannian case, when $\mathcal{R} = 2$, $\dim \mathcal{K}(W, j_{|W}) = 3$.

The lorentzian case

In this case, as we assume that $\mathcal{R} = 2$ in W, for every point $p \in W$ there exists a curvature collineation X of $(W, j_{|W})$ such that $X_p \neq 0$ and $X^a X_a(p) < 0$. By continuity, and restricting W if necessary, we can therefore assume the existence on W of a curvature collineation X which is everywhere timelike. Following then a method similar to that of the preceding case, it is easy to prove the existence in W (restricting again W if necessary) of coordinates (s, t) such that $X = \partial_s$ and

$$j = e^{2(\Phi+A)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where A is a differentiable function on W depending only on t and Φ is a solution of the equation $\partial_s f = \phi$, where ϕ is the conformal scalar of X.

A simple computation gives then:

$$\mathbf{R} = -2e^{-2(\Phi+A)}(\partial_t^2 A + \Delta\Phi), \tag{5.2.27}$$

where:

$$\Delta\Phi = -\partial_s^2\Phi + \partial_t^2\Phi. \tag{5.2.28}$$

The condition that X is a curvature collineation implies then, as in the preceding case, that

$$\partial_s \Delta\Phi = 0. \tag{5.2.31}$$

Let then $Y = \alpha\partial_s + \beta\partial_t$ be another curvature collineation in W. By a method similar to that of the preceding case, we get the following system of equations:

$$\begin{aligned} \mathcal{L}_Y\Phi + \beta\partial_t A + \partial_s\alpha &= \psi, \\ -\partial_t\alpha + \partial_s\beta &= 0, \\ \partial_s\alpha - \partial_t\beta &= 0. \end{aligned} \tag{5.2.29}$$

If one defines then

$$\Lambda = -2e^{-2A}(\partial_t^2 A + \Delta\Phi),$$

using (5.2.31) we get

$$\beta(\partial_t\Lambda + 2\Lambda\partial_t A) + 2\Lambda\partial_t\beta = 0.$$

This equation allows us to conclude, as in the preceding case, that β has the form

$$\beta = C(s)D(t).$$

This then shows that the functions C and D satisfy again the equations (5.2.27) and (5.2.28), and so we are led to the same conclusions as in the riemannian case.■

One deduces from this result that:

THEOREM 5.2.5.*If (V, j) is a connected 2-dimensional pseudo-riemannian manifold whose Riemann tensor does not vanish on open subsets of V, then the Lie algebra $\mathcal{K}(V, j)$ is finite dimensional and its dimension is at most 3.*

Proof. From our assumptions it follows that there exists an open subset W of V such that for all $p \in W$, $\mathbf{R}(p) \neq 0$ ($\mathbf{R}$ is the curvature scalar). For each $p \in W$ consider then the vector space $\mathcal{K}_p$ (cf. §.5.1). If for all $p \in W$ one has $\dim \mathcal{K}_p = 0$, then obviously $\mathcal{K}(W, j_{|W}) = \{0\}$; otherwise there exists at least a point p_1 in W such that $\mathcal{K}_{p_1}$ is not trivial. This leads to two possibilities: either for all $p \in W$ one has $\dim \mathcal{K}_p = 1$ or there exists $p_2 \in W$ for which $\dim \mathcal{K}_{p_2} = 2$. Due to the upper semicontinuity of the function $\mathcal{R}$, it follows that there exists an open subset U of V such that the 2-dimensional pseudo-riemannian manifold $(U, j_{|U})$ is $\mathcal{K}$-regular and such that its Riemann tensor is nowhere zero. The preceding theorem tells us then that $\mathcal{K}(U, j_{|U})$ is finite dimensional, its dimension being at most 3.

Let $s : \mathcal{K}(V, j) \to \mathcal{K}(U, j_{|U})$ be the map associating with $X \in \mathcal{K}(V, j)$ its restriction to U. This map is clearly linear over the reals. We prove now that it is injective.

To prove this it is sufficient to prove that if $X \in \mathcal{K}(V, j)$ is such that $s(X) = 0$ then X is identically zero in V. Let then q be a point in V such that $\mathbf{R}(q) \neq 0$, and let $r \in U$. Since V is connected, there exists a differentiable curve $\gamma : [0, 1] \to V$ such that $\gamma(0) = r$, $\gamma(1) = q$. Now, as $K = \gamma([0, 1])$ is a compact subset of V, we can find a finite open covering of K, $(A_1, ..., A_m)$ such that $r \in A_1$, $q \in A_m$ and, for every $1 \leq i \leq (m-1)$ $A_i \cap A_{i+1} \neq \emptyset$. Furthermore each of the A_i can chosen to be the domain of an "isothermal" coordinate system [79], that is, a coordinate system (u, v) with respect to which j takes the form $j = f\eta$ where η is the pseudo-euclidean metric of the same signature as j and f is some positive function on A_i.

Consider then $X_{|A_i}$; since the Riemann tensor does not vanish on open subsets of A_i, $X_{|A_i}$ is a conformal vector field of $(A_i, j_{|A_i})$. Thus, its components with respect to the isothermal coordinates are harmonic functions of these coordinates. Consequently, if these components vanish on some open subset of A_i they vanish everywhere in A_i. The conditions imposed on the family (A_i) and the fact that $A_1 \cap U \neq \emptyset$ show then that $X_{|A_i} = 0$ for all i, hence that $X_q = 0$. Since the set of zeros of the Riemann tensor of V does not contain open subsets of V it follows, by continuity, that X vanishes everywhere.

Thus, s is injective, and since it is linear over the reals, it follows that $\mathcal{K}(V, j)$ is isomorphic to its image by s. This proves the theorem ■

EXAMPLES.I-The riemannian case

One can use the preceding calculations to find some examples. We describe here a family of 2-dimensional riemannian manifolds $(V,j)$for which the Lie algebra $\mathcal{K}(V,j)$ is 3-dimensional.

Let us consider an open subset V of $\mathbf{R}^2$ together with coordinates (u,v). Let $c,d \in \mathbf{R}$ $(c \neq 0)$ be such that $cv+d$ is strictly positive at all points of V.

Let $\Phi : V \longrightarrow \mathbf{R}$ be a harmonic function.

Define:

$$j = e^{2\Phi(u,v)} \begin{pmatrix} (cv+d)^{\frac{1}{c^2}} & 0 \\ 0 & (cv+d)^{\frac{1}{c^2}} \end{pmatrix}$$

A simple computation gives as the only significant components of the Riemann tensor:

$$\mathbf{R}^1{}_{212} = -\mathbf{R}^2{}_{112} = \frac{1}{2(cv+d)^2},$$

whilst the Ricci tensor has components:

$$\mathbf{R}_{11} = \mathbf{R}_{22} = \frac{1}{2(cv+d)^2}.$$

This gives for the Ricci scalar:

$$\mathbf{R} = e^{-2\Phi}(cv+d)^{-\frac{1}{c^2}}(cv+d)^{-2}.$$

So:

$$D(v) = cv+d,$$

thus, $K=0$ in (5.2.28). Consequently, one has :

$$\beta(u,v) = (cv+d)(A_1u+A_2),$$

where A_1 and A_2 are real constants. Using then (5.2.20) we get:

$$\alpha(u,v) = \frac{cA_1}{2}(u^2-v^2) + cA_2u - dA_1v + A_3,$$

where A_3 is a real constant. Thus, the following curvature collineations span $\mathcal{K}(V,j)$:

$$\begin{aligned} X_1 &= \partial_u; \\ X_2 &= [\frac{c}{2}(u^2 - v^2) - dv]\partial_u + u(cv+d)\partial_v; \\ X_3 &= cu\partial_u + (cv+d)\partial_v. \end{aligned}$$

EXAMPLES.II-The lorentzian case

Consider an open subset U of $\mathbf{R}^2$ with coordinates (s,t) and let c, d be real numbers such that $t \mapsto ct + d$ is strictly positive at all points of U.

Let $\Phi : U \to \mathbf{R}$ be such that $\Delta\Phi = 0$. Then the metric j defined on U by

$$j = e^{2\Phi} \begin{pmatrix} -(ct+d)^{\frac{1}{c^2}} & 0 \\ 0 & (ct+d)^{\frac{1}{c^2}} \end{pmatrix},$$

has a 3-dimensional Lie algebra of curvature collineations; this Lie algebra is spanned by the following vector fields:

$$\begin{aligned} X_1 &= \partial_s \\ X_2 &= [\frac{c}{2}(s^2 + t^2) + dt]\partial_s + s(ct+d)\partial_t\,. \\ X_3 &= cs\partial_s + (ct+d)\partial_t \end{aligned}$$

EXAMPLES.III. From the above examples we get the following space-time (M,g), where M is an open subset of $\mathbf{R}^4$ with coordinates (t,s,u,v) such that $t \neq 0$ and $v \neq 0$ in M, and g is given by

$$g = \begin{pmatrix} -te^{2st} & 0 & 0 & 0 \\ 0 & te^{2st} & 0 & 0 \\ 0 & 0 & ve^{2uv} & 0 \\ 0 & 0 & 0 & ve^{2uv} \end{pmatrix}.$$

The vector fields

$$\begin{aligned} X_1 &= \partial_s, \\ X_2 &= \frac{(s^2+t^2)}{2}\partial_s + st\partial_t, \\ X_3 &= s\partial_s + t\partial_t, \\ X_4 &= \partial_u, \\ X_5 &= \frac{(u^2-v^2)}{2}\partial_u + uv\partial_v, \\ X_6 &= u\partial_u + v\partial_v, \end{aligned}$$

form a basis of $\mathcal{K}(M,g)$. The Riemann tensor of this space-time is given by

$$\mathbf{R}_{1212} = \frac{e^{2st}}{2t},$$

$$\mathbf{R}_{3434} = \frac{e^{2uv}}{2v}.$$

As for the Ricci tensor, it is given by

$$\mathbf{R}_{11} = -\mathbf{R}_{22} = \frac{1}{2t^2},$$

$$\mathbf{R}_{33} = \mathbf{R}_{44} = \frac{1}{2v^2},$$

and so it has Segre type $\{(1,1)(11)\}$. Thus, it may represent a non-null electromagnetic field (cf. Ch.2, p.51). Since its Weyl tensor is non-zero, its Petrov type is D (see TABLE at the end of Ch.2).

As a corollary of T.5.2.5 and T.1.8.9, we have

THEOREM 5.2.6. *If (M,g) is connected, simply connected and geodesically complete class (I) space-time, the Lie-algebra $\mathcal{K}(M,g)$ is finite-dimensional and its dimension is at most 6.*

5.3. Space-times of class II(a).

In this section, we consider space-times (M,g) with the property that there exists an open dense subset O of M such that for every $p \in O$ the set of solutions of the equation given at p by:

$$\mathbf{R}^a{}_{bcd}v^d = 0, \tag{5.3.1}$$

is a 1-dimensional non-null subspace of $T_p(M)$. We denote by $\mathcal{V}_p$ this subspace.

In all that follows we make the assumption that the distribution $p \mapsto \mathcal{V}_p$ is in fact defined in the whole of M, is differentiable and, furthermore, that there exists a nowhere null, nowhere zero, vector field u on M such that for every $p \in M$ u_p spans $\mathcal{V}_p$. We assume u scaled in such a way that $u^a u_a = \epsilon = \pm 1$.

A space-time satisfying these conditions will be called a class II(a) space-time in the sequel.

General observations

The following result is due to Collinson, [11]:

THEOREM 5.3.1. *If (M, g) is a class II(a) space-time and u is the solution of (5.3.1) described above, then u is a geodesic vector field.*

Proof. One has $R^a{}_{bcd}u_a = 0$, consequentely:

$$R^a{}_{bcd;k}u_a + R^a{}_{bcd}u_{a;k} = 0.$$

This gives

$$R^a{}_{b[cd;k]}u_a + R^a{}_{b[cd|}u_{a;|k]} = 0,$$

and the Bianchi identities give, therefore:

$$R^a{}_{b[cd|}u_{a;|k]} = 0.$$

Contracting with u^k we get:

$$R^a{}_{bcd}(u_{a;k}u^k) = 0,$$

and this proves that $u_{a;k}u^k = \zeta u_a$ for some function ζ on M. In fact if at some point p of M $u_{a;k}u^k$ is not parallel to u then this holds in some open neighbourhood of p. Now in this neighbourhood either the Riemann tensor is zero or its rank is at most 1. This contradicts our assumptions about the rank of the Riemann tensor.

As u has constant length one has in fact $\zeta = 0$ ■

THEOREM 5.3.2. *Let (M, g) be a class II(a) space-time, u the vector field on M defined at the beginning of this §. Then if $X \epsilon \mathcal{K}(M, g)$ there exist functions ψ^X, ρ^X in M such that the following relations hold (cf. §.5.1):*

(a). $h^X{}_{ab} = \psi^X g_{ab} + \rho^X u_a u_b$;

(b). $f^X{}_{ab}u^b = X^b u_{a;b}$;

(c). $[u, X] = (\psi^X + \epsilon\rho^X)u$.

Except when confusion may arise, we shall drop the superscript X in the objects defined above.

Proof . Relation (a) follows from T.2.3.1.(b). Now, if C denotes contraction one has $C(u \otimes Riemann) = 0$. Hence,

$$\mathcal{L}_X(C(u \otimes Riemann)) = 0.$$

As Lie derivation commutes with contractions, this gives

$$C(\mathcal{L}_X u \otimes Riemann) + C(u \otimes \mathcal{L}_X Riemann) = 0.$$

Since X is a curvature collineation it follows that :

$$C(\mathcal{L}_X u \otimes Riemann) = 0,$$

so one has, for some differentiable function $\sigma : M \to \mathbf{R}$:

$$\mathcal{L}_X u = \sigma u.$$

Now one has,

$$[u, X]_a = -\sigma u_a = u^b X_{a;b} - X^b u_{a;b} =$$

$$u^b(\psi g_{ab} + \rho u_a u_b + f_{ab}) - X^b u_{a;b} =$$

$$(\psi + \epsilon\rho)u_a + f_{ab}u^b - X^b u_{a;b}.$$

Contracting this with u^a and using $u^a u_{a;b} = f_{ab}u^a u^b = 0$, we deduce that $-\sigma = \psi + \epsilon\rho$; this proves (c); (b) follows immediately ■

We deduce from this that:

THEOREM 5.3.3. *Let (M, g) be a class II(a) space-time, uthe vector field on M defined at the beginning of this §. For $X \epsilon \mathcal{K}(M, g)$ define $\alpha^{X} = \epsilon u^a X_a$ ($\epsilon = u^a u_a$), $Z = X - \alpha^{X} u$, so that $u^a Z_a = 0$.Then*
(a).$u^b \alpha^{X}_b = \psi + \epsilon\rho$;
(b).$[u, Z] = 0$.

Again, the superscript X in α^{X} will be dropped whenever no confusion is possible.

Proof. As u is geodesic, we have :

$$\epsilon(\psi^{X} + \epsilon\rho^{X}) = u^a[u, X]_a = u^a(u^b X_{a;b} - X^b u_{a;b}) =$$

$$u^a u^b(\alpha_b u_a + \alpha u_{a;b} + Z_{a;b}) = \epsilon u^b \alpha_b + u^a u^b Z_{a;b} =$$

$$\epsilon u^b \alpha_b - u^b Z^a u_{a;b} = \epsilon u^b \alpha_b.$$

This proves (a). Using (a) and T.5.3.2.(c) we have

$$[u, Z] = [u, X - \alpha u] = [u, X] - [u, \alpha u] = 0,$$

and the proposition follows ■

It follows that:

THEOREM 5.3.4. *Let (M, g) be a class II(a) space-time. If $X \epsilon \mathcal{K}(M, g)$ then :*
(a).$\epsilon \alpha_m = (\psi^X + \epsilon \rho^X) u_m + Z^a (u_{a;m} - u_{m;a})$.
(b).Either $\psi^X + \epsilon \rho^X$ is constant or the vector field u is hypersurface orthogonal .

Proof. We have $X_a = \alpha u_a + Z_a$, hence:

$$X^m{}_{;b} = \alpha_b u^m + \alpha u^m{}_{;b} + Z^m{}_{;b}.$$

Now, as X is a curvature collineation , contracting (5.1.2) with u_a we get, using (5.3.1):

$$X^m u_a R^a{}_{bcd;m} - u_a X^a{}_{;m} R^m{}_{bcd} = 0.$$

Now we have, using the above decomposition for X:

$$(\alpha u^m + Z^m) u_a R^a{}_{bcd;m} - u_a(\alpha_m u^a + \alpha u^a{}_{;m} + Z^a{}_{;m}) R^m{}_{bcd} = 0,$$

that is :

$$-\epsilon \alpha_m R^m{}_{bcd} - \alpha u_a u^a{}_{;m} R^m{}_{bcd} - u_a Z^a{}_{;m} R^m{}_{bcd} + \alpha u^m u_a R^a{}_{bcd;m} + Z^m u_a R^a{}_{bcd;m} = 0.$$

The second term in this expression vanishes as $u_a u^a{}_{;m} = 0$, so it becomes:

$$-\epsilon \alpha_m R^m{}_{bcd} - u_a Z^a{}_{;m} R^m{}_{bcd} + \alpha u^m u_a R^a{}_{bcd;m} + Z^m u_a R^a{}_{bcd;m} = 0. \quad (5.3.2)$$

Consider separately the last three terms of this expression :

$$-u_a Z^a{}_{;m} R^m{}_{bcd} = Z^a u_{a;m} R^m{}_{bcd},$$

as $u^a Z_a = 0$ gives $u_a Z^a{}_{;m} + Z^a u_{a;m} = 0$. From $u_a R^a{}_{bcd} = 0$ we get $u_{a;m} R^a{}_{bcd} + u_a R^a{}_{bcd;m} = 0$, so the second term becomes:

$$\alpha u^m (u_a R^a{}_{bcd;m}) = -\alpha u^m u_{a;m} R^a{}_{bcd} = 0,$$

as $u^m u_{a;m} = 0$. Again using the relation above we have:

$$Z^m u_a R^a{}_{bcd;m} = -Z^m u_{a;m} R^a{}_{bcd},$$

and so replacing in (5.3.2) we get :

$$-\epsilon\alpha_m R^m{}_{bcd} + Z^a u_{a;m} R^m{}_{bcd} - Z^m u_{a;m} R^a{}_{bcd} = 0.$$

Interchanging a and m in the last term this gives:

$$(-\epsilon\alpha_m + Z^a(u_{a;m} - u_{m;a}))R^m{}_{bcd} = 0.$$

Consequentely, there exists a function $\zeta : M \to \mathbf{R}$ such that

$$-\epsilon\alpha_m + Z^a(u_{a;m} - u_{m;a}) = -\zeta u_m.$$

Since $\epsilon\alpha_m u^m = \epsilon\zeta$ we deduce that $\zeta = (\psi + \epsilon\rho)$. This proves (a).

It follows that :

$$\epsilon\alpha_{mp} = (\psi^\times + \epsilon\rho^\times)_p u_m + (\psi^\times + \epsilon\rho^\times)u_{m;p} + Z^a{}_{;p}(u_{a;m} - u_{m;a}) + Z^a(u_{a;mp} - u_{m;ap}).$$

Contracting this relation with u^m, doing the same for $\epsilon\alpha_{pm}$ and substracting we get:

$$\epsilon(\psi^\times + \epsilon\rho^\times)_p + u^m Z^a u_{a;mp} - u^m Z^a u_{m;ap} - u^m(\psi^\times + \epsilon\rho^\times)_m u_p - u^m Z^a{}_{;m} u_{a;p}$$
$$+u^m Z^a{}_{;m} u_{p;a} - Z^a u^m u_{a;pm} + Z^a u^m u_{p;am} = 0. \qquad (*)$$

The second and fifth terms of $(*)$ give:

$$u^m Z^a(u_{a;mp} - u_{a;pm}) = 0,$$

since u contracts the Riemann tensor to zero. Therefore $(*)$ reduces to:

$$\epsilon(\psi^\times + \epsilon\rho^\times)_p - u^m Z^a u_{m;ap} - u^m(\psi^\times + \epsilon\rho^\times)_m u_p$$
$$-u^m Z^a{}_{;m} u_{a;p} + u^m Z^a{}_{;m} u_{p;a} + Z^a u^m u_{p;am} = 0. \qquad (**)$$

The fifth term of $(**)$ gives:

$$u^m Z^a{}_{;m} u_{p;a} = Z^m u^a{}_{;m} u_{p;a},$$

since $[u, Z] = 0$. As $u^a u_{p;a} = 0$ we have :

$$u^m Z^a{}_{;m} u_{p;a} = -Z^m u^a u_{p;am} = -Z^a u^m u_{p;ma},$$

so the fifth term cancels with the sixth in (**). Similarly, using again $u^m u_{m;a} = 0$ and $[u, Z] = 0$, we prove that the second and the fourth term cancel each other. (b) follows ■

NOTE.5.3.1. This last result justifies a division in the study of this type of space-times according to whether u is or not hypersurface orthogonal .

NOTE.5.3.2. T.5.3.4 is closely related to a similar result due to Collinson, [11], which was, however, obtained in a different setting; in fact, Collinson proves that, if, under the same conditions on u, $\psi g_{ab} + \rho u_a u_b$ is a metric with the same Riemann tensor as g_{ab} then either u is hypersurface orthogonal or the function $\psi + \epsilon\rho$ is constant.

The case when there exists a curvature collineation solution of (5.3.1)

Lemma 5.3.5. *Let (M, g) be a class II(a) space-time and u be the prefered vector field defined at the beginning of this §. If there exists a differentiable function $\lambda : M \to \mathbb{R}$ such that $\lambda u \epsilon \mathcal{K}(M, g)$, then u is hypersurface orthogonal.*

Proof.. Setting $X = \lambda u$, we have $\alpha = \lambda$, $Z = 0$; so we have, from T.5.3.4(a):

$$\lambda_m = \epsilon(\overset{X}{\psi} + \epsilon\overset{X}{\rho})u_m, \tag{5.3.3}$$

thus, either λ is constant or u is hypersurface orthogonal .

Let us, therefore, analyse the case when λ is constant. We set $\lambda = 1$.

Consider the bivector of u, $F_{ab} = u_{[a;b]}$. F is simple since u contracts it to zero. Therefore, there exists a vector field x orthogonal to u and such that $F_{ab}x^b = 0$. Let us assume there exists a point $p \epsilon M$ such that F does not vanish at p. In the case when x is non-null we can then, in any sufficiently small open neighbourhood U of p in M, choose an orthonormal basis of vector fields of the form (u, v, x, y) where, setting $x^a x_a = \epsilon'$, we have $v^a v_a = -\epsilon\epsilon'$, $y^a y_a = 1$. There exists then a function $\mu : U \to \mathbb{R}$, which is nowhere zero and such that $F_{ab} = 2\mu v_{[a}y_{b]}$; thus, we have $u_{a;b} = \psi g_{ab} + \rho u_a u_b + 2\mu v_{[a}y_{b]}$. Using this relation to compute $u_{a;bc}$ and then contracting $u_{a;[bc]} = 0$ with $x^a u^b$ we get $\mu x^a u^b v_{a;b} = \mu x^a u^b y_{a;b} = 0$. This, and $x^a v_a = x^a y_a = 0$ prove that $v^a[u, x]_a = y^a[u, x]_a = 0$ in U. Therefore, $p \to span(u_p, x_p)$ is an integrable distribution.

In the case when $x = l$ is null in U we can construct a null tetrad of the form (u, l, n, y) (where n, l are the null vectors; u is necessarily spacelike in this case). There exists then a function $\mu : U \to R$ which is nowhere zero and such that $F_{ab} = 2\mu l_{[a}y_{b]}$, and this gives $u_{a;b} = \psi g_{ab} + \rho u_a u_b + 2\mu l_{[a}y_{b]}$. Using this to compute $u_{a;bc}$ and contracting $u_{a;[bc]} = 0$ with l^a we get :

$$-u^b\psi_b - \epsilon\psi^2 + \mu l^a u^b y_{a;b} = 0.$$

Contracting $u_{a;[bc]} = 0$ with $n^a u^b l^c$ we get:

$$-u^b\psi_b - \epsilon\psi^2 - \mu u^b l^a y_{a;b} = 0.$$

Comparing these two relations we deduce that $\mu l^a u^b y_{a;b} = 0$. Thus, the distribution $p \to span(u_p, x_p)$ is again integrable.

Notice that, in both cases, $[u, x]$ is parallel to x.

From the above results we deduce the existence in U (restricting U if necessary) of a coordinate basis of vector fields of the form (u, S, T, W), where $S = Bx$,B being a solution of $\mathcal{L}_u B = \psi$. Considering then the function $A = u^a T_a$ we prove quite easily that, since $[u, T] = 0$, we have $u^b A_b = 0$; similarly, using $[S, T] = 0$. $u^a S_a = 0$ and $F_{ab}S^b = 0$, we prove that $S^b A_b = 0$. It follows that, if $T' = Au - \epsilon T$. then $[u, T'] = [S, T'] = 0$ and $u^a T'_a = 0$. Completing then (u, S, T') into a coordinate basis, we see that we can assume from the start that $A = 0$ in U. Similarly. we prove that the function $C = u^a W_a$ satisfies $u^a C_a = S^a C_a = 0$.

Denoting then by (r, s, t, w) the corresponding coordinates, g has, in these coordinates, the following form:

$$g = \begin{pmatrix} \epsilon & 0 & 0 & C \\ 0 & \epsilon' B^2 & D & F \\ 0 & D & H & L \\ C & F & L & M \end{pmatrix},$$

where $B, D, .., M$ are differentiable functions on U, C depending only on the variables t, w. This gives, as $u' = g(u, .) = \epsilon du + C dw$ and $h^u = \psi g + \rho u' \otimes u'$:

$$h^u = \psi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon' B^2 & D & F \\ 0 & D & H & L \\ 0 & F & L & -\epsilon C^2 + M \end{pmatrix}.$$

On the other hand, computing h^u directly we get:

$$h^u = 1/2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\epsilon' B^2 \partial_r B & \partial_r D & \partial_r F \\ 0 & \partial_r D & \partial_r H & \partial_r L \\ 0 & \partial_r F & \partial_r L & \partial_r M \end{pmatrix}.$$

Comparing this two matrices and defining $\Psi : U \longrightarrow \mathbf{R}$ as being a solution of $\partial_r \Psi = \psi$, we find that $B = e^{\Psi} b$, $D = e^{2\Psi} d$, $F = e^{a\Psi} f$, $H = e^{2\Psi} h$, $L = e^{2\Psi} l$ and $M = \epsilon C^2 + e^{2\Psi} m$, where the functions b, d, f, h, l, m do not depend on the variable r.

Notice that, as follows from the above expression for u', we have :

$$2F = (\partial_t C) dt \wedge dw.$$

As u is the first coordinate vector field, the conditon $R^a{}_{bcd} u^d = 0$ translates, in these coordinates, to $R^a{}_{bc1} = 0$.

In both cases (null, non-null) a simple computation gives:

$$R^2{}_{121} = -(\partial_r^2 \Psi + (\partial_r \Psi)^2).$$

Using the fact that this must vanish, we get then, in the non-null case:

$$R^1{}_{141} = -\epsilon' b^2 C e^{2\Psi} (\partial_t C)^2 / 4\Delta,$$

where Δ is the determinant of the matrix of g.

In the null case, we find that:

$$R^2{}_{131} = -\epsilon d e^{2\Psi} (\partial_t C)^2 / 4\Delta,$$

and

$$R^2{}_{141} = -\epsilon f e^{2\Psi} (\partial_t C)^2 / 4\Delta.$$

It follows immediately that, in both cases, we must have $\partial_t C = 0$; this contradicts the fact that the bivector F is non-zero and thus proves the lemma.(For any simple bivector H on M denote by N_H the set of points of M where H is non-zero, by Z_H the interior of its complement; denote then by I_H the set of points of M where H is non-null and by L_H the interior of its complement. Then $I_H \cup L_H \cup Z_H$ is open dense in M . We have proved above that in the case of F, both A_H and B_H are empty; by continuity F vanishes everywhere) ■

We deduce that:

THEOREM 5.3.6. *Let (M,g) be a class II(a) space-time, u the prefered vector field on M defined at the beginning of this §. If there exists $X \epsilon \mathcal{K}(M,g)$ which is parallel to u then:*

(a).If u is a global gradient, $\mathcal{K}(M,g)$ is infinite dimensional over the reals;

(b).If the set of zeros of X has empty interior, (M,g) is locally conformally equivalent to a decomposable space-time.

Proof. By assumption, there exists a differentiable function $\lambda : M \to \mathbf{R}$ such that $X = \lambda u$. From L.5.3.5 we deduce that u is hypersurface orthogonal and from T.5.3.4.(a) we deduce that $\lambda_{;m}$ is then parallel to u_m, or λ is constant.

Let $\phi \in \mathcal{F}(M)$ be such that $\phi_{;m}$ is parallel to u_m. Consider then $W = \phi\lambda u$; we have

$$W^a{}_{;b} = \phi_{;b}\lambda u^a + \phi(\lambda u^a)_{;b}.$$

Replacing this for X in (5.1.2) and using (5.3.1) we see that W is a curvature collineation . This proves that $\mathcal{K}(M,g)$ is infinite dimensional. In fact, take the function ϕ above and for every integer k consider the function ϕ^k; the gradient of this function is still parallel to u and $\phi^k X$ is therefore a curvature collineation. Now consider the vector subspace of $\mathcal{F}(M)$ spanned by the ϕ^k, for all integers k. This vector subspace is infinite dimensional whenever ϕ is not constant. This proves (a).

From the preceding lemma we know that u is locally a gradient. The above argument can therefore be applied in the neighbourhood of every point of M. Take then a point p in M such that $X_p \neq 0$. Then, at p, the function λ does not vanish; consequently, in some open neighbourhood of p, $u = (1/\lambda)X$ is a curvature collineation. Since this can be done for all points in an open dense subset of M we see that $u \in \mathcal{K}(M,g)$. This implies that there exist $\psi, \rho \in \mathcal{F}(M)$ such that, as the bivector of u is identically zero (L.5.3.5):

$$u_{a;b} = \psi g_{ab} + \rho u_a u_b, \tag{5.3.4}$$

and in fact we have $\epsilon\rho = -\psi$ since $u^a u_{a;b} = 0$. Computing then $u_{a;bc}$ and using the fact that $u_{a;[bc]} = 0$, we get:

$$\psi_{;a} = \epsilon u^b \psi_{;b} u_a = -\epsilon\psi^2 u_a. \tag{5.3.5}$$

Thus, $\psi_{;a}$ is parallel to u_a.

Consider then the 1-form σ on M given by $\sigma_a = -\epsilon\psi u_a$. The preceding result on ψ shows that $d\sigma = 0$; the Poincaré lemma (cf. Ch.1) tells us then that for every point p in M we can find an open connected neighbourhood U of p in M and a function $\Phi \in \mathcal{F}(U)$ such that $\Phi_{;a} = \sigma_a$.

Consider then the metric $\hat{g}$ on U given by:

$$\hat{g} = e^{2\Phi} g.$$

If one denotes by $\hat{\Gamma}$ the Christoffel symbols of the Levi-Civita connection of $\hat{g}$ and by Γ those of the Levi-Civita connection of g, then we have, in U, with respect to every coordinate system:

$$\hat{\Gamma}^a_{bc} = -\epsilon\psi u_c \delta^a{}_b - \epsilon\psi u_b \delta^a{}_c + \epsilon\psi u^a g_{bc} + \Gamma^a{}_{bc}.$$

Let then $Y = e^{-\Phi} u$; denoting by $\shortparallel$ the covariant differentiation with respect to the Levi-Civita connection of $\hat{g}$, one has then:

$$e^{\Phi} Y^a{}_{\shortparallel b} = \epsilon\psi u_b u^a + u^a{}_{\shortparallel b} = \epsilon\psi u_b u^a + u^a{}_{;b} + u^c[-\epsilon\psi u_c \delta^a{}_b - \epsilon\psi u_b \delta^a{}_c + \epsilon\psi u^a g_{bc}] = 0.$$

Thus, $(U, \hat{g})$ admits a covariantly constant non-null vector field. From our remarks at the beginning of Ch.4, we deduce that $(U, \hat{g})$ is decomposable and this proves (b) ■

Keeping the same assumptions and the same notation as in the preceding theorem, consider the vector field Y; this vector field is covariantly constant with respect to the metric $\hat{g}$. Setting then $\hat{Y}_a = \hat{g}_{ab} Y^b$, the tensor field

$$\hat{k}_{ab} = \hat{g}_{ab} - \epsilon \hat{Y}_a \hat{Y}_b = e^{2\Phi}(g_{ab} - \epsilon u_a u_b) = e^{2\Phi} k_{ab},$$

is covariantly constant with respect to $\hat{g}$. For every point p on U let $u_p^\perp$ denote the subspace of $T_p(M)$ orthogonal to u, and consider the distribution $p \mapsto u_p^\perp$ on U; this distribution is integrable. Take then any point p_o in U and denote by V the integral manifold of this distribution through p_o. Then for every $p \in V$, $\hat{k}_p$ is just the evaluation at p of the metric induced on V by $\hat{g}$. Restricting U if necessary we can then assume that $(U, \hat{g}_{|U})$ is isometric to a product of the form $(I \times V, i \oplus \hat{k})$, where (I, i) is a 1-dimensional pseudo-riemannian manifold of the convenient signature.

Let now $X \in \mathcal{K}(U, g_{|U})$ and let $\phi = \psi^{^X}$, $\rho = \rho^{^X}$; setting $\alpha = \epsilon u^a X_a$ and $Z = X - \alpha u$, we have proved in T.5.3.3 that $[u, Z] = 0$. This means that X is projectable (cf. §.1.2) with respect to the above decomposition. On the other hand, we have:

$$(\mathcal{L}_Z k)_{ab} = Z^c k_{ab;c} + Z^c{}_{;a} k_{cb} + Z^c{}_{;b} k_{ac}$$
$$= -\epsilon u_b Z^c u_{a;c} - \epsilon u_a Z^c u_{b;c} + Z_{b;a} - \epsilon u_b u_c Z^c{}_{;a} + Z_{a;b} - \epsilon u_a u_c Z^c{}_{;b}$$

Using then T.5.3.3.(b) and since $\alpha_b = \epsilon(\phi + \epsilon\rho)u_b$ (cf. T.5.3.4.(a)), this gives, since Φ_a is parallel to u_a:

$$\mathcal{L}_Z \hat{k} = 2\phi\hat{k}.$$

The above remarks on the projectability of X together with this relation show therefore that Z is a conformal vector field of $(V, \hat{k})$. Thus we have the following theorem (part of which is due to Hall):

THEOREM 5.3.7. *Let (M, g) be a space-time satisfying the conditions of the preceding theorem. Then every curvature collineation of (M, g) is locally of the form $X = \alpha u + Z$, where the function α has its gradient parallel to u and Z is a conformal vector field of the pseudo-riemannian manifold $(V, \hat{k})$.*

Finally, let us consider the case when u is covariantly constant in (M, g). The preceding results hold obviously, but now with g in the place of $\hat{g}$ and k in the place of $\hat{k}$.

Let then $X \in \mathcal{K}(M, g)$, and consider the vector field Z defined as above. Then Z is a conformal vector field in (V, k). On the other hand, since α_a is parallel to u_a and u is covariantly constant, the relation (5.1.2), when applied to X, reduces to exactly the same relation but with Z in the place of X. Now as the Riemann tensor of (M, g) can also be considered as the Riemann tensor of (V, k), this implies that Z is a curvature collineation of (V, k).

This tells us that along the integral curves of Z the Riemann tensor of (V, k) does not vary. However, this vector field is conformal with respect to this metric; thus the metric k is scalled along this same integral curves by the function ϕ. Now it has been proved by Hall [35] that under these circumstances, the gradient of ϕ, if non-zero at some point of V, must contract the Riemann tensor to zero in some open connected neighbourhood of that point. Our assumptions on the Riemann tensor tell us that this is only possible if the gradient of ϕ is zero (there are no such vectors tangent to V).

Thus:

THEOREM 5.3.8. *If u is covariantly constant, every curvature collineation X of (M,g) is locally of the form $X = \alpha u + Z$, where the function α has its gradient parallel to u and Z is a homothetic vector field of the pseudo-riemannian manifold (V,k).*

NOTE.5.3.3. If we start with a space-time satisfying the conditions of T.5.3.6, with u covariantly constant, then our assumptions tell us that the only vector fields satisfying (5.3.1) are parallel to u. When u is not covariantly constant we can conformally scale the metric so that, with respect to the new metric $\hat{g}$, u is parallel to a covariantly constant vector field. However, it does not follow, now, that the vector fields satisfying (5.3.1), with respect to $\hat{g}$, are necessarily parallel to u. To see this take an open connected subset M of $\mathbb{R}^4$ together with coordinates (t,u,v,x), and consider the metric g defined on U by:

$$g = e^{(au+b)} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix}.$$

where $f : U \to \mathbb{R}$ is strictly positive and depends only on the variables v, x and a, b are real constants ($a \neq 0$). A simple calculation shows that the non-identically zero components of the Riemann tensor of g are:

$$\mathbf{R}_{1313} = \mathbf{R}_{1414} = \frac{a^2}{4}\ f\ e^{(au+b)},$$

and

$$\mathbf{R}_{3434} = -\frac{e^{(au+b)}}{4f}(2f\partial_v^2 f - 2(\partial_v f)^2 + 2f\partial_x^2 f - 2(\partial_x f)^2 + a^2 f^3).$$

This shows that, in general, the Riemann tensor of this metric has rank 3; consequentely equation (5.3.1) has (in general), for this metric, at most one independent solution. In fact, the above expressions for the Riemann tensor show that if v satisfies (5.3.1) then necessarily $v = \alpha\partial_u$. Moreover, there are no covariantly constant vectors for g. In fact one such vector has to satisfy (5.3.1), so it has the general form above; a simple computation shows then that in the above coordinates $v_{1;1} = -a\alpha e^{au+b}/2$ and this shows that v cannot be covariantly constant unless if it is identically zero.

Notice now that the metric $j = e^{-(au+b)}g$ is "1+1+2"-decomposable (∂_t and ∂_u are both covariantly constant with respect to j).

The condition on u imposed in T.5.3.6.(a) is actually too strong a condition as we now prove:

THEOREM 5.3.9. *Let (M, g) be a class II(a) space-time, u be the prefered vector field defined at the beginning of this §. Assume that (M, g) admits a curvature collineation parallel to u. Then the Lie algebra $\mathcal{K}(M, g)$ is infinite dimensional if and only if there exists a non constant function $\theta \in \mathcal{F}(M)$ whose gradient is everywhere parallel to u_a.*

NOTE.5.3.4. Notice that, as proved previously (L.5.3.5), a function such as θ always exists locally. The above statement on θ is a statement about the global properties of u.

Proof. By assumption, there exists $X \in \mathcal{K}(M, g)$ which is parallel to u.

Assume first the existence of the function θ. Then for every integer k the vector field $\theta^k X$ is a globally defined curvature collineation (as follows from the condition on the gradient of θ) and the subspace of $\mathcal{K}(M, g)$ spanned by all such elements when k runs through the integers is infinite dimensional (as follows from the non constancy of θ).

Conversely, assume that $\mathcal{K}(M, g)$ is infinite dimensional.

Let then $Y = \alpha^{Y} u + Z$ be a curvature collineation .

Since the existence of X guarantees that u is hypersurface orthogonal (cf. L.5.3.5), it follows that the gradient of α^{Y} is everywhere parallel to u. Thus, if this gradient does not vanish identically in M we set $\alpha^{Y} = \theta$.

Otherwise α^{Y} is constant, since M is assumed connected. In particular the vector field X defined above is a constant multiple of u.

The relations (5.3.4) and (5.3.5) (cf. Proof of T.5.3.6) show then that either u is covariantly constant or the function ψ can be chosen to play the role of θ.

Thus, the only case that remains to be analysed is the case when u is covariantly constant and every curvature collineation is of the form $Y = ru + Z$ where r is a real constant and Z is orthogonal to u. In this case however, Z satisfies:

$$\mathcal{L}_Z h = 2\phi h,$$

where $h_{ab} = g_{ab} + \epsilon u_a u_b$, and we have proved in T.5.3.8 that ϕ is constant. This shows that in this last case $\mathcal{K}(M, g)$ is finite dimensional, thus contradicting our initial assumption ■

NOTE.5.3.5. The special case when u is covariantly constant has been considered by Katzin et al [54]. These authors noticed that for every function α whose gradient is parallel to u the vector field αu is a curvature collineation. However, these authors did not study the general case.

EXAMPLES.

Consider an open connected subset U of $\mathbf{R}^4$ with coordinates (x^1, x^2, x^3, x^4) and let g be a metric on U for which the space of solutions of (5.3.1) is spanned by $u = \partial_1$.

The assumption that there exists $X \in \mathcal{K}(U, g)$ parallel to u implies, by T.5.3.6.(b), that (if we choose U small enough) we can assume the existence of $\Phi \in \mathcal{F}(U)$ such that

$$g = e^{2\Phi} h,$$

where h is a metric on U for which u is parallel to a covariantly constant vector field. This implies the existence of functions $h_{\alpha\beta} \in \mathcal{F}(U)$ such that $\partial_1 h_{\alpha\beta} = 0$ and:

$$h = \epsilon dx^1 \otimes dx^1 + h_{\alpha\beta} dx^\alpha \otimes dx^\beta,$$

where $\alpha, \beta = 2, 3, 4$. Now we can consider the hypersurfaces S_c of U given by $x^1 = c$, where $c \in \mathbf{R}$, and, in them, the metric $j = h_{\alpha\beta} dx^\alpha \otimes dx^\beta$; since this is a 3-dimensional metric it is known that we can find coordinates in S_c (restricting S_c if necessary) for which j is diagonal [68]. Thus, we may assume from the start that the coordinates have been so chosen that:

$$h = \epsilon dx^1 \otimes dx^1 - \epsilon F(x^\alpha) dx^2 \otimes dx^2 + G(x^\alpha) dx^3 \otimes dx^3 + H(x^\alpha) dx^4 \otimes dx^4.$$

Hence:

$$g = e^{2\Phi}(\epsilon dx^1 \otimes dx^1 - \epsilon F dx^2 \otimes dx^2 + G dx^3 \otimes dx^3 + H dx^4 \otimes dx^4).$$

Since we assume that u spans the space of solutions of (5.3.1), T.5.3.1 tells us that u is geodesic. Defining then $v_a = u_{a;b} u^b$, v_a is parallel to u_a. A direct computation gives in the above coordinates $v_1 = \epsilon e^{2\Phi} \partial_1 \Phi$ and $0 = v_\alpha = -\epsilon e^{2\Phi} \partial_\alpha \Phi$

for $\alpha = 2, 3, 4$. This shows that Φ depends only on x^1. A simple computation gives then:

$$\mathbf{R}^2{}_{112} = \partial_1^2 \Phi;$$

and so, as this component of the Riemann tensor vanishes by (5.3.1), we deduce the existence of real constants a, b such that:

$$g = e^{2(ax^1+b)}(\epsilon dx^1 \otimes dx^1 - \epsilon F dx^2 \otimes dx^2 + G dx^3 \otimes dx^3 + H dx^4 \otimes dx^4).$$

This is, therefore, the general local form of a class II(a) metric which admits a curvature collineation which is a solution of (5.3.1). Any other curvature collineation is of the form $f(x^1)\partial_1 + Z$, where Z is a vector field on each hypersurface S_c which is conformal with respect to the metric j defined above.

Recently [76] some attention has been paid to the particular case of spherically symmetric metrics [57] which admit proper curvature collineations. When we ask furthermore that such metrics belong to class II(a) with the solutions to (5.3.1) orthogonal to the orbits of the $SO(3)$ group of motions, the above results tell us that the general form of such a metric is:

$$g = e^{2(ax^1+b)}(\epsilon dx^1 \otimes dx^1 - \epsilon e^{2\nu(x^2)} dx^2 \otimes dx^2 + e^{2\mu(x^2)}(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)). \quad (*)$$

The metrics found by Tello-Llanos [76]:

$$g = -dt \otimes dt + (t + t_o)^2 r^{-2} U^2(r)(dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)),$$

and

$$g = -(r+r_o)^2 U^{-2}(t) \frac{dV}{dt}^2 dt \otimes dt + dr \otimes dr + (r+r_o)^2 V^2(t)(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)),$$

are obviously members of the family decribed by $(*)$ (modulo a change of coordinates). They contain in particular the Robertson-Walker models:

$$g = -dt \otimes dt - (1 + Ct^2)(1 + \frac{r^2}{4R_o^2})(dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)).$$

That the Einstein static model given by the above expression with $C = 0$ had a proper curvature collineation was noticed by Katzin *et al* [53].

Having analysed the case of class II(a) space-times for which there exists a curvature collineation which satisfies (5.3.1), we have now to analyse the case of class II(a) space-times for which no such curvature collineation exists.

The general case.

In what follows we shall split the study according to whether the vector field u is or is not hypersurface orthogonal . Since some remarks and definitions are common to both cases we start by introducing them.

In what follows we shall assume that the function $\mathcal{R}$ (defined in §.5.1) is constant (see §.5.1).

Consider then a point $p_o \in M$ and let $\mathcal{R}(p_o) = r$; let U be an open connected neighbourhood of p_o in M which is regular (cf. §.5.1). We can then find a family $X_1, ..., X_r$ of curvature curvature collineations defined in U such that the vectors $X_{i|p_o}$ span a r-dimensional subspace of $T_{p_o}(M)$. By continuity and by the constancy of $\mathcal{R}$, we deduce (restricting U if necessary) that every curvature collineation defined in U is a linear combination of the above vector fields with coefficients in the ring $\mathcal{F}(U)$.

Finaly, restricting again U if necessary, we may assume that there exists a family of vector fields on U, of the form (u, v, x, y) which at every point p of U defines an orthonormal basis of $T_p(M)$.

A regular open subset of M where the above constructions are possible will be called a **nice** open set from now on.

Let then $p_o \in M$ and let $\mathcal{R}(p_o) = r$. Let U be a nice neighbourhood of p_o and let $\mathcal{F} = \{X_1, ..., X_r\}$ be a family of curvature collineations in U with the properties of the above family. We shall say that $\mathcal{F}$ is a $\mathcal{K}(U, g_{|U})$-**spanning family** (or simply a **spanning family**). Every curvature collineation on U is then a linear combination of the elements of $\mathcal{F}$ with coeficients in the ring $\mathcal{F}(U)$. Thus, if $Y \in \mathcal{K}(U, g_{|U})$, there exist unique functions $A^I_Y \in \mathcal{F}(U)$, $1 \leq I \leq r$,such that

$$Y = \sum_{1 \leq I \leq r} A^I_Y X_I. \tag{5.3.6}$$

In what follows we shall keep this notation for the components of Y with respect to the spanning family $\mathcal{F}$; however, if no confusion is likely, we may drop the subscript Y.

On the other hand, since each X_I is an element of $\mathcal{K}(U, g_{|U})$ we can find $\psi_I, \rho_I \in \mathcal{F}(U)$ such that (cf. T.5.3.2), setting $h^{X_I} = h^I$, we have

$$h^I_{ab} = \psi_I g_{ab} + \rho_I u_a u_b. \tag{5.3.7}$$

Since U is a nice open set consider an orthonormal tetrad $\mathcal{B} = (u, v, x, y)$ on U (cf. Ch.2). Then, we can find functions $\alpha_I, \beta_I, \gamma_I, \delta_I \in \mathcal{F}(U)$, $1 \leq I \leq r$, such that

$$X_I = \alpha_I u + \beta_I v + \gamma_I x + \delta_I y. \tag{5.3.8}$$

This allows us to define a map $\nu : U \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^4)$, by defining for $p \in U$, $\nu(p)$ as the linear map from $\mathbb{R}^r$ to $\mathbb{R}^4$ given in the canonical basis of these spaces by the matrix

$$\mathcal{M} = \begin{pmatrix} \alpha_1 & \alpha_2 & . & . & . & \alpha_r \\ \beta_1 & \beta_2 & . & . & . & \beta_r \\ \gamma_1 & \gamma_2 & . & . & . & \gamma_r \\ \delta_1 & \delta_2 & . & . & . & \delta_r \end{pmatrix}$$

where the functions $\alpha_I, \beta_I, \gamma_I, \delta_I$ have been evaluated at p.

Since the columns of this matrix are precisely the components of the X_I, with respect to the basis (u, v, x, y), and since the family $\mathcal{F}$ has rank r at every point of U, we see that $\nu(p)$ is an injective linear map for all values of p (notice that $r \leq 4$).

Let now $y \in \mathcal{K}(U, g_{|U})$, and let the functions A^I_Y be given by (5.3.6). We set then

$$A^I_{Y;b} = A^{I1}_Y u_b + A^{I2}_Y v_b + A^{I3}_Y x_b + A^{I4}_Y y_b. \tag{5.3.9}$$

Define then, for $1 \leq j \leq 4$ the map

$$\Omega^j(Y) : U \to \mathbb{R}^r,$$

by setting for $p \in U$:

$$\Omega^j(Y)(p) = (A^{1j}_Y, ..., A^{rj}_Y),$$

where the functions A^{Ij}_Y have been evaluated at p.

We can then consider, for $1 \leq j \leq 4$, the map

$$\Lambda^j : U \times \mathcal{K}(U, g_{|U}) \to \mathbb{R}^4,$$

given for $p \in U$ and $Y \in \mathcal{K}(U, g_{|U})$ by

$$\Lambda^j(p;Y) = \nu(p)(\Omega^j(Y)(p)). \tag{5.3.11}$$

In what follows we are going to study these maps in detail.

First notice that Λ^j, $1 \le j \le 4$, depends both on the spanning family $\mathcal{F}$ and on the orthonormal tetrad $\mathcal{B}$. Let $\hat{\mathcal{F}}$ and $\hat{\mathcal{B}}$ be nother spanning family and orthonormal tetrad. Then we can in the same way define the maps $\hat{\Lambda}^j$ corresponding to $\hat{\mathcal{F}}$ and $\hat{\mathcal{B}}$. Writing $\mathcal{B} = (z_1, z_2, z_3, z_4)$ and $\hat{\mathcal{B}} = (\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$, we have, in U:

$$z_k = U_k{}^m \hat{z}_m,$$

where the $U_k{}^m$ are differentiable functions in U.

Define for $Y \in \mathcal{K}(U, g_{|U})$, $\Lambda^j(p;Y) = (\sigma^{1j}, ..., \sigma^{4j})$, $\hat{\Lambda}^j(p;Y) = (\hat{\sigma}^{1j}, ..., \hat{\sigma}^{4j})$ and, setting $\hat{\mathcal{F}} = \{\hat{X}_1, ..., \hat{X}_r\}$, let $\Lambda^j(p;\hat{X}_L) = (\sigma_L^{1j}, ..., \sigma_L^{4j})$. Using these notations one then proves easily that

$$\hat{\sigma}^{ij} = \sum_{1 \le k,m \le 4} U_k{}^i U_m{}^j (\sigma^{ij} - \sum_{1 \le L \le r} \hat{A}^L \sigma_L^{km}), \tag{5.3.12}$$

If $Y \in \mathcal{K}(U, g_{|U})$ and if $\mathcal{F}$ is a $\mathcal{K}(U, g_{|U})$-spanning family and $\mathcal{B}$ an orthonormal tetrad in U, we have, with the preceding notation, for $1 \le j \le 4$

$$\Lambda^j(p;Y) = (\sum_{1 \le I \le r} \alpha_I A^{Ij}, \sum_{1 \le I \le r} \beta_I A^{Ij}, \sum_{1 \le I \le r} \gamma_I A^{Ij}, \sum_{1 \le I \le r} \delta_I A^{Ij}). \tag{5.3.13}$$

Consider now the bivector $F_{ab} = 2u_{[a;b]}$. We have shown (T.5.3.1) that this bivector is either zero or simple at any given point of M. In what follows we shall assume that if non-zero this bivector has the same type everywhere.

NOTE.5.3.6. These assumptions are somewhat justified by the fact that in any case we can find a finite family of open submanifolds of M each of which has the above properties and whose union is dense in M.

We start by considering the case when $\mathcal{R} = 1$. We have the following result due to Hall:

Lemma 5.3.10. *Assume there exists $X \in \mathcal{K}(M, g)$ whose set of zeros does not contain open subsets of M and $\lambda \in \mathcal{F}(M)$ such that $\lambda X \in \mathcal{K}(M, g)$. Then either λ is constant or X is parallel to u.*

Proof. Since $Y = \lambda X$ we have:

$$2\psi^{Y} g_{ab} + 2\rho^{Y} u_a u_b = \lambda_b X_a + \lambda_a X_b + 2\lambda\psi^{X} g_{ab} + 2\lambda\rho^{X} u_a u_b,$$

so we have:

$$2(\psi^{Y} - \lambda\psi^{X})g_{ab} + 2(\rho^{Y} - \lambda\rho^{X})u_a u_b = \lambda_b X_a + \lambda_a X_b.$$

At any given point $p \in U$ the tensor field on the left handside of this relation has possible ranks 0,1,3 or 4; the tensor field on the right handside has possible ranks 0,1 or 2. Thus either both have rank 0 or both have rank 1. In the first case the gradient of λ vanishes at the point in consideration; in the second case X is necessarily parallel to u at the point under consideration. The conclusion of the lemma is then immediate ■

We assume from now on that $\mathcal{R} = r \geq 2$.

The case when u is not hypersurface orthogonal

Here we assume that the set of points of M where the bivector F does not vanish is open dense in M. Recall that, as u is not hypersurface orthogonal , for any given $X \in \mathcal{K}(M, g)$ the function $\psi^{X} + \epsilon\rho^{X}$ is constant (cf. T.5.3.4.(b)).

Let then U be a nice subset of M and, keeping the preceding notation for the spanning families, set $c_I = \psi_I + \epsilon\rho_I$.

Let $\mathcal{S}(U)$ denote the set of those $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^{Y} + \epsilon\rho^{Y} = 0$. Then:

Lemma 5.3.11. *$\mathcal{S}(U)$ is a Lie subalgebra of $\mathcal{K}(U, g_{|U})$, and $\dim\mathcal{S}(U) \geq (r-1)$ $(r = \mathcal{R})$. Furthermore, if there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^{Y} + \epsilon\rho^{Y} \neq 0$, then any other element W of $\mathcal{K}(U, g_{|U})$ is of the form $W = aY + bX$, where $a, b \epsilon \mathbf{R}$, and $X \epsilon \mathcal{S}(U)$.*

Proof. From T.5.3.2.(b) we deduce that $X \epsilon \mathcal{S}(U)$ if and only if $[u, X] = 0$. This immediately shows that $\mathcal{S}(U)$ is a vector subspace of $\mathcal{K}(U, g_{|U})$; the Jacobi identity proves that if $X, Y \epsilon \mathcal{S}(U)$, then $[u, [X, Y]] = 0$, so $\mathcal{S}(U)$ is a Lie sub-algebra of $\mathcal{K}(U, g_{|U})$.

Suppose that $\dim\mathcal{S}(U) \leq (r-1)$. Then, for $p \epsilon U$, the subspace $\mathcal{S}_p$ of $\mathcal{K}_p$ spanned by the elements of $\mathcal{S}(U)$ evaluated at p has at most dimension $(r-1)$;

consequently, at least one of the constants c_I, $1 \leq I \leq q$ is non-zero. Reordering the X_I, we can assume that $c_q \neq 0$. For $1 \leq I \leq (r-1)$ define then:

$$Y_I = c_r X_I - c_I X_r.$$

The family $Y_1, .., Y_{(r-1)}, X_r$ is then a $\mathcal{K}(U, g_{|U})$-spanning family and $[u, Y_I] = 0$ for $1 \leq I \leq (r-1)$.

For $W \epsilon \mathcal{K}(U, g_{|U})$, define

$$X = (\psi^X + \epsilon\rho^X)W - (\psi^W + \epsilon\rho^W)Y;$$

then $X \epsilon \mathcal{S}(U)$, and this proves the lemma ■

This lemma allows us to assume from now on that $c_I = 0$ for $1 \leq I \leq (r-1)$. It also shows that $\mathcal{K}(U, g_{|U})$ is finite dimensional if and only if $\mathcal{S}(U)$ is finite dimensional.

Define now, for $p \in U$, $\mathcal{S}_p$ as the vector subspace of $T_p(M)$ spanned by all the elements of $\mathcal{S}(U)$ evaluated at p. $\mathcal{S}_p$ is a vector subspace of $\mathcal{K}_p$ by definition and we have, by the above result, $\dim \mathcal{S}_p = r$ or $r-1$. In the first case it is clear that the spanning family $\mathcal{F}$ can be chosen such that $c_r = 0$ as well, so that, in particular $\dim \mathcal{S}_q = r$ for all $q \in U$. This shows that the dimension of $\mathcal{S}_p$ is the same at every point of U. If this dimension is $r-1$, the spanning family $\mathcal{F}$ chosen above (with $c_1 = ... = c_{r-1} = 0$) is such that the elements of $\mathcal{S}(U)$ are linear combinations of the $r-1$ first elements of $\mathcal{F}$ with coefficients in $\mathcal{F}(U)$.

To account for these two possibilities let s be defined as $s = 0$ if $\dim \mathcal{S}_p = r$ and $s = 1$ if $\dim \mathcal{S}_p = r-1$. Then the family $\mathcal{F}' = \{X_1, ..., X_{r-s}\}$ is a spanning family of $\mathcal{S}(U)$.

We can then construct, as precedingly, the maps ν, $\Omega^j(Y)$ and Λ^j, but using now the Lie algebra $\mathcal{S}(U)$ instead of $\mathcal{K}(U, g_{|U})$.

A - The case when F is non-null.

In this case, we can choose the orthonormal tetrad $\mathcal{B} = (u, v, x, y)$ in such a way that v, y span the blade of F. To account for the type of F and as $u^a u_a = \epsilon$, we set $x^a x_a = \epsilon'$ and $v^a v_a = -\epsilon\epsilon'$, $y^a y_a = 1$. there exists then a function $\mu \in \mathcal{F}(U)$ such that

$$F_{ab} = 2\mu v_{[a} y_{b]}. \tag{5.3.14}$$

Choose then a spanning family $\mathcal{F} = \{X_1, ..., X_r\}$ as above (i.e. so that the the family $\mathcal{F}' = \{X_1, ..., X_{(r-s)}\}$ is a spanning family for $\mathcal{S}(U)$). Take then $Y \in \mathcal{S}(U)$, so that we have

$$Y = \sum_{1 \le I \le (r-s)} A^I X_I. \tag{5.3.15}$$

From (5.3.14) and (5.3.8) one deduces that:

$$\alpha_{I;b} = c_I u_b - \epsilon\mu\delta_I v_b - \epsilon'\mu\beta_I y_b, \tag{5.3.16}$$

where $c_I = \epsilon\psi_I + \rho_I$. We have then

Lemma 5.3.12. *Let $Y \epsilon \mathcal{S}(U)$. Then:*
(a).

$$\sum_{1 \le I \le (r-s)} (\alpha_{I;p} A^I_{Y;m} - \alpha_{I;m} A^I_{Y;p}) = 0;$$

(b). For every $Y \in \mathcal{K}(U, g_{|U})$ there exists a differentiable function $\mathcal{P}_Y : U \to \mathbf{R}$ such that $\Lambda^1(p;Y) = \Lambda^3(p;Y) = 0$, $\Lambda^2(p;Y) = (0,0,0,-\mathcal{P}_Y)$ and $\Lambda^4(p;Y) = (0, \mathcal{P}_Y, 0, 0)$.

Proof. We have (cf. §.5.1):

$$\alpha^Y = \sum_{1 \le I \le (r-s)} A^I_Y \alpha_I. \tag{5.3.17}$$

Differentiating this relation using (5.3.16) and comparing with the value of $\alpha_{;m}$ given by T.5.3.4.(a), we get

$$\sum_{1 \le I \le (r-s)} \alpha_I A^I_{;b} = 0. \tag{5.3.18}$$

Taking again covariant deriveatives and antisymmetrising this leads to (a).

Define then

$$\mathcal{P}_Y = \sum_{1 \le I \le (r-s)} \beta_I A^{I4}_Y. \tag{5.3.19}$$

If no confusion is possible, we shall denote this function simply by $\mathcal{P}$.

Since:

$$2\psi^{Y} g_{ab} + 2\rho^{Y} u_a u_b = \sum_{1\leq I\leq (r-s)} (A^{I}_{Y;b} X_{Ia} + A^{I}_{Y;a} X_{Ib}) + 2 \sum_{1\leq I\leq (r-s)} A^{I}_{Y}(\psi_I g_{ab} + \rho_I u_a u_b),$$

we have:

$$\sum_{1\leq I\leq (r-s)} (A^{I}_{Y;b} X_{Ia} + A^{I}_{Y;a} X_{Ib}) = 2\mathcal{O}_Y g_{ab} + 2\mathcal{N}_Y u_a u_b, \tag{5.3.20}$$

where the functions $\mathcal{O}_Y, \mathcal{N}_Y$ are given by

$$\mathcal{O}_Y = \psi^{Y} - \sum_{1\leq I\leq (r-s)} A^{I}_{Y}\psi_I,$$

$$\mathcal{N}_Y = \rho^{Y} - \sum_{1\leq I\leq r} A^{I}_{Y} X_I.$$

Contracting this relation in all possible ways with the elements of the orthonormal basis and combining the resulting relations with those that result from (a) in the same manner, we get (b) (for the detailed computations, see Appendix 5.I)■

Let $X'_1, ..., X'_q$ be another $\mathcal{K}(U, g_{|U})$-spanning family, and (u, v', x, y') be another orthonormal basis under the same conditions as the basis (u, v, x, y), so that there exist differentiable functions b, d, r, t on U such that $v' = bv + dy$ and $y' = rv + ty$. If $Y \epsilon \mathcal{K}(U, g_{|U})$, then denoting by B^I the components of Y with respect to the new spanning family, we find , after a lengthy but simple calculation, that, with respect to the new spanning family and the new orthonormal basis we have, using (5.3.12)

$$\mathcal{P}'_Y = [\sum_I B^I \mathcal{P}_I - \mathcal{P}_Y]/(dr - tb),$$

where $\mathcal{P}_I = \mathcal{P}_{X_I}$.

This shows that, despite the fact that $\mathcal{P}_Y$ depends on the orthonormal basis and the spannning family chosen, the following assertions ,where $p_o \epsilon U$, do not depend on such a choice:

(A).There exists at least one $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\mathcal{P}_Y(p_o) \neq 0$;

(B).There exists an open neighbourhood V of p_0 in U such that if Y is any curvature collineation on U, then the restriction of $\mathcal{P}_Y$ to V is identically zero.

In fact the above formula shows that if (B) is true with respect to some choice of orthonormal basis and spanning family, then it remains true for any other such choice; on the other hand, it is obvious that if (A) is true for some choice of spanning family and orthonormal basis, then for no other can (B) be true.

A point $p \epsilon M$ is called **regular** if and only if it satisfies either condition (A) or condition (B) above. The set of regular points of M is dense in M. In fact let $p \in M$ be non regular, U be a nice neighbourhood of p and $\mathcal{F}$ a $\mathcal{K}(U, g_{|U})$-spanning family. For all $Y \epsilon \mathcal{K}(U, g_{|U})$ one has then $\mathcal{P}_Y(p) = 0$, otherwise p would satisfy (A); however there exists at least one $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\mathcal{P}_Y$ is not identically zero in U, otherwise p would satisfy (B); this shows that there exists at least a point in U satisfying (A).

This leads to the:

THEOREM 5.3.13..*Let (M, g) be a class II(a) space-time for which the vector field u is not hypersurface orthogonal, its bivector being non-null and assume that $\mathcal{R}$ is constant, with $\mathcal{R} = r \geq 2$. If p_o is a regular point of M then :*
(a). If p_o satisfies (A), there exists a nice neighbourhood U of p_o in M such that $\mathcal{K}(U, g_{|U})$ is finite dimensional. Moreover, $dim\mathcal{K}(U, g_{|U}) = r+2$ or $r+1$, according to whether or not there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^Y + \epsilon\rho^Y \neq 0$;
(b). If p_o satisfies (B) there exists a nice neighbourhood U of p_o in M such that $\mathcal{K}(U, g_{|U})$ is finite dimensional. Moreover, $dim\mathcal{K}(U, g_{|U}) = r+1$ or r, according to whether or not there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^Y + \epsilon\rho^Y \neq 0$.

Proof. Let U be a nice neighbourhood of p_o in M, (u, v, x, y) the orthonormal basis defined above and $\mathcal{F} = (X_I)$ be a $\mathcal{K}(U, g_{|U})$-spanning family. In case (b), choosing U to coincide with the neighbourhood V of p_o whose existence is guaranteed by (B), we see that we have, for all $Y \epsilon \mathcal{K}(U, g_{|U})$ (cf. L.5.3.12)

$$\Lambda^j(p; Y) = 0.$$

The definition of Λ^j gives then $\nu(p)(\Omega^j(Y)(p)) = 0$, and since $\nu(p)$ is injective, we deduce that $\Omega^j(Y) = 0$ for $1 \leq j \leq 4$. The definition of Ω^j and (5.3.9) show then that $A^I_{Y;b} = 0$ hence that the A^I_Y are constants and this proves (b).

In case (a), let $Y_o \epsilon \mathcal{K}(U, g_{|U})$ be such that $\mathcal{P}_{Y_o}(p_o) \neq 0$.

Restricting the U if neccessary, we can assume that $\mathcal{P}_{Y_o}$ (which we denote from now on by $\mathcal{P}_o$) never vanishes on U. Setting $A^I_{Y_o} = A_o{}^I$, $\Omega^j_{Y_o} = \Omega_o{}^j$, we deduce from L.5.3.12 that:

$$\Lambda^j(p;Y) = \lambda\Lambda^j(p;Y_o),$$

where $\lambda = \mathcal{P}_Y/\mathcal{P}_o$.

Using the definition of Λ^j and both the linearity and injectivity of $\nu(p)$, we deduce from this relation that we have:

$$\Omega^j(Y) = \lambda\Omega^j(Y_o),$$

for all j. The definition of the Ω^j gives then

$$A^I_{Y;b} = \lambda A_{o;b}{}^I, \qquad (*)$$

and, taking covariant derivatives and antisymmetrising, this gives:

$$\lambda_{;a}A_o{}^I{}_{;b} - \lambda_{;b}A_o{}^I{}_{;a} = 0. \qquad (**)$$

It is clear from the preceding lemma that $\nu(p)(\Omega^2(Y_o)(p) - \Omega^4(Y_o)(p)) \neq 0$, and this proves that the two elements of $\mathbf{R}^{(r-s)}$, $\Omega^2(Y_o)(p)$ and $\Omega^4(Y_o)(p)$, are linearly independent at all points of U. Using their definition one deduces that the matrix:

$$P = \begin{pmatrix} A^{12} & . & . & . & A^{(r-s)2} \\ A^{14} & . & . & . & A^{(r-s)4} \end{pmatrix}$$

has rank 2 at all points of U. This in turn shows that given any point p in U we can find indices J, L in $\{1, ..., (r-s)\}$ such that the gradients of A^J_o and A^L_o are independent at p. The relation $(**)$ shows then that the gradient of λ must vanish at p. As p is arbitrary this shows that λ is constant. From $(*)$ one concludes then the existence of real constants ζ^I, $1 \leq I \leq (r-s)$ such that

$$A^I = \lambda A^I_o + \zeta^I,$$

and this proves (a) ■.

B - The case when F is null.

In this case, placing ourselves in a nice open set U of M, we consider again a $\mathcal{K}(U, g_{|U})$-spanning family $\mathcal{F} = \{X_1, ..., X_r\}$ such that $\mathcal{F}' = \{X_1, ..., X_{(r-s)}\}$ is a spanning family for $\mathcal{S}(U)$. However, as F is null it is now more appropriate to use a null tetrad $\mathcal{B} = (n, l, x, y)$ (where l, n are the null vectors) such that l, v span the blade of F. There exists then a function $\theta \in \mathcal{F}(U)$ such that

$$F_{ab} = 2\theta l_{[a} v_{b]}. \tag{5.3.21}$$

With respect to this new tetrad each X_I can be written

$$X_I = \tau_I n + \zeta_I l + \alpha_I u + \gamma_I v, \tag{5.3.22}$$

where $\tau_I, \zeta_I, \alpha_I, \gamma_I$ are differentiable functions in U.

Using then (5.3.21) and T.5.3.4.(a) we get, for $1 \leq I \leq r - s$:

$$\alpha_{I;b} = \theta \tau_I v_b - \theta \gamma_I l_b. \tag{5.3.23}$$

As for the previous case, given then $Y \in \mathcal{S}(U)$, there exist unique $A^I_Y \in \mathcal{F}(U)$ such that $Y = \sum_{1 \leq I \leq (r-s)} A^I_Y X_I$; we define then the functions $A^{IJ} \in \mathcal{F}(U)$ by:

$$A^I_{;b} = A^{I2}_Y n_b + A^{I1}_Y l_b + A^{I3}_Y u_b + A^{I4}_Y v_b. \tag{5.3.24}$$

The result of L.5.3.12.(a), as well as (5.3.18) and (5.3.20) being still valid in this case we get, after some computations similar to those in the proof of L.5.3.12.(b) (cf. Appendix 5.III), at the following:

Lemma 5.3.14..*For every $Y \in \mathcal{K}(U, g_{|U})$ there exist differentiable functions, $\mathcal{O}_Y$ and $\mathcal{T}_Y$ on U such that*

$$\Lambda^1(p, Y) = (-\mathcal{O}_Y, 0, 0, \mathcal{T}_Y)$$
$$\Lambda^2(p, Y) = (0, \mathcal{O}_Y, 0, 0)$$
$$\Lambda^3(p, Y) = (0, 0, 0, 0)$$
$$\Lambda^4(p, Y) = (0, -\mathcal{T}_Y, 0, \mathcal{O}_Y)$$

NOTE.5.3.7. Here we defined $\nu(p)$ as being given by the matrix:

$$\mathcal{M} = \begin{pmatrix} \tau_1 & \tau_2 & . & . & . & \tau_{(r-s)} \\ \zeta_1 & \zeta_2 & . & . & . & \zeta_{(r-s)} \\ \alpha_1 & \alpha_2 & . & . & . & \alpha_{(r-s)} \\ \gamma_1 & \gamma_2 & . & . & . & \gamma_{(r-s)} \end{pmatrix},$$

and the functions $\mathcal{O}_Y$ and $\mathcal{T}_Y$ are given by:

$$\mathcal{O}_Y = \psi^Y - \sum_{1 \leq I \leq (r-s)} A^I_Y \psi_I,$$

$$\mathcal{T}_Y = \sum_{1 \leq I \leq (r-s)} \gamma_I A_Y^{I1}.$$

Let p_o ba a point in M, U a nice neighbourhood of p_o, $\mathcal{F}$ a spanning family in U and $\mathcal{B}$ a null tetrad in U.

A simple but lengthy calculation using (5.3.12) shows that the following statements do not depend on the choices of $\mathcal{F}$ and $\mathcal{B}$, provided such choices are restricted to satisfy the conditions imposed on $\mathcal{F}$ and $\mathcal{B}$:

(A). There exists an open neighbourhood V of p_o in M such that for every $Y \epsilon \mathcal{K}(M,g)$ $\mathcal{O}_Y$ and $\mathcal{T}_Y$ are identically zero in V;

(B). There exists an open neighbourhood V of p_o in M and $Y_o \epsilon \mathcal{K}(M,g)$ such that, given any other $Y \epsilon \mathcal{K}(M,g)$:

$$\mathcal{O}_Y \mathcal{T}_{Y_o} - \mathcal{T}_Y \mathcal{O}_{Y_o},$$

is identically zero in V;

(C).There exist $Y_o, Y_1 \epsilon \mathcal{K}(M,g)$ such that :

$$(\mathcal{O}_{Y_1} \mathcal{T}_{Y_0} - \mathcal{O}_{Y_o} \mathcal{T}_{Y_1})(p_o) \neq 0.$$

Define a point $p \epsilon M$ as being **regular** if and only if it satisfies one of the above conditions. The set of regular points of M is, as in the preceding case, open dense in M.

THEOREM 5.3.15. *Let (M,g) be a class II(a) space-time for which the vector field u is not hypersurface orthogonal , its bivector being null and assume that $\mathcal{R}$ is constant and such that $\mathcal{R} = r \geq 2$. If p_o is a regular point of M then :*

(a). If p_o satisfies (A), there exists a regular neighbourhood U of p_o in M such that $\mathcal{K}(U, g_{|U})$ is finite dimensional; moreover, $\dim \mathcal{K}(U, g_{|U}) = r+1$ or r according to whether or not there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^Y + \epsilon \rho^Y \neq 0$;

(b). If p_o satisfies (B), there exists a regular neighbourhood U of p_o in M such that $\mathcal{K}(U, g_{|U})$ is finite dimensional;moreover, $\dim \mathcal{K}(U, g_{|U}) = r+2$ or $r+1$ according to whether or not there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^Y + \epsilon \rho^Y \neq 0$;

(c). If p_o satisfies (C), there exists a regular neighbourhood U of p_o in M such that $\mathcal{K}(U, g_{|U})$ is finite dimensional; moreover, $\dim \mathcal{K}(U, g_{|U}) = r+3$ or $r+2$ according to whether or not there exists $Y \epsilon \mathcal{K}(U, g_{|U})$ such that $\psi^Y + \epsilon \rho^Y \neq 0$.

Proof. The proof in cases (a) and (b) is similar to the proof of T.5.3.13. In case (c), as p_o is regular, we can choose a nice neighbourhood U of p_o in M, a spanning family $\mathcal{F}$ and an orthonormal family $\mathcal{B}$ such that there exist Y_o and Y_1 in $\mathcal{K}(U, g_{|U})$ satisfying (c) at p_o. By continuity, and restricting U if necessary, we may assume that (c) is satisfied by Y_o and Y_1 at all points of U. It follows from the fact that $\nu(p)$ is an injective linear map that, given any $Y \epsilon \mathcal{K}(U, g_{|U})$, there exist differentiable functions a, b on some open neighbourhood (which we can assume to be U) of p_0 such that:

$$\Omega^j(Y) = a\Omega^j(Y_o) + b\Omega^j(Y_1).$$

This obviously gives:

$$A^{I}_{Y;m} = aA_0{}^{I}{}_{;m} + bA_1{}^{I}{}_{;m},$$

where $Y_o = \sum A_0{}^{I} X_I$ and $Y_1 = \sum A_1{}^{I} X_I$. Differentiating and antisymmetrising this relation we deduce that:

$$a_{;p}A_0{}^{I}{}_{;m} - a_{;m}A_0{}^{I}{}_{;p} + b_{;p}A_1{}^{I}{}_{;m} - b_{;m}A_1{}^{I}{}_{;p} = 0.$$

Contracting this relation in all possible ways with the elements of the null tetrad and using the result of L.5.5.14, we prove then quite easily that a and b are constants (see Appendix 5.III) ■

NOTE.5.3.8. These results tell us that if u is not hypersurface orthogonal then "locally" the Lie algebra of curvature collineations is finite dimensional. The question of whether this result is true for the global Lie algebra has not been solved. One sees from the above results themselves that finite dimensionality will be the case for space-times in this class which are themselves nice open sets. If such a strong condition is not imposed then it may happen that $\mathcal{K}(M, g)$ is infinite dimensional. The reason for this assertion is that, as remarked by Hall [42], a global curvature collineation is in general not determined by its values on an open subset of the manifold where it is defined (two vector fields of Minkwoski space may coincide on some open set without being equal globally; however both are curvature collineations).

Notice also that the above results show that in the analytic case $\mathcal{K}(M, g)$ is finite dimensional (assuming all curvature collineations to be analytic).

NOTE.5.3.9. Unfortunately we have not been able to find examples of space-times satisfying the conditions of this sub-section.

The case when u is hypersurface orthogonal

We keep unchanged the notions of nice set of M and of generating family. We recall that it is assumed that $\mathcal{R} = r \geq 2$ in M.

Let U be a regular subset of M, (u, v, x, y) an orthonormal family of vector fields and $\mathcal{F} = \{X_1, ..., X_r\}$ a $\mathcal{K}(U, g_{|U})$-generating family. Using the same notation as in the non hypersurface orthogonal case with F non-null, and a similar method of proof we get (see Appendix 5.IV):

Lemma 5.3.16. *Given $Y \epsilon \mathcal{K}(U, g_{|U})$ there exist differentiable functions $\mathcal{O}_Y$, $\mathcal{N}_Y$, $\mathcal{T}_Y$, $\mathcal{V}_Y$ and $\mathcal{W}_Y$ on U such that, for $1 \leq j \leq 4$*

$$\Lambda^j(p; Y) = \Theta^j,$$

where the vectors Θ^j are given by:

$$\begin{aligned}
\Theta^1 &= (\mathcal{N}_Y + \epsilon\mathcal{O}_Y, 0, 0, 0),\\
\Theta^2 &= (0, -\epsilon\mathcal{O}_Y, \mathcal{T}_Y, \mathcal{V}_Y),\\
\Theta^3 &= (0, -\mathcal{T}_Y, \mathcal{O}_Y, \mathcal{W}_Y),\\
\Theta^4 &= (0, -\mathcal{V}_Y, -\mathcal{W}_Y, \mathcal{O}_Y).
\end{aligned}$$

NOTE.5.3.10. The functions defined above are given by:

$$\begin{aligned}
\mathcal{O}_Y &= \psi^Y - \sum_{1\leq I\leq r} A^I_Y \psi_I,\\
\mathcal{N}_Y &= \rho^Y - \sum_{1\leq I\leq r} A^I_Y X_I,\\
\mathcal{T}_Y &= \sum_{1\leq I\leq r} \gamma_I A^{I2}_Y,\\
\mathcal{V}_Y &= \sum_{1\leq I\leq r} \delta_I A^{I2}_Y,\\
\mathcal{W}_Y &= \sum_{1\leq I\leq r} \delta_I A^{I3}_Y.
\end{aligned}$$

This lemma shows that one can, at most, find five elements Y_a of $\mathcal{K}(U, g_{|U})$ giving independent functions $\mathcal{O}_Y$, etc. - in the sense that, given any other curvature collineation , the functions that it defines through L.5.3.16, are functional linear combinations of the functions defined by the Y_{a}. In other words, defining

$$Y_{\mathsf{a}} = \sum_{1 \le I \le r} \Phi^{I}_{\mathsf{a}} X_{I},$$

given any other $Y \epsilon \mathcal{K}(U, g_{|U})$, there exist differentiable functions a^{a} on M such that:

$$A^{I}_{Y;b} = \sum_{1 \le \mathsf{a} \le m} a^{\mathsf{a}} \Phi^{I}_{\mathsf{a};b}. \qquad (5.3.25)$$

(where m is the maximum number of Y_{a} in the above sense).

Taking covariant derivatives and antisymmetrising the above relation we get:

$$\sum_{1 \le \mathsf{a} \le m} (a^{\mathsf{a}}{}_{;p} \Phi^{I}_{\mathsf{a};k} - a^{\mathsf{a}}{}_{;k} \Phi^{I}_{\mathsf{a};p}) = 0.$$

Contracting this relation with $u^p u^k, ..., y^p y^k$, taking each of the relations thus obtained, multiplying it, in turn, by $\alpha_I, \beta_I, \gamma_I, \delta_I$ and then summing over I we get the following system (where we have set $\mathcal{O}_Y = \mathcal{O}_{\mathsf{b}}$, etc.) (cf. Appendix 5.V):

$$\sum_{\mathsf{b}} (\epsilon \mathcal{O}_{\mathsf{b}} + \mathcal{N}_{\mathsf{b}}) a^{\mathsf{b}2} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{T}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} (\epsilon \mathcal{O}_{\mathsf{b}} + \mathcal{N}_{\mathsf{b}}) a^{\mathsf{b}3} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} (-\epsilon \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}3} + \mathcal{T}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{T}_{\mathsf{b}} a^{\mathsf{b}3} - \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{V}_{\mathsf{b}} a^{\mathsf{b}3} - \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (-\epsilon \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}4} + \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{T}_{\mathsf{b}} a^{\mathsf{b}4} + \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{V}_{\mathsf{b}} a^{\mathsf{b}4} - \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}}(-\mathcal{T}_{\mathsf{b}}a^{\mathsf{b}4} + \mathcal{V}_{\mathsf{b}}a^{\mathsf{b}3}) = 0,$$

$$\sum_{\mathsf{b}}(\mathcal{O}_{\mathsf{b}}a^{\mathsf{b}4} + \mathcal{W}_{\mathsf{b}}a^{\mathsf{b}3}) = 0,$$

$$\sum_{\mathsf{b}}(\mathcal{W}_{\mathsf{b}}a^{\mathsf{b}4} - \mathcal{O}_{\mathsf{b}}a^{\mathsf{b}3}) = 0.$$

Suppose now that there exists $W \epsilon \mathcal{K}(U, g_{|U})$ which is not a linear combination of the X_I with constant coefficients; then, for every point p_o in U, we can find real numbers $\lambda^0, \lambda^1, ..., \lambda^r$ such that:

$$W' = \lambda^0 W + \sum_{1 \le I \le r} \lambda^I X_I,$$

satisfies:

$$W'_{p_o} = 0.$$

Writing

$$W = \sum_{1 \le I \le r} \mu^I X_I,$$

this gives:

$$\lambda_0 \mu^I(p_0) + \lambda^I = 0,$$

for $1 \le I \le r$. A simple computation shows that one has then

$$\psi^{W'}(p_0) = \lambda^0 \mathcal{O}_W(p_0).$$

This leads to the consideration of two cases:

(A). For all $W \epsilon \mathcal{K}(U, g_{|U})$, $\mathcal{O}_W$ is identically zero;

(B). For every $p_o \in U$ there exists $W \epsilon \mathcal{K}(U, g_{|U})$ such that $\mathcal{O}_W(p_0) \neq 0$.

NOTE.5.3.11. I am indebted to my supervisor, Dr. G. S. Hall, for suggesting the consideration of the fixed point case which ultimately lead to the analysis in separate of the above two cases.

Case (A)

In this case, we have $\mathcal{O}_{\mathsf{a}} = 0$ in U for all a; using the relations in the previous page we deduce then that the functions a^{b}, $1 \le \mathsf{b} \le m$ have their gradients parallel to u (see Appendix 5.VI)

Let us consider then for $Y \in \mathcal{K}(U, g_{|U})$ and $1 \le I \le r$, the functions

$$\eta^{I} = A^{I} - \sum_{\mathsf{b}} a^{\mathsf{b}} \Phi^{I}_{\mathsf{b}}. \tag{5.3.26}$$

Taking covariant derivatives, and using (5.3.25), we get

$$\eta^{I}_{;k} = -\sum_{\mathsf{b}} \Phi^{I}_{\mathsf{b}} a^{\mathsf{b}}{}_{;k},$$

so, as the gradients of the a^{b} are parallel to u, there exists functions $\mu^{I} \in \mathcal{F}(U)$ such that:

$$\eta^{I}_{;k} = \mu^{I} u_k. \tag{5.3.26}$$

Suppose then that U is sufficiently small for a coordinate system (r, s, t, w) such that $\partial_r = u$ to exist on U (recall that u is hypersurface orthogonal and so that its bivector is zero, as has been shown before).

For $1 \le I \le r$ let $Z_I = X_I - \alpha_I u$. We analyse separately the following two cases:

(A1). The Z_I are linearly independent at every point of U;

(A2). The rank of the family $\{Z_1, ..., Z_r\}$ is $< r$.

Case (A1)

In this case, if $Y = \sum_I A^I X_I \in \mathcal{K}(U, g_{|U})$, we have

$$Z = Y - \alpha^{Y} u = \sum_{I} A^{I} Z_{I}.$$

From T.5.3.3.(b) we deduce that:

$$0 = [u, Z] = \sum_{I} (\mathcal{L}_u A^{I}) Z_{I},$$

hence that for $1 \le I \le r$

$$\mathcal{L}_u A^{I} = 0. \tag{5.3.27}$$

Thus, the functions A^{I} do not depend on r. Similarly, the functions Φ^{I}_{b} do not depend on r. One has then, in the above coordinate system

$$\eta^{I}(r) = A^{I}(s, t, w) - \sum_{\mathsf{b}} a^{\mathsf{b}}(r) \Phi^{I}_{\mathsf{b}}(s, t, w). \tag{5.3.28}$$

Let us introduce a definition.

Let I be an open interval of the real line, $f^1, ..., f^k$ differentiable functions from I to $\mathbf{R}$. We assume that one at least of the f^i is not identically zero.

For any integer l with $1 \leq l \leq k$, any family $i_1, ..., i_l \in \{1, ..., k\}$ and any family $r_1, ..., r_l \in$ I we define:

$$M(f^{i_1}, .., f^{i_l}; r_1, .., r_l) = \begin{pmatrix} f^{i_1}(r_1) & . & . & . & f^{i_l}(r_1) \\ f^{i_1}(r_2) & . & . & . & f^{i_l}(r_2) \\ . & & . & . & . \\ . & & . & . & . \\ f^{i_1}(r_l) & . & . & . & f^{i_l}(r_l) \end{pmatrix}$$

We say that the family $(f^1, ..., f^k)$ has rank m_o if and only if m_o is the maximum value of l for which there exists one of the above matrices with non-zero determinant.

Notice that a family of functions such as $(f^1, ..., f^k)$ always has a rank (this can be easily seen by induction). Furthermore, assume that $(f^1, ..., f^k)$ has rank l. Then, reordering the f^i we can assume that there exist $r_1, ..., r_l \in$ I such that $\det M(f^1, ..., f^l; r_1, ..., r_l) \neq 0$. Then, for $1 \leq p \leq (k - l)$ the matrix $M(f^1, ..., f^l, f^{l+p}; r_1, ..., r_l, r)$ has zero determinant for all values of $r \in$ I. As the first l columns are independent, this means that the last column of this matrix is a linear combination of the other columns; thus we have:

$$f^{l+p}(r_j) = \sum_{1 \leq i \leq l} C_i^p f^i(r_j),$$

for $1 \leq j \leq l$, and

$$f^{l+p}(r) = \sum_{1 \leq i \leq l} C_i^p f^i(r),$$

and a classical argument shows that the C_i^p are real constants. Thus, if a family $(f^1, ..., f^k)$ has rank $l < k$, then there exists a subfamily of $(f^1, ..., f^k)$ such that every other element of the family is a linear combination of the elements of the sub-family with constant coefficients.

This notion together with (5.3.28) allows us to prove the

THEOREM 5.3.17. *In case (A1)* $\mathcal{K}(U, g_{|U})$ *is finite dimensional and we have* $\dim \mathcal{K}(U, g_{|U}) = r + m \leq 8$.

NOTE.5.3.12. As we assume that the Z_I are all linearly independent we have $r \leq 3$, since the Z_I are orthogonal to u. As the maximum number of Y_a is $m \leq 5$, this gives the above bound on $\dim \mathcal{K}(U, g_{|U})$.

Proof. It is sufficient to prove that if $Y \in \mathcal{K}(U, g_{|U})$, there exist real constants k^I, $1 \leq I \leq r$, and D^{b}, $1 \leq \mathsf{b} \leq m$ such that:

$$Y = \sum_I k^I X_I + \sum_{\mathsf{b}} D^{\mathsf{b}} Y_{\mathsf{b}}.$$

Consider then the maps a^{b} given by (5.3.25).

Let n be the rank of the family $(a^1, ..., a^m)$. Reordering the Y_{b} we can therefore assume that for $1 \leq \mathsf{b} \leq m-n$ the function $a^{n+\mathsf{b}}$ is of the form

$$a^{n+\mathsf{b}} = \sum_{1 \leq \mathsf{i} \leq n} C^{\mathsf{b}}_{\mathsf{i}} a^{\mathsf{i}},$$

where the $C^{\mathsf{b}}_{\mathsf{i}}$ are real constants for $1 \leq \mathsf{b} \leq (m-n)$ and $1 \leq \mathsf{i} \leq n$. (5.3.28) gives then, after some computations

$$\eta^I = A^I - \sum_{1 \leq \mathsf{i}\, n} a^{\mathsf{i}} (\Phi^I_{\mathsf{i}} - \sum_{1 \leq \mathsf{e} \leq (m-n)} C^{\mathsf{e}}_{\mathsf{i}} \Phi^I_{n+\mathsf{e}}). \tag{5.3.29}$$

Consider then the family $(1, a^1, ..., a^n)$. If this family has rank $n+1$ we can find $r_1, ..., r_{n+1}$ such that

$$\det M(1, a^1, ..., a^n; r_1, ..., r_{n+1}) \neq 0.$$

Denoting this matrix by M and setting, for $1 \leq I \leq r$:

$$L^I = \begin{pmatrix} \eta^I(r_1) \\ \eta^I(r_2) \\ . \\ . \\ \eta^I(r_{n+1}) \end{pmatrix},$$

and

$$V^I = \begin{pmatrix} A^I(s,t,w) \\ \Phi^I_1(s,t,w) - \sum C^{\mathsf{e}}_1 \Phi^I_{n+\mathsf{e}}(s,t,w) \\ \cdots\cdots \\ \cdots\cdots \\ \Phi^I_n(s,t,w) - \sum C^{\mathsf{e}}_n \Phi^I_{n+\mathsf{e}}(s,t,w) \end{pmatrix},$$

we get, from the expression for η^I, $M.V^I = L^I$. Since M is invertible this gives $V^I = M^{-1} L^I$; since L^I and M are constants, we deduce that the A^I are constants.

Otherwise the family $(1, a^1, ..., a^n)$ has rank n (the rank cannot be smaller than n). This implies that 1 is a linear combination of $a^1, .., a^n$ with constant coefficients:

$$1 = \sum_{1 \leq \mathsf{e} \leq n} E_{\mathsf{e}} a^{\mathsf{e}}.$$

Replacing in the expression for η^{I}, (5.3.29), we get

$$\eta^{I} = \sum_{1\leq i\leq n} a^{i}(E_{i}A^{I} - \Phi_{i}^{I} - \sum_{1\leq e\leq(n-m)} C_{i}^{e}\Phi^{I}_{n+e}). \tag{5.3.30}$$

Since the family a^{i} has rank n this shows, by the same process as in the preceding case, that for every $1 \leq i \leq n$ and every $1 \leq I \leq r$, there exists a real constant k_{i}^{I} such that:

$$E_{i}A^{I} - \Phi_{i}^{I} - \sum_{1\leq e\leq m-n} C_{i}^{e}\Phi^{I}_{n+e} = k_{i}^{I}.$$

Since not all the E_{i} are zero, choose an index a such that $E_{a} \neq 0$. Then we have

$$A^{I} = \frac{1}{E_{a}}\Phi^{I}_{a} + \sum_{1\leq e\leq(m-n)} \frac{C_{a}^{e}}{E_{a}}\Phi^{I}_{n+e} + \frac{1}{E_{a}}k_{i}^{I},$$

and this proves the proposition ■

Case (A2)

In this case, as the Z_{I} are not all independent, we can, by reordering them assume that there exist functions Θ^{I} on U such that

$$Z_{r} = \sum_{1\leq I\leq(r-1)} \Theta^{I}Z_{I}. \tag{5.3.31}$$

Furthermore, as the family $\{X_1, ..., X_r\}$ spanns at each point in U a r-dimensional subspace of the tangent space, we see that the $Z_1, ..., Z_{r-1}$ are independent.

Let now $Y \in \mathcal{K}(U, g_{|U})$, so that we have, using the above relations

$$Z = Y - \alpha^{Y}u = \sum_{1\leq I\leq r} A^{I}Z_{I} = \sum_{1\leq I\leq(r-1)} (A^{I} + \Theta^{I}A^{r})Z_{I}. \tag{5.3.32}$$

From T.5.3.3, we deduce that

$$0 = [u, Z] = \sum_{1\leq I\leq(r-1)} (\mathcal{L}_{u}(A^{I} + \Theta^{I}A^{r}))Z_{I},$$

so that, for $1 \leq I \leq (r-1)$, we have

$$\mathcal{L}_{u}(A^{I} + \Theta^{I}A^{r}) = 0. \tag{5.3.33}$$

Define then

$$D^I = A^I + \Theta^I A^r, 1 \leq I \leq (r-1)$$
$$D^I_{\mathsf{i}} = \Phi^I_{\mathsf{i}} + \Theta^I \Phi^r_{\mathsf{i}}, 1 \leq I \leq (r-1), 1 \leq \mathsf{i} \leq m. \tag{5.3.34}$$

Then the functions D^I, D^I_{i} do not depend on r. From (5.3.26) we have then, for $1 \leq I \leq (r-1)$

$$\eta^I = D^I - \Theta^I A^r - \sum_{1 \leq \mathsf{i} \leq m} a^{\mathsf{i}}(D^I_{\mathsf{i}} - \Theta^I \Phi^r_{\mathsf{i}}).$$

Using again (5.3.26) with $I = r$, we get, from the above expression, for $1 \leq I \leq (r-1)$

$$\eta^I = D^I - \Theta^I \eta^r - \sum_{1 \leq \mathsf{i} \leq m} a^{\mathsf{i}} D^I_{\mathsf{i}}. \tag{5.3.35}$$

(5.3.31) together with T.5.3.3 and the fact that the Z_I $(1 \leq I \leq (r-1)$ are linearly independent, give

$$\mathcal{L}_u \Theta^I = 0. \tag{5.3.36}$$

Consequently, we have from (5.3.35), for $1 \leq I \leq (r-1)$

$$\eta^I(r) = D^I(s,t,w) - \Theta^I(s,t,w)\eta^r(r) - \sum_{1 \leq \mathsf{i} \leq m} a^{\mathsf{i}}(r) D^I_{\mathsf{i}}(s,t,w). \tag{5.3.37}$$

Using then the same kind of argument as in case (A2) we can prove that in this case either $\mathcal{K}(U, g_{|U})$ is finite dimensional or there exists a curvature collineation parallel to u (see Appendix 5.VII for a complete proof).

Thus:

THEOREM 5.3.18. *In case (A2) if there does not exist a curvature collineation parallel to u the Lie algebra $\mathcal{K}(U, g_{|U})$ is finite dimensional and its dimension is ≤ 9.*

Case (B)

In this case, for every point $p \in U$ there exists a curvature collineation W such that $W_p = 0$ and $\psi^W(p) \neq 0$.

Let us then consider the Riemann tensor as a linear endomorphism of the space of bivectors. As the Riemann tensor has at most rank 3 and at least rank 2, and as the space of bivectors is 6-dimensional, we deduce that the Riemann tensor has at least 3 real eigenvalues ($= 0$); since complex eigenvalues always come in pairs, we deduce the existence of a fourth real eigenvalue. Denoting by G_{ab} this eigenbivector we have

$$\mathbf{R}_{abcd}G^{cd} = \lambda G_{ab},$$

for some real function λ on M. Contracting this relation with u^b, we see that

$$G_{ab}u^b = 0.$$

Thus, G is a simple bivector whose blade is orthogonal to u.

Consider now the Ricci tensor. Since this tensor is obtained from the Riemann tensor by contraction it is clear that if W is a curvature collineation then

$$\mathcal{L}_W Ricci = 0.$$

This translates locally into

$$W^c\mathbf{R}_{ab;c} + W^c{}_{;a}\mathbf{R}_{cb} + W^c{}_{;b}\mathbf{R}_{ac} = 0.$$

Some simple computations using the form of h^W reduce this expression to

$$W^c\mathbf{R}_{ab;c} + 2\psi^W\mathbf{R}_{ab} + f^c{}_a\mathbf{R}_{cb} + f^c{}_b\mathbf{R}_a c = 0, \tag{5.3.38}$$

where f is the bivector of W.

Thus, if $W_{p_o} = 0$ and $\psi^W(p_o) \neq 0$, we have at p_o

$$2\psi^W(p_o)\mathbf{R}_{ab}(p_o) + f^c{}_a(p_o)\mathbf{R}_{cb}(p_o) + f^c{}_b(p_o)\mathbf{R}_{ac}(p_o) = 0. \tag{5.3.39}$$

Using this relation we can prove that

Lemma 5.3.19. *Case (B) can occur only when u is spacelike and the Riemann tensor has a null eigenbivector with non-zero eigenvalue.*

Proof. In the remaining cases the bivector G described above is non-null. This allows us to choose an orthonormal basis (u, v, x, y) in such a way that $G_{ab} = 2v_{[a}x_{b]}$. Defining $H_{ab} = 2v_{[a}y_{b]}$ and $L_{ab} = 2x_{[a}y_{b]}$, and using the fact that u satisfies (5.3.1) and the fact that G is an eigenbivector, it is easy to prove that there exist functions $A, B, C, D \in \mathcal{F}(U)$ such that

$$\mathbf{R}_{abcd} = AG_{ab}G_{cd} + BH_{ab}H_{cd} + C(H_{ab}L_{cd} + H_{cd}L_{ab}) + DL_{ab}L_{cd}. \tag{5.3.40}$$

One can then compute the Ricci tensor from the above expression. Replacing in (5.3.39) one proves then that the Riemann tensor has at most rank 1, which contradicts the fact that (M, g) is a class (II) space-time (see the Appendix VIII to this chapter for the detailed computations)■

Finally we consider the case when u is spacelike and the Riemann tensor has a null eigenbivector whose blade is orthogonal to u.

Choose then a null tetrad (l, n, u, v) in such a way that $G_{ab} = 2l_{[a}v_{b]}$.

Setting $H_{ab} = 2n_{[a}v_{b]}$ and $L_{ab} = 2n_{[a}l_{b]}$, we have the following general form for the Riemann tensor

$$\mathbf{R}_{abcd} = AG_{ab}G_{cd} + B(G_{ab}H_{cd} + G_{cd}H_{ab}) + C(G_{ab}L_{cd} + G_{cd}L_{ab}) + DL_{ab}L_{cd}. \tag{5.3.41}$$

This gives for the Ricci tensor

$$\mathbf{R}_{ab} = Al_al_b + 2(B - D)l_{(b}n_{d)} + 2Cl_{(b}v_{d)} + 2Bv_bv_d. \tag{5.3.42}$$

Using this relation and contracting (5.3.39) with $2n^{(a}l^{b)}$ and defining $\zeta = f_{ab}l^av^b$, we get

$$2\psi^W(p_o)B(p_o) - 2\psi^W(p_o)D(p_o) - \zeta(p_o)C(p_o) = 0.$$

Contracting (5.3.39) with v^av^b we get

$$2\psi^W(p_o)B(p_o) + \zeta(p_o)C(p_o) = 0.$$

Summing we get $2B(p_o) = D(p_o)$; since in fact the point p_o is arbitrary, we deduce that $2B = D$ in U.

On the other hand, a contraction of (5.3.39) with $2l^{(a}v^{b)}$ gives $\zeta(p_o)(B + D)(p_o) = 0$; using $2B = D$ this gives $\zeta(p_o)B(p_o) = 0$. Assume that $B(p_o) \neq 0$. Then one has $\zeta(p_o) = 0$ and the first of the relations above gives then $\psi^W(p_o) = 0$, which is absurd (cf. p.45). Consequently, we have $B = D = 0$ in U. Thus the Ricci tensor reduces to

$$\mathbf{R}_{ab} = Al_al_b + 2Cl_{(a}v_{b)}, \tag{5.3.43}$$

(and, in particular, it has Segre type $\{(211)\}$). As $B = D = 0$ in U, we have, from (5.3.41)

$$\mathbf{R}_{abcd} = AG_{ab}G_{cd} + C(G_{ab}L_{cd} + G_{cd}L_{ab}). \tag{5.3.44}$$

Now let w be such that $\mathbf{R}_{ab}w^b = 0$. From (5.3.43) we deduce that $w = au + bl$ for some $a, b \in \mathcal{F}(U)$. In particular, since $\mathbf{R}_{ab}l^b = 0$, given a curvature collineation X in U we have, as $\mathcal{L}_X Ricci = 0$:

$$\mathcal{L}_X l = au + bl, \tag{5.3.45}$$

for some $a, b \in \mathcal{F}(U)$.

Contracting (5.3.43) with v^b and with n^b we get

$$\begin{aligned} \mathbf{R}_{ab}v^b &= Cl_a, \\ \mathbf{R}_{ab}n^b &= Al_a + Cv_a. \end{aligned} \tag{5.3.46}$$

Let then $c, ..., m \in \mathcal{F}(U)$ be given by

$$\begin{aligned} \mathcal{L}_X v &= cu + dv + en + fl, \\ \mathcal{L}_X n &= hu + in + jl + mv. \end{aligned} \tag{5.3.47}$$

Taking the Lie derivative of $\mathbf{R}_{ab}v^b$ along X and using (5.3.45) and (5.3.46) we get, after some simple computations

$$\begin{aligned} \mathcal{L}_X C + C(b-d) + eA &= 0, \\ aC &= 0, \\ eC &= 0, \end{aligned} \tag{5.3.48}$$

Suppose that $a(p) \neq 0$ for some $p \in U$. Then C vanishes in some open neighbourhood V of p in U. (5.3.44) shows then that in V the rank of the Riemann tensor is at most 1. This contradicts our assumption on the rank of the Riemann tensor made at the beginning of this §. Thus a (and e) vanishes identically in U.

Similarly, if we compute the Lie derivative of $\mathbf{R}_{ab}n^b$ along X and we use (5.3.45) and (5.3.46), we get

$$\begin{aligned} \mathcal{L}_X A + A(b-i) + C(f-m) &= 0, \\ \mathcal{L}_X C + C(d-i) &= 0, \\ cC &= 0. \end{aligned} \tag{5.3.49}$$

Hence, c vanishes identically in U.

Going now back to (5.3.44), we get

$$\mathbf{R}^a{}_{bcd}n^c l^d = -CG^a{}_b. \tag{5.3.50}$$

Using the above relations for $\mathcal{L}_X l$ and $\mathcal{L}_X v$, we get, as $(\mathcal{L}_X g)_{ab} = 2\psi^X g_{ab} + 2\rho^X u_a u_b$

$$\begin{aligned} \mathcal{L}_X l_b &= (2\psi^X + b)l_b, \\ \mathcal{L}_X v_b &= (2\psi^X + d)v_b + fl_b. \end{aligned}$$

Taking then the Lie derivative of (5.3.50) along X and using the above relations we get

$$\mathcal{L}_X C + C(2\psi^X + d - i) = 0.$$

Comparing this relation with the second relation in (5.3.49) we get $\psi^X C = 0$. This shows that ψ^X is identically zero in U. As X is arbitrary this must, in particular, hold for the curvature collineation W we started with. Thus, we get a contradiction since we assumed that ψ^W was not identically zero.

We conclude therefore that space-times in class (B) do not exist.

NOTE.5.3.13. The advantages of the non-vanishing of ψ^W at the fixed point of W were pointed out to me by my supervisor, Dr. G. S. Hall. The techniques used here are similar to those employed by Hall in the study of fixed points of homotheties [38].

EXAMPLES. Let M be a connected open subset of $\mathbb{R}^4$ and (r, s, t, w) coordinates on M. Let $m, p, q \in \mathcal{F}(M)$ be such that m depends only on s, w and p, q depend only on t. Furthermore, assume that m and $2pr + q$ are strictly positive in M.

Consider then the metric g defined on M by:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 m \\ 0 & 0 & (2rp+q)^2/4p & 0 \\ 0 & r^2 m & 0 & 0 \end{pmatrix}.$$

A simple computation shows that the components of the Riemann tensor which are not identically zero are the following (taking into account the symmetries)

$$\mathbf{R}^2{}_{224} = -\frac{m\partial_s\partial_w m - \partial_s m \partial_w m + m^3}{m^2},$$
$$\mathbf{R}^2{}_{323} = -\frac{2pr+q}{2r},$$
$$\mathbf{R}^3{}_{234} = -\frac{2rpm}{2pr+q}.$$

The covariant Riemann tensor has then as its only non-zero components

$$\mathbf{R}_{2334} = \frac{mr(2pr+q)}{2}$$
$$\mathbf{R}_{2424} = \frac{r^2(m\partial_s\partial_w m - \partial_s m \partial_w m + m^3)}{m}$$
$$\mathbf{R}_{3423} = \frac{mr(2pr+q)}{2}.$$

Our assumptions show that the first of these components is never zero; the Riemann tensor has therefore, at least rank 2 everywhere (the rank being 3 whenever the second of these components is non-zero).

This shows that in this case the dimension of the space of solutions of (5.3.1) is at most 1; since obviously $u = \partial_r$ is a solution of this equation, we see that (M, g) is a class (II) space-time. (that ∂_r is a solution of (5.3.1) follows from the fact that in none of the above components of the Riemann tensor figures the index 1).

We study now the Lie algebra $\mathcal{K}(M, g)$.

Let $X = A\partial_r + B\partial_s + C\partial_t + D\partial_w \in \mathcal{K}(M, g)$.

We have then $\overset{X}{\alpha} = A$ and $Z = B\partial_s + C\partial_t + D\partial_w$. Thus, from T.5.3.3.(b) and T.5.3.4.(a) we deduce that A depends only on r and B, C, D do not depend on r.

Since $\overset{X}{h}_{ab} = \overset{X}{\psi} g_{ab} + \overset{X}{\rho} u_a u_b$, we get, using the above form of the metric (and noticing that u is spacelike)

$$\overset{X}{h} = \begin{pmatrix} \overset{X}{\psi} + \overset{X}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overset{X}{\psi} r^2 m \\ 0 & 0 & \overset{X}{\psi}(2pr+q)^2/4p & 0 \\ 0 & \overset{X}{\psi} r^2 m & 0 & 0 \end{pmatrix}.$$

On the other hand, a direct computation gives

$$\begin{aligned} h_{11} &= \partial_r A, \\ h_{22} &= mr^2 \partial_s D, \\ h_{23} &= \frac{(2pr+q)^2 \partial_s C + 4mpr^2 \partial_t D}{8p}, \\ h_{24} &= \frac{r(mr(\partial_s B + \partial_w D) + rB\partial_s m + rD\partial_w m + 2Am)}{2}, \\ h_{33} &= \frac{\nu}{8p^2} \\ h_{34} &= \frac{(2pr+q)^2 \partial_w C + 4mpr^2 \partial_t B}{8p}, \\ h_{44} &= mr^2 \partial_w B, \end{aligned}$$

where

$$\nu = 2p(2pr+q)^2 \partial_t C + (4p^2r^2 - q^2)C\partial_t p + 2(2pr+q)pC\partial_t Q + 8Ap^3 r + 4Ap^2 q).$$

Comparing these expressions we see that $\partial_s D = 0$ (from h_{22}) and that $\partial_w B = 0$ (from h_{44}).

Now computing $(\mathcal{L}_X Riemann)^2{}_{323}$ we get, denoting by $S^a{}_{bdc}$ the components of the tensor $\mathcal{L}_X Riemann$

$$S^2{}_{323} = -\frac{((4pr^2 + 2qr)\partial_t C + 2Cr^2\partial_t p + Cr\partial_t q - Aq)}{2r^2}.$$

This shows that if $q \neq 0$ and $B = C = D = 0$, X cannot be a curvature collineation. In fact in such case we have $S^2{}_{323} = Aq/2r^2 \neq 0$. Hence, if $q \neq 0$ there are no curvature collineations parallel to u.

Computing now $S^4{}_{243}$ we get after some simplifications

$$[4p^2\partial_s C + 4pm\partial_t D]r^2 + [4pq\partial_s C]r + q^2\partial_s C = 0.$$

Since the expressions in bracketts do not depend on r we deduce that they have to be identically zero. Assuming $q \neq 0$, we deduce that $\partial_s C = \partial_t D = 0$.

Similarly, computing $S^2{}_{423}$, we get, using a similar argument, $\partial_t B = \partial_w C = 0$ (these relations can in fact be obtained directly from h_{34}).

After some lengthy but simple computations the remaining $S^a{}_{bcd}$ give

$$A(r) = \alpha r^2 + \beta r,$$

$$4p\partial_t C + 2C\partial_t p - \alpha q = 0,$$

$$2q\partial_t C + C\partial_t q - \beta q = 0, \qquad (*)$$

$$2p[\partial_s(mB) + \partial_w(mD)] + \alpha mq = 0,$$

$$m\zeta[\partial_s B + \partial_w D) + (m\partial_s\zeta - 2\zeta\partial_s m)D + (m\partial_w\zeta - 2\zeta\partial_w m)B = 0,$$

where α, β are real constants and ζ is given by

$$\zeta = m\partial_s\partial_w m - \partial_s m\partial_w m + m^3.$$

The fourth equation in $(*)$ shows that if p, q are independent functions of t then we must have $\partial_s(mB) + \partial_w(mD) = \alpha m = 0$. Thus, we must have $\alpha = 0$, hence $A(r) = \beta r$. Now this gives

$$h_{24} = Amr = \beta mr^2,$$

thus $\psi^X = \beta$. Since $h_{11} = \beta = \psi^X + \rho^X$, we deduce that $\rho^X = 0$, and so, when the family $\{p, q\}$ has rank 2 every curvature collineation is a homothetic vector field.

SUMMARY. In this section we have analysed class II(a) space-times. In the first part we studied the case when there exists (at least) a curvature collineation which contracts the Riemann tensor to zero and we have proved that, in this case, the vector field u (defined at the beginning of this section) is hypersurface orthogonal (L.5.3.5). Locally, this fact leads to an infinite-dimensional Lie algebra of curvature collineations (T.5.3.6.(a)). This result can in fact be extended to a global result when a supplementary (global) condition is imposed on u (T.5.3.9). A (local) characterization of this type of space-times (T.5.3.6.(b)) as well as a (local) characterization of all curvature collineations was also obtained (T.5.3.7. and T.5.3.8).

We then analysed the general case, that is without assuming the existence of a curvature collineation parallel to u. To proceed we have been forced to impose some further regularity conditions on (M, g). These concerned the constancy of the function $\mathcal{R}$ (defined in §.5.1) and, in the case when we assumed u non hypersurface orthogonal , the type of its bivector (assumed to be the same everywhere). In these cases no global results where obtained but we have proved that, locally, either $\mathcal{K}(M, g)$ is finite -dimensional or there exists a curvature collineation parallel to u (T.5.3.13, T.5.3.15, T.5.3.17 and T.5.3.18).

5.4.Space-times in class (III)(a)

In this section, we consider space-times (M, g) with the property that there exists an open dense subset O of M such that for every $p \in O$ the set of solutions of the equation given at p by:

$$\mathbf{R}^a{}_{bcd} v^d = 0, \tag{5.3.1}$$

is a 2-dimensional non-null subspace of $T_p(M)$. We denote this subspace by $\mathcal{V}_p$.

In all that follows we make the assumption that the distribution $\mathcal{V} : p \mapsto \mathcal{V}_p$ is in fact globally defined and differentiable. Furthermore, we assume that there exist nowhere null, nowhere zero, vector fields u and v on M such that for every $p \in M$ (u_p, v_p) spans $\mathcal{V}_p$. We assume u and v scaled in such a way that $u^a u_a = 1$, $v^a v_a = \epsilon = \pm 1$ and $u^a v_a = 0$.

A space-time satisfying these conditions will be called a class III(a) spacetimein the sequel.

In this section we keep the notation of the preceding one.

General observations

Lemma 5.4.1. *There exist functions* $\pi, \sigma \in \mathcal{F}(M)$ *such that* $u^b u_{a;b} = \sigma v_a$, $v^b u_{a;b} = \pi v_a$, $v^b v_{a;b} = -\epsilon \pi u_a$ *and* $u^b v_{a;b} = -\epsilon \sigma u_a$.

Proof. From $u_a \mathbf{R}^a{}_{bcd} = 0$ we get, taking covariant derivatives, antisymmetrising and using Bianchi's identities

$$u_{a;k}\mathbf{R}^a{}_{bcd} + u_{a;d}\mathbf{R}^a{}_{bkc} + u_{a;c}\mathbf{R}^a{}_{bdk} = 0.$$

Contracting with u^k we get $u^k u_{a;k}\mathbf{R}^a{}_{bcd} = 0$, and this shows that $u^k u_{a;k}$ lies in the plane spanned by u and v. As $u^a u_a$ is constant this vector is orthogonal to u, thus it is parallel to v.

The remaining of the proof is similar■

Using the results of Ch.2, T.2.3.1.(d), and a method of proof similar to that of T.5.3.2, we get

THEOREM 5.4.2. *If* $X \epsilon \mathcal{K}(M, g)$ *there exist functions* $\psi^{X}, \rho^{X}, \lambda^{X}, \mu^{X}$ *in* M *such that the following relations hold (cf. §.5.1):*
(a).$h^{X}{}_{ab} = \psi^{X} g_{ab} + \rho^{X} u_a u_b + \lambda^{X}(u_a v_b + u_b v_a) + \mu^{X} v_a v_b$;
(b).$[u, X] = au + bv$;
(c).$[v, X] = cu + dv$,
where the functions a, b, c, d *are given by*

$$a = \psi^{X} + \rho^{X},$$

$$b = \lambda^{X} + \epsilon f_{ab} u^b v^a - X^b v^a u_{a;b},$$

$$c = \epsilon\lambda^{X} + f_{ab} v^b u^a - X^b u^a v_{a;b},$$

$$d = \psi^{X} + \epsilon\mu^{X}.$$

It follows then that

THEOREM 5.4.3.*Let* $X \epsilon \mathcal{K}(M, g)$ *and define* $\alpha^{X} = u^a X_a$, $\beta^{X} = \epsilon v^a X_a$ *and* $Z = X - \alpha^{X} u - \beta^{X} v$, *so that* $u^a Z_a = v^a Z_a = 0$.*Then*
(a).$u^b \alpha^{X}_b = a + \epsilon\sigma\beta^{X}$, $v^b \beta^{X}_b = d - \alpha^{X}\pi$;

$(b).[u, Z] = (b - u^b\beta_b^X + \pi\beta^X)v$, $[v, Z] = (c - v^b\alpha_b^X - \epsilon\alpha^X\sigma)u$.

Again the subscript X in α^X, etc., will be dropped whenever no confusion is possible.

Recalling that the function $\mathcal{R}$ was defined in §.5.1, we consider first

The case $\mathcal{R} = 1$.

We get here a result similar to that of the preceding §

THEOREM 5.4.4. *Let (M, g) be a class (III)(a) space-time such that $\mathcal{R} = 1$ in M. Then for every $p \in M$ there exists a nice neighbourhood U of p in M such that, either $\mathcal{K}(U, g_{|U})$ is finite dimensional, and its dimension is 1, or it is infinite dimensional. This latter case occurs only when there exists a curvature collineation which at every point p of U lies in the subspace of $T_p(M)$ spanned by u_p and v_p.*

Proof. Let $p \in M$. As $\mathcal{R} = 1$ we can find $X \in \mathcal{K}(M, g)$ which does not vanish at p. By continuity X does not vanish in some open connected neighbourhood U of p in M. Let then $Y \in \mathcal{K}(U, g_{|U})$. Again because $\mathcal{R} = 1$, Y is at every point of U parallel to X, and so there exists $\theta \in \mathcal{F}(U)$ such that $Y = \theta X$.

Since both Y and X are curvature collineations, it follows (cf. T.5.4.2) that we have

$$h^X{}_{ab} = \psi^X g_{ab} + \rho^X u_a u_b + \lambda^X (u_a v_b + u_b v_a) + \mu^X v_A v_b;$$

and a similar expression for Y. On the other hand, since $Y = \theta X$, we have

$$2h^Y_{ab} = \theta_a X_b + \theta_b X_a + 2\theta h^X_{ab}.$$

Combining these expressions we get

$$\theta_a X_b + \theta_b X_a =$$

$$2(\psi^Y - \theta\psi^X)g_{ab} + 2(\rho^Y - \theta\rho^X)u_a u_b + 4(\lambda^Y - \theta\lambda^X)u_{(a}v_{b)} + 2(\mu^Y - \theta\mu^X)v_a v_b. \tag{5.4.4}$$

Now we can write (5.1.2) for both X and Y. Using these expressions we get, from $\mathcal{L}_Y Riemann - \theta\mathcal{L}_X Riemann = 0$,

$$\theta_b X^m \mathbf{R}^a{}_{mcd} + \theta_c X^m \mathbf{R}^a{}_{bmd} + \theta_d \mathbf{R}^a{}_{bcm} - \theta_m X^a \mathbf{R}^m{}_{bcd} = 0. \tag{5.4.5}$$

Contracting this last relation with u_a we get $(u_a X^a)\theta_m \mathbf{R}^m{}_{bcd} = 0$. Thus, we are led to consider the two possibilities

(i). θ_m does not lie in the (u, v)-plane;

(ii). θ_m lies in the (u, v)-plane.

Notice that the first of these statements if it holds at some point, then it holds in some open neighbourhood of that point.

Case (i)

In this case, we deduce that $\theta_m \mathbf{R}^m{}_{bcd} \neq 0$, and so we have $u_a X^a = 0$. Similarly, a contraction of (5.4.5) with v_a shows that we also have $v_a X^a = 0$.

Contracting then (5.4.4) with u^a we get $(\lambda^{Y} - \theta\lambda^{X}) = 0$ and $(\psi^{Y} - \theta\psi^{X}) + (\rho^{Y} - \theta\rho^{X}) = 0$.

Contracting (5.4.4) with v^a we get $(\psi^{Y} - \theta\psi^{X}) + \epsilon 2(\mu^{Y} - \theta\mu^{X}) = 0$.

This shows that, setting $\Gamma = (\psi^{Y} - \theta\psi^{X})$, we have

$$\theta_a X_b + \theta_b X_a = 2\Gamma(x_a x_b - \epsilon y_a y_b). \tag{5.4.6}$$

This relation shows immediately that whenever $-\epsilon = 1$ (that is, whenever the plane (u, v) is timelike) we must have $\theta_m = 0$. In the case when (u, v) is spacelike, this relation shows that θ_m and X_p are both null and non-orthogonal. Thus, in this case, setting $l_b = (x_b - y_b)/\sqrt{2}$ and $n_b = (x_b + y_b)/\sqrt{2}$, we have

$$X = Sl,$$

and

$$\theta_b = Tn_b,$$

for some $S, T \in \mathcal{F}(U)$. Now as u, v are solutions of (5.4.1), there exists $\Omega \in \mathcal{F}(U)$ such that

$$\mathbf{R}_{abcd} = 4\Omega l_{[a} n_{b]} l_{[c} n_{d]}. \tag{5.4.7}$$

Using this relation and contracting (5.4.5) with $l_a n^b n^d$ we get $\Omega ST = 0$ in U. Since Ω does not vanish on open subsets of U and since S never vanishes we deduce again that θ is constant.

Case (ii).

In this case we have $x^b\theta_b = y^b\theta_b = 0$. Contracting then (5.4.4) with x^b we get, using these relations

$$(x^bX_b)\theta_a = 2(\psi^{Y} - \theta\psi^{X})x_a,$$

and this shows that either θ is constant or $x^bX_b = 0$. Similarly, a contraction of (5.4.4) with y^a shows that either θ is constant or $y^bX_b = 0$. Thus, if θ is not constant, then X lies in the (u,v) plane at all points of U. Clearly in such case for every integer k, θ^kX is also a curvature collineation. Thus $\mathcal{K}(U, g_{|U})$ is infinite dimensional ■

The cases $\mathcal{R} \geq 2$.

We assume from now on that (M,g) is regular (in the sense of §.5.1.) and we consider a $\mathcal{K}(U, g_{|U})$-generating family $\mathcal{F} = \{X_1, ..., X_r\}$ (U being a nice subset of M) and we let (u, v, x, y) be an orthonormal family. We set, as in the preceding section

$$X_I = \alpha_I u + \beta_I v + \gamma_I x + \delta_I y, \tag{5.4.8}$$

and

$$h^{I}_{ab} = \psi_I g_{ab} + \rho_I u_a u_b + 2\lambda_I u_{(a}v_{b)} + \mu_I v_a v_b. \tag{5.4.9}$$

Given then $Y \in \mathcal{K}(U, g_{|U})$, we have

$$Y = \sum_{1\leq I\leq r} A^{I}_{Y} X_I. \tag{5.4.10}$$

We set then

$$\begin{aligned}
\mathcal{O}_Y &= \psi^{Y} - \sum_{1\leq I\leq r} A^{I}_{Y}\psi_I,\\
\mathcal{R}_Y &= \rho^{Y} - \sum_{1\leq I\leq r} A^{I}_{Y}\rho_I,\\
\mathcal{L}_Y &= \lambda^{Y} - \sum_{1\leq I\leq r} A^{I}_{Y}\lambda_I,\\
\mathcal{S}_Y &= \mu^{Y} - \sum_{1\leq I\leq r} A^{I}_{Y}\mu_I.
\end{aligned} \tag{5.4.11}$$

A process similar to that used in the preceding section shows that the components of the gradients of the functions A^{I}_{Y} with respect to the orthonormal family above satisfy

$$\begin{aligned}\Lambda^1(p;Y) &= (\mathcal{R}_Y, \mathcal{A}_Y, 0, 0),\\ \Lambda^2(p;Y) &= (2\mathcal{L}_Y - \mathcal{A}_Y, \mathcal{S}_Y, 0, 0),\\ \Lambda^3(p;Y) &= (0,0,0,\mathcal{F}_Y),\\ \Lambda^4(p;Y) &= (0,0,-\mathcal{F}_Y,0).\end{aligned} \tag{5.4.12}$$

where

$$\mathcal{F}_Y = \sum_{1\leq I\leq r} \delta_I A_Y^{I3},$$

and

$$\mathcal{A}_Y = \sum_{1\leq I leq r} \beta_I A_Y^{I1}.$$

This shows that there exist at most five curvature collineations Y_a of $\mathcal{K}(U, g_{|U})$ giving independent functions $\mathcal{R}_Y$, etc. - in the sense that, given any other curvature collineation Y, the functions $\mathcal{R}_Y, \mathcal{L}_Y, \mathcal{S}_Y, \mathcal{F}_Y$ and $\mathcal{A}_Y$, that it defines, are functional linear combinations of the functions defined by the Y_a. In other words, defining

$$Y_a = \sum_{1\leq I\leq r} \Phi_a^I X_I, \tag{5.4.13}$$

given any other $Y \epsilon \mathcal{K}(U, g_{|U})$, there exist unique differentiable functions a^a on U such that:

$$A^I_{Y;b} = \sum_{1\leq\leq m} a^a \Phi^I_{a;b}. \tag{5.4.14}$$

(where m is the maximum number of Y_a in the above sense). Taking covariant derivatives and antisymmetrising the above relation we get:

$$\sum_{1\leq a\leq m} (a^a{}_{;p}\Phi^I_{a;k} - a^a{}_{;k}\Phi^I_{a;p}) = 0.$$

Contracting this relation with $u^p u^k, ..., y^p y^k$, taking each of the relations thus obtained, multiplying it, in turn, by $\alpha_I, \beta_I, \gamma_I, \delta_I$ and then summing over I we get the following system, where we have set $\mathcal{R}_Y = \mathcal{R}_b$, etc. :

$$\sum_b a^{bk}\mathcal{C}_b = 0,$$

whenever $k = 3, 4$ and $\mathcal{C}_{\mathsf{b}} = \mathcal{L}_{\mathsf{b}}, \mathcal{R}_{\mathsf{b}}, \mathcal{S}_{\mathsf{b}}, \mathcal{A}_{\mathsf{b}}$ or $\mathcal{F}_{\mathsf{b}}$, and

$$\begin{aligned} \sum_{\mathsf{b}} (a^{\mathsf{b}1}(2\mathcal{L}_{\mathsf{b}} - \mathcal{A}_{\mathsf{b}}) - a^{\mathsf{b}2}\mathcal{R}_{\mathsf{b}}) &= 0, \\ \sum_{\mathsf{b}} (a^{\mathsf{b}1}\mathcal{S}_{\mathsf{b}} - a^{\mathsf{b}2}\mathcal{A}_{\mathsf{b}}) &= 0, \\ \sum_{\mathsf{b}} a^{\mathsf{b}1}\mathcal{F}_{\mathsf{b}} &= 0, \\ \sum_{\mathsf{b}} a^{\mathsf{b}2}\mathcal{F}_{\mathsf{b}} &= 0. \end{aligned} \tag{5.4.15}$$

These relations show that $a^{\mathsf{b}3} = a^{\mathsf{b}4} = 0$ for $1 \leq I \leq m$, that is

$$x^c a^{\mathsf{b}}_{;c} = y^c a^{\mathsf{b}}_{;c} = 0. \tag{5.4.16}$$

An immediate conclusion is

THEOREM 5.4.5. *If there is no hypersurface orthogonal vector field in the distribution $span(u, v)$, $\mathcal{K}(U, g_{|U})$ is finite dimensional and its dimension is bounded by 9.*

Proof. In fact each $a^{\mathsf{b}}_{;c}$ lies in $span(u, v)$. The above condition implies that $a^{\mathsf{b}}_{;c} = 0$ so that the a^{b} are constants. (5.4.6) shows then that the A^{I}_{Y} satisfy

$$A^{I}_{Y} = \sum_{\mathsf{b}} a^{\mathsf{b}} \Phi^{I}_{\mathsf{b}} + \sigma^{I},$$

where the σ^{I} are constants ■

NOTE.5.4.1. The fact that $span(u, v)$ is integrable does not imply by itself that this distribution contains hypersurface orthogonal vector fields. In fact it may contain none or just one (in the sense that every other one is parallel to this one). As examples of these facts take an open connected subset U of $\mathbb{R}^4$ with coordinates (r, s, t, w) and a let A, B be differentiable functions in U which depend only on s and r respectively; assume furthermore that $B^2 < 1$. Then the metric g given on U by

$$g = \begin{pmatrix} -1 & 0 & 0 & A \\ 0 & 1 & B & 0 \\ 0 & B & 1 & 0 \\ A & 0 & 0 & 1 \end{pmatrix},$$

has signature $(3, 1)$. When $A(s) = s^2$ and $B(r) = r^2$ one can prove that there are no hypersurface orthogonal vector fields in the distribution $span(\partial_r, \partial_s)$. When

$A(s) = s^2$ and $B(r) = r^2$ one can prove that there are no hypersurface orthogonal vector fields in the distribution $span(\partial_r, \partial_s)$. When $A(s) = s^2$ and $B = 0$ one can prove that all hypersurface orthogonal vector fields which belong to $span(\partial_r, \partial_s)$ are parallel to ∂_s.

In fact, as it was pointed out to me by my supervisor, Dr. G. S. Hall, there exists more than one independent hypersurface orthogonal vector field in $span(u, v)$ if and only if the distribution defined by the orthogonal complement of $span(u, v)$ is itself integrable.

Define now for $1 \leq I \leq r$ (the X_I being the elements of the spanning family),

$$Z_I = X_I - \alpha_I u - \beta_I v. \tag{5.4.17}$$

Then

Lemma 5.4.6. *Assume the Z_I independent. If for every $p \in U$ the subspace of $T_p(M)$ spanned by the Z_I at p has dimension r, $\mathcal{K}(U, g_{|U})$ is finite-dimensional and* $\dim \mathcal{K}(U, g_{|U}) = r + m \leq 3$.

Proof. If $Y \in \mathcal{K}(U, g_{|U})$ and $Z = Y - \alpha^Y u - \beta^Y v$, then we have

$$Z = \sum_{1 \leq I \leq r} A^I_Y Z_I.$$

This gives

$$[u, Z] = \sum_I (\mathcal{L}_u A^I_Y) Z_I + \sum_I A^I_Y [u, Z_I].$$

As $[u, Z]$ and the $[u, Z_I]$ lie in the (u, v)-plane (cf. T.5.4.3.(b)) and as the Z_I are independent and orthogonal to this plane, this proves that

$$\mathcal{L}_u A^I_Y = 0. \tag{5.4.18}$$

for all I. Similarly, we have

$$\mathcal{L}_v A^I_Y = 0. \tag{5.4.19}$$

This shows that $A^{I1}_Y = A^{I2}_Y = 0$ for all $Y \in \mathcal{K}(U, g_{|U})$ and all I, and so that for all Y we have $\mathcal{R}_Y = \mathcal{A}_Y = \mathcal{S}_Y = \mathcal{L}_Y = 0$ (cf. (5.4.12)). It follows that in (5.4.12) the maximum number of independent functions is 1. If this number is 0 then $\mathcal{F}_Y = 0$ and so (5.4.14) shows that the A^I_Y are constants. Since the maximum for r is in this case 2, we see that $\mathcal{K}(U, g_{|U})$ has at most dimension 2 (notice that r is the dimension of the subspace spanned by the Z_I and the maximum dimension this subspace can have is 2. If the maximum number of independent functions is 1 then we can, for every $p \in U$ find $Y_o \in \mathcal{K}(U, g_{|U})$ such that $\mathcal{F}_Y(p) \neq 0$. We apply then the same type of argument as in the preceding section ■

The case when $span(u, v)$ contains hypersurface orthogonal vector fields.

Here we can assume that the family Z_l has at most rank $r - 1$ (the case when this rank is r has been dealt with in the preceding lemma). Using then arguments in all aspects similar to those used in the study of the hypersurface orthogonal case in the last section it is not difficult to prove that

THEOREM 5.4.7. *Let (M, g) be a class III(a) space-time $span(u, v)$ be the distribution defined at the beginning of this section. Let U be a nice open set of M. Then $\mathcal{K}(U, g_{|U})$ is infinite dimensional if and only if $span(u, v)$ contains at least a hypersurface orthogonal vector field and a curvature collineation .*

This case contains in particular the case when one (or both) of u, v is covariantly constant.

When both u, v are covariantly constant one proves quite easily that every curvature collineation X is locally a vector field projectable with respect to the decomposition of M defined by u and v (cf. Ch.4) and the component Z of X orthogonal to $span(u, v)$ is a curvature collineation of the leaves of the decomposition orthogonal to $span(u, v)$. The results of §.5.2. tell us then that Z is in fact a conformal vector field of these manifolds. Thus, we have the following result, due to Hall

THEOREM 5.4.8.*Let (M, g) be a strictly 1+1+2-decomposable space-time, u, v be the covariantly constant vector fields of M. Then every curvature collineation X of M is locally of the form*

$$X = \alpha u + \beta v + Z,$$

where the functions α and β have their gradients tangent to $span(u, v)$ and Z is a curvature collineation of the integral manifolds of the distribution orthogonal to $span(u, v)$.

Proof. To see this take a point p in M and choose a sufficiently small open neighbourhood U of p such that $(U, g_{|U})$ is isometric to $(I \times J \times V, i \oplus j \oplus f)$, where I, J are open intervals of the real line and V is a 2-dimensional manifold, i, j, f their respective metrics. We choose coordinates (r, s, t, w) in U and i, j with the appropriate signature so as to identify the integral curves of u (resp. v) with the manifolds I (resp. J).

Now if $X = \alpha u + \beta v + Z$ is a curvature collineation, then we have, from (5.1.2) and (5.3.1)

$$(\alpha u^m + \beta v^m + Z^m)\mathbf{R}^a{}_{bcd;m} + Z^m{}_{;b}\mathbf{R}^a{}_{mcd}$$
$$+Z^m{}_{;c}\mathbf{R}^a{}_{bmd} + Z^m{}_{;d}\mathbf{R}^a{}_{bcm} - (\alpha_m u^a + \beta_m v^a + Z^a{}_{;m})\mathbf{R}^m{}_{bcd} = 0. \qquad (*)$$

Contracting this relation with u_a we get

$$\alpha_m \mathbf{R}^m{}_{bcd} = 0.$$

Similarly, contracting with v_a we get

$$\beta_m \mathbf{R}^m{}_{bcd} = 0.$$

These relations show that the gradients of α and β lie in the (u, v) plane. On the other hand we have from T.5.4.3.(b), as u and v are covariantly constant (see also L.5.4.1)

$$[u, Z] = (b - u^a\beta_a)v.$$

Now, we have

$$X_{a;b} = \alpha_b u_a + \beta_a v_b + Z_{a;b},$$

and (T.5.4.2.(a))

$$X_{a;b} = \psi g_{ab} + \rho u_a u_b + 2\lambda(u_{(a}v_{b)}) + \mu v_a v_b + f_{ab}.$$

Contracting both expressions with $v^a u^b$ we get

$$u^b \beta_b = \lambda + \epsilon f_{ab} v^a u^b.$$

From T.5.4.2.(c) we deduce then that $b - u^a\beta_a = 0$, that is $[u, Z] = 0$. Similarly, we get $[v, Z] = 0$. This shows that X is projectable.

Since $\mathbf{R}^a{}_{bcd}u^d = 0$, we get, taking covariant derivatives

$$\mathbf{R}^a{}_{bcd;e}u^d = 0.$$

Using then the Bianchi identities this shows that we have

$$\mathbf{R}^a{}_{bcd;e}u^e = 0.$$

Using these results, we see that in $(*)$ the only terms that remain are those with Z and this shows then (5.1.2) holds for Z. As Z can be considered as a vector field in the manifold V and as $\mathbf{R}$ can be considered as the Riemann tensor of (V, f)

we deduce that Z is a curvature collineation of this last manifold. We can then apply the results of section §.5.2. ■

EXAMPLES. Given a space-time (M, g), a special conformal vector field of M is a conformal vector field X whose conformal scalar ϕ (cf. §.1.9) satisfies

$$\phi_{;ab} = 0.$$

These vector fields appear in Katzin et al. [52]. It has been remarked by Katzin et al. [52], Hall [37] and McIntosh [63] that such vector fields are curvature collineations.

Recently some attention has been paid to this type of vector fields by Coley and Tupper [10], Hall [37] and Carot [8]. Hall, in particular, has made a thorough analysis in the cases when $\phi_{;a}$ is non-null. Noticing that $\phi_{;a}$ defines a covariantly constant vector field we see that such space-times are decomposable. Define then a new vector field ψ_a on M by setting

$$\psi_a = F_{ba}\phi^b,$$

where $F_{ab} = (X_{a;b} - X_{b;a})/2$ is the bivector of the special conformal vector field X. Hall [37] has proved that this new vector field is in fact a global gradient, is not parallel to ϕ_a and that it satisfies

$$\mathbf{R}^a{}_{bcd}\psi^d = 0.$$

This together with the fact that ϕ_a is covariantly constant shows that if a space-time admits a special conformal vector field which is not a homothety, then (M, g) is a class III space-time (it should be noticed that $\phi_{;a}$ is the unique covariantly constant 1-form on M (up to constant multiples). A complete analysis of such space-times can be found in [37]. A complete description of the metrics of this type which are physically significant (in the sense that they satisfy the energy conditions of Ch.2) can be found in [10], [8].

Thus, the work of the above authors provides us with examples of class III(a) space-times which admit proper curvature collineations.

As an example, take an open subset U of $\mathbb{R}^4$ with coordinates (t, x, y, z). Consider the metric on U given in these coordinates by

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 f(y,z) \end{pmatrix},$$

where f is a differentiable function of y, z only. A simple computation shows that the only non-identically zero component of the covariant Riemann tensor in these coordinates is

$$R_{3434} = -\frac{x^2(2f\partial_y^2 f - (\partial_y f)^2 + 4f^2)}{4f}.$$

Thus (U, g) is a class III(a) space-time. This example is due to Coley and Tupper [10]. They showed that the vector field given by

$$X = \frac{-x^2 - t^2}{2}\partial_t - tx\partial_x,$$

is a special conformal vector field.

Using the results and examples of §.5.2 and the remarks in the preceding theorem we can construct examples in the case when both u and v are covariantly constant. For instance if M is an open connected subset of $\mathbb{R}^4$ with coordinates (x, y, s, t), the metric defined in M by

$$g = \begin{pmatrix} e^{2\phi}(cx+d)^{\frac{1}{c^2}} & 0 & 0 & 0 \\ 0 & e^{2\phi}(cx+d)^{\frac{1}{c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

is a "1+1+2"-decomposable metric ($span(u, v) = span(\partial_s, \partial_t)$ is timelike in this case). The vector fields X_1, X_2 and X_3 defined in EXAMPLES.I of §.5.2 together with the vector fields

$$Y = f(s,t)\partial_s + h(s,t)\partial_t,$$

wher f, h are arbitrary functions of s, t in M span the Lie algebra $\mathcal{K}(M, g)$ (in the sense that every other curvature collineation is a linear combination of the above ones with coefficients in the ring $\mathcal{F}(M)$).

5.5.The null cases.

This last section is concerned with the cases when the distribution spanned by the vector fields that contract the Riemann tensor to zero is null. Mainly because in this case there is no well defined projection operator over this distribution, the methods of the preceding sections do not apply and general results seem therefore much harder to obtain.

Several authors have analysed in the past these cases in some special situations. We mention now some of their results.

Collinson [12] in his analysis of the problem of the existence of proper curvature collineations in empty space-times proved that these can only exist when the space-time in question is of Petrov type N. As for vacuum the Riemann and the Weyl tensors coincide, such space-times belong to class II(b), since there exists a null vector field l such that

$$\mathbf{R}^{a}{}_{bcd}l^{d} = 0.$$

Aichelburg [1], [2] studied the problem for generalised pp-waves. A complete list of all curvature collineations for several special classes of such space-times can be found in [1]. In these cases the null vector field l is covariantly constant and it follows from the work of Aichelburg that $\mathcal{K}(M, g)$ is infinite dimensional.

Class II(b) empty space-times of the Robinson-Trautmann class have been analysed by Collinson and Vaz [14], [78]. Under the assumption that the space-times in consideration admitted at least one curvature collineation, these authors obtained canonical forms for the functions characterizing such space-times. In particular thay showed that in some particular cases [78] the Lie algebra $\mathcal{K}(M, g)$ can be finite-dimensional (see also [77]).

Vaz and Collinson [78] and Hall [35] have also shown that type-N empty space-times with twisting geodesic rays do not admit proper curvature collineations.

The general study of curvature collineations of non-expanding and twist free vacuum type N metrics has been made by Halford et al [27]. These authors have shown that for some cases in this class the Lie algebra $\mathcal{K}(M, g)$ can be infinite-dimensional.

APPENDIX TO CHAPTER 5.

I. Proof of L.5.3.12

From (a) we have:

$$\sum_{1\leq I\leq(r-s)} (\alpha_{I;p}A^{I}_{Y;m} - \alpha_{I;m}A^{I}_{Y;p}) = 0;$$

Contracting this with u^m we get

$$\sum_{1\leq I\leq(r-s)} A^{I1}_{Y}\alpha_{I;p} = 0.$$

Contracting successively with v^p, x^p and y^p we get

$$\sum_{1\leq I\leq(r-s)} A^{I1}_{Y}\beta_{I} = 0, \qquad (I.1)$$

$$\sum_{1\leq I\leq(r-s)} A^{I1}_{Y}\delta_{I} = 0. \qquad (I.2)$$

Contracting with v^m we get

$$\sum_{1\leq I\leq(r-s)} (\epsilon A^{I2}_{Y}\alpha_{I;p} + \mu\delta_{I}A^{I}_{Y;p}) = 0.$$

Contracting successively with the remaining elements of the orthonormal basis we get

$$\sum_{1\leq I\leq(r-s)} A^{I3}_{Y}\delta_{I} = 0, \qquad (I.3)$$

$$\sum_{1\leq I\leq(r-s)} (-\epsilon\epsilon' A^{I2}_{Y}\beta_{I} + A^{I4}_{Y}\delta_{I}) = 0, \qquad (I.4)$$

$$\sum_{1\leq I\leq(r-s)} A^{I3}_{Y}\beta_{I} = 0. \qquad (I.5)$$

Now we consider relation (5.3.20)

$$\sum_{1\leq I\leq(r-s)} (A^{I}_{Y;b}X_{Ia} + A^{I}_{Y;a}X_{Ib}) = \mathcal{O}_{Y}g_{ab} + \mathcal{N}_{Y}u_a u_b.$$

Contracting this relation with u^a we get

$$\sum_{1\leq I\leq(r-s)} (\epsilon\alpha_{I}A^{I}_{Y;b} + \epsilon A^{I1}_{Y}X_{Ib}) = 2(\mathcal{O}_{Y} + \epsilon\mathcal{N}_{Y})u_b.$$

Contracting successively with u^b, v^b, x^b and y^b, we get

$$\sum_{1\leq I\leq(r-s)} A_Y^{I1}\alpha_I = \epsilon(\mathcal{O}_Y + \epsilon\mathcal{N}_Y), \tag{I.6}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I2}\alpha_I + A_Y^{I1}\beta_I) = 0, \tag{I.7}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I3}\alpha_I + A_Y^{I1}\gamma_I) = 0, \tag{I.8}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I4}\alpha_I + A_Y^{I1}\delta_I) = 0. \tag{I.9}$$

Contracting with v^a, we get

$$\sum_{1\leq I\leq(r-s)} (-\epsilon\epsilon'\beta_I A^I_{Y;b} - \epsilon\epsilon' A_Y^{I2} X_{Ib}) = 2\mathcal{O}_Y v_b.$$

Contracting successively with the remaining elements of the orthonormal basis we get

$$\sum_{1\leq I\leq(r-s)} A_Y^{I2}\beta_I = -\epsilon\epsilon'\mathcal{O}_Y, \tag{I.10}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I3}\beta_I + A_Y^{I2}\gamma_I) = 0, \tag{I.11}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I4}\beta_I + A_Y^{I2}\delta_I) = 0. \tag{I.12}$$

Contracting with x^a we get

$$\sum_{1\leq I\leq(r-s)} (-\epsilon'\gamma_I A^I_{Y;b} + \epsilon' A_Y^{I3} X_{Ib}) = 2\mathcal{O}_Y x_b.$$

Contracting with the x^b, y^b we get

$$\sum_{1\leq I\leq(r-s)} A_Y^{I3}\gamma_I = \epsilon\mathcal{O}_Y, \tag{I.13}$$

$$\sum_{1\leq I\leq(r-s)} (\epsilon' A_Y^{I4}\gamma_I + \epsilon' A_Y^{I3}\delta_I) = 0. \tag{I.14}$$

Finnaly, contracting with y^a we get

$$\sum_{1\leq I\leq(r-s)} (\delta_I A^I_{Y;b} + A_Y^{I4} X_{Ib}) = 2\mathcal{O}_Y y_b.$$

Contracting with y^b

$$\sum_{1\leq I\leq(r-s)} A_Y^{I4}\delta_I = \mathcal{O}_Y. \tag{I.15}$$

(I.1) and (I.7) show that

$$\sum_{1\le I\le (r-s)} A_Y^{I2}\alpha_I = \sum_{1\le I\le (r-s)} A_Y^{I1}\beta_I = 0.$$

Similarly, (I.2) and (I.9) show that

$$\sum_{1\le I\le (r-s)} A_Y^{I4}\alpha_I = \sum_{1\le I\le (r-s)} A_Y^{I1}\delta_I = 0.$$

Considering (I.3) and (I.14) and then (I.5) and (I.11) we prove that

$$\sum_{1\le I\le (r-s)} A_Y^{I3}\delta_I = \sum_{1\le I\le (r-s)} A_Y^{I4}\gamma_I = \sum_{1\le I\le (r-s)} A_Y^{I3}\beta_I = \sum_{1\le I\le (r-s)} A_Y^{I2}\gamma_I = 0$$

(I.10), (I.15) and (I.4) show that we have $\mathcal{O}_Y = 0$. This leads then to

$$\sum_{1\le I\le (r-s)} (A_Y^{I3}\alpha_I + A_Y^{I1}\gamma_I) = \sum_{1\le I\le (r-s)} A_Y^{I3}\gamma_I = \sum_{1\le I\le (r-s)} A_Y^{I4}\delta_I = \sum_{1\le I\le (r-s)} A_Y^{I2}\beta_I = 0$$

Finnaly, using (5.3.18) we get

$$\sum_{1\le I\le (r-s)} A_Y^{I1}\alpha_I = \sum_{1\le I\le (r-s)} A_Y^{I3}\alpha_I = \sum_{1\le I\le (r-s)} A_Y^{I1}\gamma_I = 0,$$

and this proves the lemma.

II.Proof of L.5.3.14

Keeping the notation of the text, we have

$$X_I = \tau_I n + \zeta_I l + \alpha_I u + \gamma_I v,$$

and

$$A^I_{;b} = A_Y^{I2} n + A_Y^{I1} l + A_Y^{I3} u + A_Y^{I4} v.$$

Now, if $Y \in \mathcal{S}(\mathcal{U})$, we have, with $Y = \sum_{1\le I\le (r-s)} A^I_Y X_I$, and using the general form for h^Y

$$\sum_{1\le I\le (r-s)} (A^I_{Y;b} X_{Ia} + A^I_{Y;a} X_{Ib}) = 2\mathcal{O}_Y g_{ab} + 2\mathcal{N}_Y u_a u_b. \qquad (*)$$

Contracting this relation with u^a we get

$$\sum_{1\le I\le (r-s)} (A^I_{Y;b}\alpha_I + A_Y^{I3} X_{Ib}) = 2(\mathcal{O}_Y + \mathcal{N}_Y) u_b.$$

Contracting this relation in turn with u^b, v^b, l^b and n^b we get

$$\sum_{1\leq I\leq(r-s)} \alpha_I A_Y^{I3} = (\mathcal{O}_Y + \mathcal{N}_Y), \tag{II.1}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I4}\alpha_I + A_Y^{I3}\gamma_I) = 0, \tag{II.2}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I2}\alpha_I + A_Y^{I3}\tau_I) = 0, \tag{II.3}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I1}\alpha_I + A_Y^{I3}\zeta_I) = 0. \tag{II.4}$$

Contracting $(*)$ with v^a we get

$$\sum_{1\leq I\leq(r-s)} (A^I_{Y;b}\gamma_I + A_Y^{I4}X_{Ib}) = 2\mathcal{O}_Y v_b.$$

Contracting this relation in turn with v^b, l^b and n^b we get

$$\sum_{1\leq I\leq(r-s)} A_Y^{I4}\gamma_I = \mathcal{O}_Y, \tag{II.5}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I2}\gamma_I + A_Y^{I4}\tau_I) = 0, \tag{II.6}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I1}\gamma_I + A_Y^{I4}\zeta_I) = 0. \tag{II.7}$$

Contracting $(*)$ with l^a we get

$$\sum_{1\leq I\leq(r-s)} (A^I_{Y;b}\tau_I + A_Y^{I2}X_{Ib}) = 2\mathcal{O}_Y l_b.$$

Contracting this relation in turn with l^b and n^b we get

$$\sum_{1\leq I\leq(r-s)} \tau_I A_Y^{I2} = 0, \tag{II.8}$$

$$\sum_{1\leq I\leq(r-s)} (A_Y^{I1}\tau_I + A_Y^{I2}\zeta_I) = 2\mathcal{O}_Y. \tag{II.9}$$

Finally, contracting $(*)$ with $n^a n^b$, we get

$$\sum_{1\leq I\leq(r-s)} A_Y^{I1}\zeta_I = 0. \tag{II.10}$$

As in the preceding case, we have here

$$\sum_I \alpha_I A^I_{Y;m} = 0, \tag{**}$$

and this gives

$$\sum_I \alpha_I A_Y^{I1} = \sum_I \alpha_I A_Y^{I2} = \sum_I \alpha_I A_Y^{I3} = \sum_I \alpha_I A_Y^{I4} = 0, \qquad (II.11).$$

(**) gives, taking covariant derivatives and antisymmetrising

$$\sum_{1\le I\le(r-s)} (\alpha_{I;p} A^I_{Y;m} - \alpha_{I;m} A^I_{Y;p}) = 0; \qquad (***)$$

ontracting this relation in all possible ways with the elements of the null tetrad we get using (5.3.23)

$$\sum_{1\le I\le(r-s)} A_Y^{I3} \tau_I = 0, \qquad (II.12)$$

$$\sum_{1\le I\le(r-s)} A_Y^{I3} \gamma_I = 0, \qquad (II.13)$$

$$\sum_{1\le I\le(r-s)} (A_Y^{I4} \gamma_I + A_Y^{I1} \tau_I) = 0, \qquad (II.14)$$

$$\sum_{1\le I\le(r-s)} \gamma_I A_Y^{I2} = 0, \qquad (II.15)$$

Combining these equations we get the result of L.5.3.14.

III.Proof of T.5.3.15.(c).

Starting from

$$a_{;p} A_0{}^I{}_{;m} - a_{;m} A_0{}^I{}_{;p} + b_{;p} A_1{}^I{}_{;m} - b_{;m} A_1{}^I{}_{;p} = 0.$$

we get, by contraction in all possible ways with the elements of the null tetrad

$$a^3 A_0{}^{I4} - a^4 A_0{}^{I3} + b^3 A_1{}^{I4} - b^4 A_1{}^{I3} = 0, \qquad (III.1)$$

$$a^3 A_0{}^{I2} - a^2 A_0{}^{I3} + b^3 A_1{}^{I2} - b^2 A_1{}^{I3} = 0, \qquad (III.2)$$

$$a^3 A_0{}^{I1} - a^1 A_0{}^{I3} + b^3 A_1{}^{I1} - b^1 A_1{}^{I3} = 0, \qquad (III.3)$$

$$a^4 A_0{}^{I2} - a^2 A_0{}^{I4} + b^4 A_1{}^{I2} - b^2 A_1{}^{I4} = 0, \qquad (III.4)$$

$$a^4 A_0{}^{I1} - a^1 A_0{}^{I4} + b^4 A_1{}^{I1} - b^1 A_1{}^{I4} = 0, \qquad (III.5)$$

$$a^2 A_0{}^{I1} - a^1 A_0{}^{I2} + b^2 A_1{}^{I1} - b^1 A_1{}^{I2} = 0, \qquad (III.6)$$

Since $\Lambda^3(p,Y) = (0,0,0,0)$, it follows from the injectivity of ν that we have $A_0^{\ I3} = A_1^{\ I3} = 0$. Thus the first three of these relations reduce to

$$a^3 A_0^{\ I4} + b^3 A_1^{\ I4} = 0, \tag{III.7}$$

$$a^3 A_0^{\ I2} + b^3 A_1^{\ I2} = 0, \tag{III.8}$$

$$a^3 A_0^{\ I1} + b^3 A_1^{\ I1} = 0. \tag{III.9}$$

If one of a^3, b^3 is non-zero at some point of U then these relations show that in some open neighbourhood of the point in question in U $A_{0;m}$ and $A_{1;m}$ are parallel. This contradicts our assumption that we are in case (C).

Consider now relation (III.4). Multiplying it by γ_I, summing over I and using the fact that $\sum_I \gamma_I A_i^{\ I2} = 0$ (proved in the previous section), we get

$$a^2 \mathcal{O}_{Y_0} + b^2 \mathcal{O}_{Y_1} = 0.$$

Performing similar computations with the remaining relations and using again the fact that we are in case (C), we prove that a and b are constants.

This shows that

$$A_Y^{\ I} = aA_0^{\ I} + bA_1^{\ I} + \sigma^I,$$

for every I, where σ^I is a real constant. This proves the theorem.

IV.Proof of Lemma 5.3.16.

Let $Y = \sum_I A_Y^{\ I} X_I \in \mathcal{K}(U, g_{|U})$. Then we have,

$$\alpha^Y = \sum_I \alpha_I A_Y^{\ I},$$

so, using T.5.3.4.(a) and the fact that u is hypersurface orthogonal , we get

$$\sum_I \alpha_I A_Y^{I2} = \sum_I \alpha_I A_Y^{I3} = \sum_I \alpha_I A_Y^{I4} = 0. \tag{IV.1}$$

Using now the relation (5.3.20)

$$\sum_I (A_{Y;b}^{\ I} X_{Ia} + A_{Y;a}^{\ I} X_{Ib}) = 2\mathcal{O}_Y g_{ab} + 2\mathcal{N}_Y u_a u_b, \tag{$*$}$$

we get, contracting with u^b

$$\sum_I (\epsilon\alpha_I A_{Y;a}^{\ I} + \epsilon A_Y^{I1} X_{Ia}) = 2(\mathcal{O}_Y + \epsilon\mathcal{N}_Y) u_a.$$

Contracting then with u^a, v^a, x^a and y^a we get, using (IV.1)

$$\sum_{1\leq I\leq(r-s)} A_Y^{I1}\alpha_I = \epsilon\mathcal{O}_Y + \mathcal{N}_Y, \tag{IV.2}$$

$$\sum_I A_Y^{I1}\beta_I = 0, \tag{IV.3}$$

$$\sum_I A_Y^{I1}\gamma_I = 0, \tag{IV.4}$$

$$\sum_I A_Y^{I1}\delta_I = 0. \tag{IV.5}$$

Contracting with v^a we get

$$\sum_I (\beta_I A^I_{Y;b} + A_Y^{I2} X_{Ib}) = 2\mathcal{O}_Y v_b.$$

Contracting then with v^b, x^b and y^b we get

$$\sum_I A_Y^{I2}\beta_I = -\epsilon\mathcal{O}_Y, \tag{IV.6}$$

$$\sum_I (A_Y^{I3}\beta_I + A_Y^{I2}\gamma_I) = 0, \tag{IV.7}$$

$$\sum_I (A_Y^{I4}\beta_I + A_Y^{I2}\delta_I) = 0. \tag{IV.8}$$

Contracting with x^a we get

$$\sum_I (\gamma_I A^I_{Y;b} - A_Y^{I3} X_{Ib}) = 2\mathcal{O}_Y x_b.$$

Contracting with the x^b, y^b we get

$$\sum_I A_Y^{I3}\gamma_I = \epsilon\mathcal{O}_Y, \tag{IV.9}$$

$$\sum_I (A_Y^{I4}\gamma_I + A_Y^{I3}\delta_I) = 0. \tag{IV.10}$$

Finnaly, contracting with y^a we get

$$\sum_I (\delta_I A^I_{Y;b} + A_Y^{I4} X_{Ib}) = 2\mathcal{O}_Y y_b.$$

Contracting with y^b

$$\sum_I A_Y^{I4}\delta_I = \mathcal{O}_Y. \tag{IV.11}$$

These relations lead immediately to L.5.3.16.

V.Proof of the relations in p.148.

Starting from

$$\sum_{1\le b\le m}(a^{b}{}_{;p}\Phi^{I}_{b;k}-a^{b}{}_{;k}\Phi^{I}_{b;p})=0, \qquad (*)$$

we get, by contracting with u^p

$$\sum_{1\le b\le m}(\epsilon a^{b1}\Phi^{I}_{b;k}-a^{b}{}_{;k}\Phi^{I1}_{b})=0.$$

Contracting then with v^k, x^k and y^k we get

$$\sum_{1\le b\le m}(a^{b1}\Phi^{I2}_{b}-a^{b2}\Phi^{I1}_{b})=0, \qquad (V.1)$$

$$\sum_{1\le b\le m}(a^{b1}\Phi^{I3}_{b}-a^{b3}\Phi^{I1}_{b})=0, \qquad (V.2)$$

$$\sum_{1\le b\le m}(a^{b1}\Phi^{I4}_{b}-a^{b4}\Phi^{I1}_{b})=0, \qquad (V.3)$$

Contracting $(*)$ with v^p we get

$$\sum_{1\le b\le m}(-a^{b2}\Phi^{I}_{b;k}+a^{b}{}_{;k}\Phi^{I2}_{b})=0.$$

Contracting then with x^k and y^k we get

$$\sum_{1\le b\le m}(-a^{b2}\Phi^{I3}_{b}-a^{b3}\Phi^{I2}_{b})=0, \qquad (V.4)$$

$$\sum_{1\le b\le m}(-a^{b2}\Phi^{I4}_{b}-a^{b4}\Phi^{I2}_{b})=0, \qquad (V.5)$$

Contracting $(*)$ with x^p we get

$$\sum_{1\le b\le m}(a^{b3}\Phi^{I}_{b;k}-a^{b}{}_{;k}\Phi^{I3}_{b})=0.$$

Contracting then with x^k and y^k we get

$$\sum_{1\le b\le m}(a^{b3}\Phi^{I4}_{b}-a^{b4}\Phi^{I3}_{b})=0, \qquad (V.6)$$

Consider now (V.1); multiplying by α_I and summing over I we get, using (IV.1) and (IV.2)

$$\sum_{b}(\epsilon\mathcal{O}_b+\mathcal{N}_b)a^{b2}=0. \qquad (V.7)$$

Similarly, multiplying by β_I and summing over I we get, using (IV.3) and (IV.6)

$$\sum_{\mathsf{b}} \mathcal{O}_{\mathsf{b}} a^{\mathsf{b}1} = 0. \tag{V.8}$$

The remaining relations can be obtained in the same way.

VI.

From the relations in p.148 we get, in this case,

$$\sum_{\mathsf{b}} \mathcal{N}_{\mathsf{b}} a^{\mathsf{b}2} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{T}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{N}_{\mathsf{b}} a^{\mathsf{b}3} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}1} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{T}_{\mathsf{b}} a^{\mathsf{b}2} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{T}_{\mathsf{b}} a^{\mathsf{b}3} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}2} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}4} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}3} = 0,$$

$$\sum_{\mathsf{b}} \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}4} = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{V}_{\mathsf{b}} a^{\mathsf{b}3} - \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (\mathcal{T}_{\mathsf{b}} a^{\mathsf{b}4} + \mathcal{W}_{\mathsf{b}} a^{\mathsf{b}2}) = 0,$$

$$\sum_{\mathsf{b}} (-\mathcal{T}_{\mathsf{b}} a^{\mathsf{b}4} + \mathcal{V}_{\mathsf{b}} a^{\mathsf{b}3}) = 0.$$

Summing the last two we get $\sum_b(\mathcal{W}_b a^{b2} + \mathcal{V}_b a^{b3}) = 0$, and so, summing with the one before we get $\sum_b \mathcal{V}_b a^{b3} = 0$. This shows then that we have $\sum_b \mathcal{W}_b a^{b2} = \sum_b \mathcal{T}_b a^{b4} = 0$. The conclusion that $a^{b2} = a^{b3} = a^{b4} = 0$ follows then from the fact that the matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{N}_1 & . & . & . & \mathcal{N}_m \\ \mathcal{V}_1 & . & . & . & \mathcal{V}_m \\ \mathcal{T}_1 & . & . & . & \mathcal{T}_m \\ \mathcal{W}_1 & . & . & . & \mathcal{W}_m \end{pmatrix},$$

defines an injective linear map for every point of U, and the above relations show that we have $\mathcal{M}a^{b2} = \mathcal{M}a^{b3} = \mathcal{M}a^{b4} = 0$.

VII. Proof of T.5.3.18.

Consider (5.3.37)

$$\eta^I(r) = D^I(s,t,w) - \Theta^I(s,t,w)\eta^r(r) - \sum_{1\leq i\leq m} a^i(r) D^I_i(s,t,w). \qquad (VII.1)$$

Consider the family $(\eta^r, a^1, ..., a^r)$ and let m_o be its rank. Then, reordering the Y_a if necessary, one of the following situations takes place

(i). $(\eta^r, a^1, ..., a^{m_o-1})$ has rank m_o;

(ii). $(a^1, ..., a^{m_o})$ has rank m_o.

Case (i)

In this case, we can, for each $1 \leq j \leq r - m_o + 1$, find real numbers π^j, C^j_b, where $1 \leq b \leq m_o - 1$ such that

$$a^{m_o-1+j} = \pi^j \eta^r + \sum_{1\leq b\leq m_o-1} C^j_b a^b. \qquad (VII.2)$$

Replacing in (VII.1) we get after some computations

$$\eta^I = D^I - [\Theta^I + \sum_{1\leq j\leq r-m_o+1} \pi^j D^I_{m_o-1+j}]\eta^r - \sum_{1\leq i\leq m_o-1} a^i(D^I_i + \sum_{1\leq j r-m_o+1} C^j_i D^I_{m_o-1+j}). \qquad (VII.3)$$

Consider then the family $(1, \eta^r, a^1, ..., a^{m_o-1})$. If this family has rank m_o+1 it follows from (VII.3) that the functions D^I are constants. In fact in such case we can find m_o+1 values $r_1, ..., r_{m_o+1}$ for r such that $\det M(1, \eta^r, a^1, ..., a^{m_o-1}; r_1, ..., r_{m_o+1}) \neq 0$. Evaluate then (VII.3) at these points (keeping s, t, w arbitrary). We get then a system of equations whose matrix is precisely $M(1,\eta^r,a^1,...,a^{m_o-1};r_1,...,r_{m_o+1})$ and whose second member is constant.

One has then, from (5.3.34)

$$Y=\sum_{1\leq I\leq r-1}(D^I-\Theta^I A^r)X_I+A^r X_r=\sum_{1\leq I\leq r-1}D^I X_I+A^r(X_r-\sum_{1\leq I\leq r-1}\Theta^I X_I).$$

Since the D^I are constants, the above expression shows that

$$A^r(X_r-\sum_{1\leq I\leq r-1}\Theta^I X_I),$$

is a curvature collineation . Notice that this vector field is everywhere parallel to u, as follows from the definition of the Θ^I (cf. (5.3.31)).

If there is no curvature collineation parallel to u we are forced therefore to have $A^r=0$. (5.3.34) shows then that the A^I_Y are constants, and so Y is a constant linear combination of the elements of the spanning family.

The remaining possibility is that $(1,\eta^r,a^1,...,a^{m_o-1})$ has rank m_o, in which case there exist real constants τ,μ_b, $1\leq b\leq m_o-1$, such that

$$1=\tau\eta^r+\sum_{1\leq b\leq m_o-1}\mu_b a^b. \qquad (VII.4)$$

Replacing in (VII.3) we get after simplification

$$\eta^I=[\tau D^I-\Theta^I-\sum_{1\leq j\leq r-m_o+1}\pi^j D^I_{m_o-1+j}]\eta^r$$

$$-\sum_{1\leq i\leq m_o-1}a^i(\mu_i D^I-D^I_i-\sum_{1\leq j\, r-m_o+1}C^j_i D^I_{m_o-1+j}). \qquad (VII.5)$$

By the same process as in the previous case, this shows that the functions

$$F^I=\tau D^I-\Theta^I-\sum_{1\leq j\leq r-m_o+1}\pi^j D^I_{m_o-1+j}, \qquad (VII.6)$$

and

$$H^I_i=\mu_i D^I-D^I_i-\sum_{1\leq j\, r-m_o+1}C^j_i D^I_{m_o-1+j}, \qquad (VII.7)$$

are constants in U.

If the μ_b are all zero, it follows from (VII.4) that $1=\tau\eta^r$, and so $\tau\neq 0$. One has then, setting $f^I=F^I/\tau$

$$D^I=\frac{\Theta^I}{\tau}+\sum_{1\leq j\leq r-m_o+1}\frac{\pi^j}{\tau}D^I_{m_o-1+j}+f^I. \qquad (VII.8)$$

From (5.3.34) we have then

$$Y = \sum_{1\leq I\leq r-1} (D^I - \Theta^I A^r)X_I + A^r X_r =$$

$$\frac{1}{\tau}\sum_{1\leq I\leq r-1}\Theta^I X_I + \sum_{1\leq I\leq r-1} f^I X_I + \sum_{1\leq b\leq r-m_o+1}\frac{\pi^b}{\tau}\sum_{1\leq I\leq r-1} D^I_{m_o-1+j}X_I +$$

$$A^r(X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I). \qquad (VII.9)$$

Using the second relation in (5.3.34) we have

$$Y_{m_o-1+b} = \sum_{1\leq I\leq r-1} D^I_{m_o-1+b}X_I + \Phi^r_{m_o-1+b}(X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I). \qquad (VII.10)$$

Thus, replacing in (VII.9) we get after simplification

$$Y = \sum_{1\leq I\leq r-1} f^I X_I + \sum_{1\leq j\leq r-m_o+1}\frac{\pi^j}{\tau}Y_{m_o-1+j} +$$

$$\frac{1}{\tau}\sum_{1\leq I\leq r-1}\Theta^I X_I + (A^r - \sum_{1\leq j\leq r-m_o+1}\Phi^r_{m_o-1+j}\frac{\pi^j}{\tau})(X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I). \qquad (VII.11)$$

Since both the f^I and the π^j/τ are constants, the first two expressions on the right handside of this relation define curvature collineations. We deduce therefore that

$$W = \frac{1}{\tau}\sum_{1\leq I\leq r-1}\Theta^I X_I + (A^r - \sum_{1\leq j\leq r-m_o+1}\Phi^r_{m_o-1+j}\frac{\pi^j}{\tau})(X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I). \quad (VII.12)$$

is a curvature collineation. Noticing that

$$X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I$$

is parallel to u (cf. definition of the Θ^I (5.3.31)), let us write

$$X_r - \sum_{1\leq I\leq r-1}\Theta^I X_I = au.$$

Since we also have

$$\sum_{1\leq I\leq r-1}\Theta^I X_I = bu + Z_r,$$

we see that W has the form

$$W = (\frac{b}{\tau} + a(A^r - \sum_{1\leq j\leq r-m_o+1}\Phi^r_{m_o-1+j}\frac{\pi^j}{\tau}))u + \frac{1}{\tau}Z_r.$$

Since τ is constant we have then

$$W - \frac{1}{\tau}X_r = \lambda u,$$

where λ has the obvious definition. Thus, this proves that if no curvature collineations parallel to u exist, then we have

$$\frac{b}{\tau} + a(A^r - \sum_{1\leq \mathsf{j}\leq r-m_o+1} \Phi^r_{m_o-1+\mathsf{j}}\frac{\pi^{\mathsf{j}}}{\tau}) = \frac{\alpha_r}{\tau},$$

and

$$W = \frac{1}{\tau}X_r.$$

Replacing in (VII.11) we get

$$Y = \sum_{1\leq I\leq r-1} f^I X_I + \sum_{1\leq \mathsf{j}\leq r-m_o+1} \frac{\pi^{\mathsf{j}}}{\tau}Y_{m_o-1+\mathsf{j}} + \frac{1}{\tau}X_r,$$

and so Y is a linear combination of the X_I and the Y_{a} with constant coefficients.

If the μ_{b} are not all zero we can find a b_o such that $\mu_{\mathsf{b}_o} \neq 0$. It follows from (VII.7) that

$$D^I = \frac{1}{\mu_{\mathsf{b}_o}}D^I_{\mathsf{i}} + \sum_{1\leq \mathsf{j}\, r-m_o+1} C^{\mathsf{j}}_{\mathsf{b}_o}D^I_{m_o-1+\mathsf{j}} + h^I, \qquad (VII.13)$$

where the h^I are real constants given by $h^I = H^I_{\mathsf{b}_o}/\mu_{\mathsf{b}_o}$. One has then

$$Y = \sum_{1\leq I\leq r-1} (\frac{1}{\mu_{\mathsf{b}_o}}D^I_{\mathsf{i}} + \sum_{1\leq \mathsf{j}\, r-m_o+1} C^{\mathsf{j}}_{\mathsf{b}_o}D^I_{m_o-1+\mathsf{j}} + h^I)X_I + A^r au,$$

where a is defined as in the previous case. After simplification using (5.3.34),this gives

$$Y = \sum_{1\leq I\leq r-1} h^I X_I + \frac{1}{\mu_{\mathsf{b}_o}}Y_{\mathsf{b}_o} + \sum_{1\leq \mathsf{j}\leq r-m_o+1} C^{\mathsf{j}}_{\mathsf{b}_o}Y_{m_o-1+\mathsf{j}} +$$

$$[A^r - \frac{\Phi^r_{\mathsf{b}_o}}{\mu_{\mathsf{b}_o}} - \sum_{1\leq \mathsf{j}\leq r-m_o+1} \Phi^r_{m_o-1+\mathsf{j}}\frac{C^{\mathsf{j}}_{\mathsf{b}_o}}{\mu_{\mathsf{b}_o}}]au.$$

As the h^I, μ_{b_o} and the $C^{\mathsf{j}}_{\mathsf{b}_o}$ are constants, we deduce that

$$W = [A^r - \frac{\Phi^r_{\mathsf{b}_o}}{\mu_{\mathsf{b}_o}} - \sum_{1\leq \mathsf{j}\leq r-m_o+1} \Phi^r_{m_o-1+\mathsf{j}}\frac{C^{\mathsf{j}}_{\mathsf{b}_o}}{\mu_{\mathsf{b}_o}}]au,$$

is a curvature collineation. Thus, if no curvature collineations parallel to u exist the term in brackets is identically zero, and so we have

$$Y = \sum_{1\leq I\leq r-1} h^I X_I + \frac{1}{\mu_{\mathsf{b}_o}}Y_{\mathsf{b}_o} + \sum_{1\leq \mathsf{j}\leq r-m_o+1} C^{\mathsf{j}}_{\mathsf{b}_o}Y_{m_o-1+\mathsf{j}},$$

thus proving that Y is a linear combinations of the X_I and the Y_{b} with constant coefficients.

Case (ii)

In this case we can find constants π_{i}, $1 \leq \mathrm{i} \leq m_o$, and $C_{\mathrm{i}}^{\mathrm{j}}$, $1 \leq \mathrm{j} \leq r - m_o$, $1 \leq \mathrm{i} \leq m$ such that

$$\eta^r = \sum_{1 \leq \mathrm{i} \leq m_o} \pi_{\mathrm{i}} a^{\mathrm{i}},$$

and

$$a^{m_o+\mathrm{j}} = \sum_{1 \leq \mathrm{i} \leq m_o} C_{\mathrm{i}}^{\mathrm{j}} a^{\mathrm{i}}.$$

Replacing in (5.3.37) we get, after simplification

$$\eta^I = D^I - \sum_{1 \leq \mathrm{i} \leq m_o} [\pi_{\mathrm{i}} \Theta^I - D_{\mathrm{i}}^I - \sum_{1 \leq \mathrm{j} \leq r - m_o} C_{\mathrm{i}}^{\mathrm{j}} D_{m_o+\mathrm{j}}^I] a^{\mathrm{i}}. \tag{VII.14}$$

If the family $(1, a^1, ..., a^{m_o})$ has rank $m_o + 1$ we conclude that the D^I are constants and this allows us to prove, as in the preceding case, that Y is a linear combinations of the X_I and the Y_{b} with constant coefficients.

Otherwise the above family has rank m_o, and so there exist constants μ_{i} such that

$$1 = \sum_{1 \leq \mathrm{i} \leq m_o} \mu_{\mathrm{i}} a^{\mathrm{i}}.$$

Replacing in (VII.14), we get

$$\eta^I = \sum_{1 \leq \mathrm{i} \leq m_o} a^{\mathrm{i}} [\mu_{\mathrm{i}} D^I - \pi_{\mathrm{i}} \Theta^I + D_{\mathrm{i}}^I + \sum_{1 \leq \mathrm{j} \leq r - m_o} C_{\mathrm{i}}^{\mathrm{j}} D_{m_o+\mathrm{j}}^I],$$

and this shows that the function in brackets is constant for every i, I. Since one at least of the μ_{i} is non-zero, taking it to be μ_{b_o} we have

$$D^I = \frac{\pi_{\mathrm{b}_o}}{\mu_{\mathrm{b}_o}} \Theta^I - \frac{1}{\mu_{\mathrm{b}_o}} D_{\mathrm{b}_o}^I - \sum_{1 \leq \mathrm{j} \leq r - m_o} \frac{C_{\mathrm{b}_o}^{\mathrm{j}}}{\mu_{\mathrm{b}_o}} D_{m-o+\mathrm{j}}^I + c^I,$$

where the c^I are real constants, $1 \leq I \leq r - 1$. This gives, using again (5.3.32)

$$Y = \sum_{1 \leq I \leq r-1} c^I X_I - \frac{1}{\mu_{\mathrm{b}_o}} Y_{\mathrm{b}_o} - \sum_{1 \leq \mathrm{j} \leq r - m_o} \frac{C_{\mathrm{b}_o}^{\mathrm{j}}}{\mu_{\mathrm{b}_o}} Y_{m_o+\mathrm{j}} + W,$$

where

$$W = [a(A^r - \frac{1}{\mu_{\mathrm{b}_o}} \Phi_{\mathrm{b}_o}^r - \sum_{1 \leq \mathrm{j} \leq r - m_o} \Phi_{m_o+\mathrm{j}}^r) + \frac{b \pi_{\mathrm{b}_o}}{\mu_{\mathrm{b}_o}}] u + \frac{\pi_{\mathrm{b}_o}}{\mu_{\mathrm{b}_o}} Z_r,$$

where a, b are defined as in the preceding case.

Since the c^I, μ_{b_o} and the $C^j_{b_o}/\mu_{b_o}$ are constants we deduce that W is a curvature collineation. Since

$$W - \frac{\pi_{b_o}}{\mu_{b_o}} X_r$$

is parallel to u we deduce that, as no curvature collineations parallel to u exist,

$$W = \frac{\pi_{b_o}}{\mu_{b_o}} X_r,$$

and so, after simplification, we get

$$Y = \sum_{1 \leq I \leq r-1} c^I X_I + \frac{\pi_{b_o}}{\mu_{b_o}} X_r - \frac{1}{\mu_{b_o}} Y_{b_o} - \sum_{1 \leq j \leq r-m_o} \frac{C^j_{b_o}}{\mu_{b_o}} Y_{m_o+j},$$

and so Y is again a linear combination of the X_I and the Y_b with constant coefficients. This proves T.5.3.18.

VIII.Lemma 5.3.19.

From (5.3.40) we get

$$R_{ab} = (A + \epsilon' B) v_a v_b + 2\epsilon' C v_{(a} x_{b)} + \epsilon'(D - \epsilon A) x_a x_b + (D - \epsilon\epsilon' B) y_a y_b.$$

Write

$$f_{ab} = 2p u_{[a} v_{b]} + 2q u_{[a} x_{b]} + 2r u_{[a} y_{b]} + 2s v_{[a} x_{b]} + 2t v_{[a} y_{b]} + 2w x_{[a} y_{b]}.$$

Then contracting both the above expressions with $y^a y^b$ and using (5.3.39), we get

$$2\psi^W(p_o)(D - \epsilon\epsilon' B)(p_o) = 0.$$

As $\psi^W(p_o) \neq 0$ and the point p_o is arbitrary, this gives

$$D - \epsilon\epsilon' B = 0.$$

Contracting (5.3.39) with $v^a v^b$ and with $x^a x^b$, we can prove, after some simplifications that

$$\psi^W(p_o) = 0,$$

and this gives

$$A = 0,$$

in U. We can then go back to (5.3.40); it gives

$$\mathbf{R}_{abcd} = BH_{ab}H_{cd} + C(H_{ab}L_{cd} + H_{cd}L_{ab}) + \epsilon\epsilon' BL_{ab}L_{cd}.$$

Contracting this relation with v^d and with x^d, we get

$$\mathbf{R}_{abcd}v^d = \epsilon\epsilon' BH_{ab}y_c + \epsilon\epsilon' CL_{ab}y_c,$$

and

$$\mathbf{R}_{abcd}x^d = -CH_{ab}y_c - \epsilon\epsilon' BL_{ab}y_c.$$

If $(B^2 + \epsilon\epsilon' C^2)(p_o) \neq 0$, then the above relations show that if (a, b) is a solutions of $\epsilon\epsilon' aB - bC = 0$ and $aC - bB = 0$, then

$$R_{abcd}(av^d + bx^d) = 0.$$

In such case, again because p_o is arbitrary, we see that the Riemann tensor has at most rank 1 in some open subset of U. This contradicts our assumptions. Thus we must have $B^2 + \epsilon\epsilon' C^2 = 0$ in U. If $\epsilon\epsilon' = 1$, this gives $B = C = 0$ and so the Riemann tensor zero in some open subset of U.

Otherwise, $\epsilon\epsilon' = -1$, and so $C = \pm B$. In this case, contracting (5.3.39) with $v^a x^b + v^b x^a$, we get, using the above relations

$$2sB + 2\psi^W C = 0,$$

at p_o. Contracting the same relation with $v^a v^b$ we get

$$-2sC + 2\psi^W B = 0,$$

at p_o. Since $\psi^W(p_o) \neq 0$ these relations show that the Riemann tensor is again zero at p_o. As p_o is arbitrary, this contradicts our initial assumptions.

BIBLIOGRAPHY

[1].AICHELBURG P. - JMP **11**(8), (1970), 2458.

[2].AICHELBURG P. - GRG **3**(4), (1972), 397.

[3].ALEKSEEVSKI D. - Ann. Global Anal. Geom. **3**, (1985), 59.

[4].ARNOLD V. -*Equations Differentielles Ordinaires* - MIR , Moscow, (1976).

[5].BEEM J. K. - Letters in Math.Phys. **2**, (1978), 317.

[6].BEL L. - Cahiers de Physique, No.138.

[7].BRICKELL F. & CLARK R.-*Differentiable Manifolds - An Introduction* Van Nostrand, London, (1970).

[8].CAROT J. J. - Preprint, U. of Aberdeen, (1988).

[9].CHURCHILL, R. - Trans. Am. Math. Soc. **34**, (1932), 784.

[10].COLEY A. A. & TUPPER, B.O.J. - Preprint, U. of Halifax, (1988).

[11].COLLINSON C.D. - Bonnor Festschrift, (1984).

[12].COLLINSON C.D. - JMP **11**(3), (1970), 818.

[13].COLLINSON C.D. & SMITH - GRG .... (1986),

[14].COLLINSON C.D. & VAZ E.G.L.R. - GRG **14**(1), (1982), 5.

[15].DIEUDONNE, J. - *Sur les Groupes Classiques* - Hermann, Paris, (1981).

[16].DIEUDONNE, J. - *Treatise on Analysis, vol. III* - Pure and App. Math., vol.10-III, Academic Press, New York, (1972).

[17].DIEUDONNE, J. - *Treatise on Analysis, vol. IV* - Pure and App. Math., vol.10-IV, Academic Press, New York,(1974).

[18].DIEUDONNE, J. - *Treatise on Analysis, vol.V* - Pure and App. Math., vol.10-V, Academic Press, New York, (1978).

[19].EHLERS J. & KUNDT A. - in *Gravitation: An Introduction to Current Research* - ed L. Witten, Wiley, New York, (1962).

[20].FINKBEINER, D. - *Introduction to Matrices and Linear Transformations* Freeman, S. Francisco, (1978).

[21].FLANDERS, -*Differential forms with applications to the physical sciences*- New York, Academic P., (1963).

[22].GEHENIAU, J. - C.R.Ac.Sci. de Paris **244**, (1957), 723.

[23].GEROCH, R. - JMP **9**, (1968), 1739.

[24].GOLDBERG J. & KERR R. - JMP **2**(2), (1961), 327.

[25].HALFORD W.D.- JMP **21**(1), (1980), 129.

[26].HALFORD W.D. & KERR R. - JMP **21**(1), (1980), 120.

[27].HALFORD W.D., McINTOSH C.B.G. & VAN LEEUWEN E.H. - J.Phys. A: Math Gen **13**, (1980), 2995.

[28].HALL G.S. - J. Phys. A: Math. Gen. **9**(4), (1976), 541.

[29].HALL G.S. - in Differential Geometry, Banach Center Publications, vol.12, PWN, Warsaw, (1984).

[30].HALL G.S. - Acta Phys. Pol., vol.**B17** No.2, (1986), 139.

[31].HALL G.S. - GRG **14**(11), (1982), 1035.

[32].HALL G.S. - Class. Quant.Grav **6**, (1989), 157.

[33].HALL G.S. - Preprint, U. of Aberdeen, (1989).

[34].HALL G.S. - Class. Quant. Grav. **1**, (1984), 545.

[35].HALL G.S. - GRG **15**(6), (1983), 581.

[36].HALL G.S. - J.Phys.A.-Math.Nucl.Gen. **7**(4), (1974), L42.

[37].HALL G.S. - Preprint, U. of Aberdeen, (1989).

[38].HALL G.S. - GRG **20**(7), (1988), 671.

[39].HALL G.S. - Class.Quant. Grav. **5**, (1988), L77.

[40].HALL G.S. - GRG **20**(4), (1988), 399.

[41].HALL G.S. - Arabian J. Sci.Eng. **9**(2), (1984), 87.

[42].HALL G.S. - *Symmetries in General Relativity* - in *Relativity Today* ed. Z. Perjés, World Sci., Singapore (1988).

[43].HALL G.S. & da COSTA - JMP **29**(11), (1988), 2465.

[44].HALL G.S. & KAY W. - JMP **29**(2), (1988), 428.

[45].HALL G.S. & McINTOSH C.B.G. - Int. J. Theor. Phys. **22**(5), (1983), 469.

[46].HALL G.S. & NEGM D. - Int.J.Theor.Phys. **25**(4), (1986), 405.

[47].HALL G.S. & RENDALL A.D. - Int. J. Theor. Phys. **28**(3), (1989), 365.

[48].HALL G.S. & STEELE J.D. - Preprint, Un. of Aberdeen, (1989).

[49].HAWKING S. & ELLIS G. - *The Large Scale Structure of Space-Time*- Cambridge U.P., Cambridge, (1973).

[50].HELGASON S.- *Differential Geometry, Lie groups and Symmetric Spaces*- Pure and Applied Mathematics, vol.80, Academic Press, New York, (1978).

[51].IHRIG E. - JMP **16**(1), (1975), 54.

[52].KATZIN G. H., LEVINE J. & DAVIS W. R. - JMP **10**(4), (1969), 617.

[53].KATZIN G.H., LEVINE J. & DAVIS W.R. - Tensor **21**, (1970), 51.

[54].KATZIN G.H., LEVINE J. & DAVIS W.R. - JMP **11**(5), (1970), 1578.

[55].KOBAYASHI S. & NOMIZU K.- *Foundations of Differential Geometry* - Interscience, New York, (1963).

[56].KOBAYASHI S. & NOMIZU K. - *Foundations of Differential Geometry* - vol.II, Interscience, New York, (1963).

[57].KRAMER D., STEPHANI H., MacCALLUM M. & HERLT E. -*Exact solutions of Einstein's Field Equations*- Cambridge U.P., (1980).

[58].LICHNEROWICZ A. - *Global Theory of Connections and Holonomy Groups.* - Noordhoff Int. Pub., Leyden, (1976).

[59].LUDWIG G. & SCANLON A. - Comm.Math.Phys. **20**, (1971), 291.

[60].MACLANE S. & BIRKHOFF G. - *Algebra* - MacMillan, New York, (1967).

[61].McINTOSH C.B.G. & HALFORD W.D. - J. Phys. A: Math. Gen. **14**, (1981), 2331.

[62].McINTOSH C.B.G. - Class.Quant. Grav. **2**, (1985), 87.

[63].McINTOSH C.B.G. - in *Gravitational Radiation, Collapsed Objects and Exact Solutions, Proceedings, Perth (1979)*, Lecture Notes in Physics, No.124, 469, Springer-Verlag, New York.

[64].McINTOSH C.B.G. - GRG **7**(2), (1976), 199.

[65].NOMIZU, K. - Ann.Math. **72**(1), (1960), 105.

[66].PALAIS R. - Memoirs of the Am. Math. Soc. No.22, (1957).

[67].PENROSE, R.- in *Gravitation: Problems and Prospects*, dedicated to the memory of A. Z. Petrov, Izdat "Naukova Dumka", Kiev, (1972).

[68].PETROV, A. Z. -*Einstein Spaces*- Pergamon, London, (1969).

[69].PLEBANSKI, J. - Acta Phys. Pol. **26**, (1964), 963.

[70].SCHELL, J. - JMP **2**(2), (1961), 202.

[71].SCHMIDT, B. - Comm. Maths. Phys. **29**, (1973), 55.

[72].SHAW A.B. - Quart. J. Math. Oxford **21**, (1970), 101.

[73].STEELE J.D. - PhD Thesis, U.of Aberdeen, (1989).

[74].STEPHANI H. - *General Relativity: An Introduction to the Theory of the Gravitational Field*- Cambridge, CUP, (1982).

[75].SYNGE J. - Comm. of the Dublin Inst. for Adv. Studies, Series A, No.15, (1964).

[76].TELLO-LLANOS R.A. - GRG **20**(8), (1988), 765.

[77].VAZ E.G.L.R. - GRG **18**(11), (1986), 1187.

[78].VAZ E.G.L.R. & COLLINSON C.D. - GRG **15**(7), (1983), 661.

[79].WILMORE T.J. - *An Introduction to Differential Geometry* - Clarendon Press, Oxford, (1959).

[80].WU H. - Illinois J. Math. **8**, (1964), 291.

[81].YANO K. - *The Theory of Lie Derivatives and its Applications* - Amsterdam, North-Holland, (1955).