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SHEAVES, LOCAL HOMEOMORPHISMS AND LOCAL SETS AS HILBERT LOCALE MODULES

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SHEAVES, LOCAL HOMEOMORPHISMS AND LOCAL SETS AS HILBERT LOCALE MODULES

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Abstract

We give a thorough account of the various equivalent notions for “sheaf” on a locale, namely the separated and complete presheaves, the local homeomorphisms, and the local sets, and to provide a new approach based on quantale modules whereby we see that sheaves can be identified with certain Hilbert modules in the sense of Paseka. This formulation provides us with an interesting category that has immediate meaningful relations to those of sheaves, local homeomorphisms and local sets.

The concept of B-set (local set over the locale B) present in [3] is seen as a simetric idempotent matrix with entries on B, and a map of B-sets as defined in [8] is shown to be also a matrix satisfying some conditions. This gives us useful tools that permit the algebraic manipulation of B-sets.

The main result is to show that the existing notions of “sheaf” on a locale B are also equivalent to a new concept what we call a Hilbert module with an Hilbert base. These modules are the projective modules since they are the image of a free module by a idempotent automorphism

On the first chapter, we recall some well known results about partially ordered sets and lattices.

On chapter two we introduce the category of Sup-lattices, and the category of locales, **Loc**. We describe the adjunction between this category and the category **Top** of topological spaces whose restriction to spacial locales give us a duality between this category and the category of sober spaces. We finish this chapter with the definitions of module over a quantale and Hilbert Module.

Chapter three concerns with various equivalent notions namely: sheaves of sets, local homeomorphisms and local sets (projection matrices with entries on a locale). We finish giving a direct algebraic proof that each local set is isomorphic to a complete local set, whose rows correspond to the singletons.

On chapter four we define B-locale, study open maps and local homeomorphims.

The main new result is on the fifth chapter where we define the Hilbert modules and Hilbert modules with an Hilbert and show this latter concept is equivalent to the previous notions of sheaf over a locale.

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Chapter 1

Introduction

1.1 Aims

To give a thorough account of the various equivalent notions for "sheaf" on a locale, namely the separated and complete presheaves, the local homeomorphisms, and the local sets, and to provide a new approach based on quantale modules whereby we see that sheaves can be identified with certain Hilbert modules in the sense of Paseka. This formulation provides us with an interesting category that has immediate meaningful relations to those of sheaves, local homeomorphisms and local sets.

1.2 Partially ordered sets and lattices

The purpose of the present section is just to fix some notation, terminology and recall some well known results. If the reader is familiar with partial orders we suggest him to skip this section in order to avoid a tedious reading. We just present it in order to make this work more self contained.

A *partial order* R on a set P is a binary relation which satisfies:

1. $x \leq x$, $\forall x \in P$ (R is reflexive);

2. $x \leq y \wedge y \leq x \Rightarrow x = y$, $\forall x, y \in P$ (R is antisymmetric);

3. $x \leq y \wedge y \leq z \Rightarrow x \leq z$, $\forall x, y, z \in P$ (R is transitive).

If Q is a partial order in P, we say that (P, Q) is a *partial ordered* set.

Given an order Q on the set P, the inverse relation Q^{-1} also is an order on P and is called the *inverse (or dual) order*. From now on we will use the symbol "$\leq$", to designate an arbitrary partial order and the symbol "$\geq$" to designate its dual order.

Given a partial ordered set $(P, \leq)$ and a subset $A \subseteq P$ we define:

- $x \in P$ is a *lower bound* of A if $x \leq a$, $\forall a \in A$;

- m is the *meet (or infimum)* of A if m is the greatest lower bound, i.e. m is a lower bound of A and if $x \in P$ is lower bound of A then $x \leq m$. If A has a meet we denote that element by $\bigwedge A$. When $A = \{a_i \mid i \in I\}$ for some index set I we write $\bigwedge_{i \in I} a_i$ to designate $\bigwedge A$, when it exists. If $A = \{a, b\}$ we denote by $a \wedge b$ the meet of A, when it exists.

- The *minimum* of A is an element of A that is also a lower bound.

Since $\geq$ is also a partial order on P we also have the dual concepts for a subset $A \subseteq P$:

- An *upper bound* of A on $(P, \leq)$ is a lower bound of A on $(P, \geq)$;

- s is the *join (or supremum)* of A on $(P, \leq)$ if s is the meet of A on $(P, \geq)$. when the join of A exists (relatively to a fixed order $\leq$) will be denoted it by $\bigvee A$. The join of a family $(a_i)_{i \in I}$ of elements of P is denoted by $\bigvee_{i \in I} a_i$, if it exists, and we denote $\bigvee\{a, b\}$ by $a \vee b$.

- The *maximum* of A relative to the order $\leq$ is the minimum of A on the partial ordered set $(P \geq)$.

If $(P, \leq)$ has a minimum then we denoted it by 0_P (or simply by 0 if there is no ambiguity). If it has a maximum, we denote it by 1_P (or simply by 1 if no other order is involved). Note that, if $(P, \leq)$ has minimum then $0_P = \bigwedge P = \bigvee \emptyset$, and if $(P, \leq)$ has maximum then $1_P = \bigvee P = \bigwedge \emptyset$.

A *lattice* is a partially ordered $(L, \leq)$ set where every subset $\{a, b\} \subseteq L$ has a join $a \vee b$ and a meet $a \wedge b$. So, in a lattice $(L, \leq)$, we can define two binary operations $\wedge$ and $\vee$. These operations satisfy:

1. $(L, \wedge)$ and $(L, \vee)$ are commutative semi-groups were every element is idempotent;

2. $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Moreover, given an algebra $(L, \wedge, \vee)$ satisfying these conditions[1] it also satisfy: $a \wedge b = b$ iff $a \vee b = b$, and we can define an order $\leq$ on L by $a \leq b$ if $a \wedge b = a$.

In fact, it is equivalent to give a lattice $(L, \leq)$ or to give an algebra $(L, \wedge, \vee)$ satisfying 1. and 2. Note that these conditions are equations so lattices form an equational class and therefore are closed for sub-algebras, direct products and quotients. Observe that a sub-algebra of a lattice $(L, \wedge, \vee)$ is a non-empty subset closed for the operations $\wedge$ and $\vee$ which is not the same as a non empty sub-set of L that is a lattice when considering the induced order. In other words, a sub-set of a lattice can be a lattice without being a sub-lattice, for example, the sub-vector spaces of a vector space V, ordered by set inclusion, form a lattice, but this not a sub-lattice of the power set

[1]These conditions are equations (formulas which are obtained equaling two terms and quantifying the free variables with universal quantifiers) so, lattices are universal algebras.

$\wp(V)$ because the join of two subspaces is not the union of these spaces, but, instead, it is the smallest subspace that contains that union.

A special case are the *distributive lattices*, the lattices that satisfy the following equivalent [2] distributive laws:

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

A *complete lattice* is a partially ordered set $(P, \leq)$ such that each of whose subsets $X \subseteq P$ has a join $\bigvee X$ and a meet $\bigwedge X$. Note that, in a partially ordered, if every of its subsets has a join, then it is a complete lattice because the meet of a subset is the join of all its lower bounds. Being a complete lattice his a self dual property (that is, if $(P, \leq)$ is a complete lattice then $(P, \geq)$ is also a complete lattice), so applying this result to the dual partially ordered set we get the dual property: that if every subset of a partially ordered set admits a meet then we have a complete lattice.

In a partially ordered set $(P, \leq)$, a non-empty subset $I \subseteq P$ is a *lower set* [3] if it is downward closed, i.e. $x \in I$ and $y \leq x$ implies $y \in I$. The dual concept is called a *upper set* [4], that is, an upper set of $(P, \leq)$ is a lower set of $(P, \geq)$.

An *ideal* I of $(P, \leq)$ is a lower set that satisfies, for all $x, y \in I$, there exists $z \in I$, such that $x \leq z$ and $y \leq z$.

If $(P, \leq)$ is a lattice a non-empty subset $I \subseteq P$ is an ideal of $(P, \leq)$ if,

[2] We think that, for the readers who are not familiar with lattice theory, it is an interesting exercise to verify that, in a lattice, the two distributive laws are equivalent.

[3] Sometimes, lower sets are also called *order ideals*.

[4] Also called *order filters* in some literature.

and only if, the two following conditions are satisfied:

$$x \in I \text{ and } y \leq x \;\Rightarrow\; y \in I;$$
$$x, y \in I \;\Rightarrow\; x \vee y \in I.$$

The dual concept of ideal is called a *filter*. That is, a filter of a partially ordered set $(P, \leq)$ is an ideal of $(P, \geq)$. If $(P, \leq)$ is a lattice F a non-empty subset $F \subseteq P$ is a filter if, and only if, it satisfies:

$$x \in F \text{ and } x \leq y \;\Rightarrow\; y \in F;$$
$$x, y \in F \;\Rightarrow\; x \wedge y \in F.$$

Given a partially ordered set $(P, \leq)$ and $a \in P$, we define $\downarrow a = \{x \in P \mid x \leq a\}$ and we call it the *principal ideal* generated by a. The *principal filter* generated by $a \in P$ is the set $\uparrow a = \{x \in P \mid a \leq x\}$.

An ideal I of a lattice $(P, \leq)$ is called a *prime ideal* if it satisfies the property:

$$x, y \in I \Rightarrow x \in I \text{ ou } y \in I.$$

The dual concept (an ideal of $(P, \geq)$) is called a prime filter. If I is a prime ideal of lattice $(P, \leq)$ then $P \setminus I$ is a prime filter of $(P, \leq)$.

Given a lattice L, and $p \in L$ is called a *(meet) irreducible* element if it satisfies the condition: $x \wedge y = p \Rightarrow (x = p \text{ ou } y = p)$; $p \in L$ is called a *(meet) prime element* if: $x \wedge y \leq p \Rightarrow (x \leq p \text{ ou } y \leq p)$. Any prime element is irreducible. The concepts of *join irreducible* and *join prime* are the duals concepts of meet irreducible and meet prime, respectively. In any lattice a prime element is an irreducible element. If L is also a distributive lattice then both concepts are equivalent.

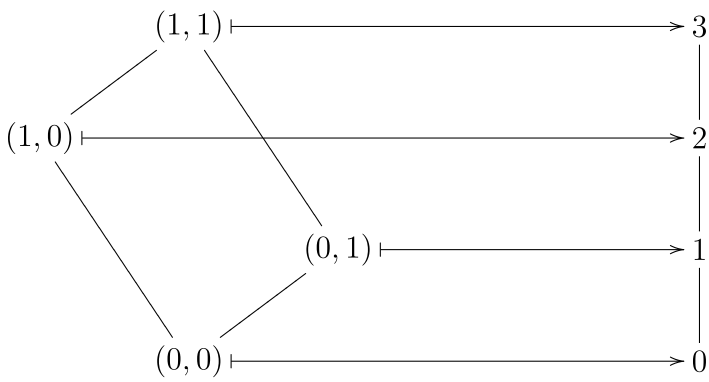
Given $a \in L$, the principal ideal $\downarrow a$ is a prime ideal iff a is a prime element.

Definition 1.2.1 An function $\varphi : P \to Q$ between the partially ordered sets $(P, \leq)$ and $(Q, \leq)$ is called *monotone* or *order preserving* if it satisfies the condition:

$$x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$$

Partially ordered sets together with monotone functions form a category. An isomorphism in this category is called an order isomorphism and is a bijective monotone function whose inverse his also monotone. Or equivalently a bijective function φ that satisfies the condition $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$.

In this category there are bijective morfisms that are not isomorfisms, the following Hasse diagram show us one of these examples.



More formally:

Example 1.2.2 Consider the set $P = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ with the order given by $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$ (P is simply the product of the chain with two elements by itself) and the set $Q = \{0, 1, 2, 3\}$ with the usual order. The function $f : P \to Q$ defined by $f(a, b) = 2a+b$ is monotone, injective and surjective, so it is a monomorphism and a epimorphism (note that we are working with in concrete category), but is not an isomorphism since its inverse is not order preserving[5].

[5]The meaning of the terms monomorphism an epimorphism will always be the cate-

Lemma 1.2.3 *Given partially ordered sets $(P, \leq)$ and $(Q, \leq)$, a surjective map $\varphi : P \to Q$ that satisfies $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y), \forall x, y \in P$ is an order isomorphism.*

Proof. We only have to show that φ is injective. If $\varphi(x) = \varphi(y)$ then $\varphi(x) \leq \varphi(y)$ and $\varphi(y) \leq \varphi(x)$. Therefore $x \leq y$ and $y \leq x$, hence $x = y$. ∎

gorical ones. In a concrete category all injective morphisms are monomorphisms and all surjective morphisms are epimorphisms. Also, in a concrete category which has a free object with one generator, being a monomorphism is equivalent to being an injective morphism. This is what usually happens when dealing with categories that come from algebra, namely this holds in universal algebra. But in some cases there are epimorphisms that are not surjective, for example in the category of rings, the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ is a epimorphism in the categorical sense! In the category of lattices, where morphisms are the maps the preserve binary meets and binary joins, there are also epimorphisms which are not surjective.

Chapter 2

Background on modules

2.1 Sup-lattices

By a *sup-lattice* L we mean a partially ordered set such that each of whose subsets $X \subseteq L$ has a join (supremum or lower upper bound) $\bigvee X \in L$. A *sup-lattice homomorphism* is a function $h : L \to L'$ between sup-lattices that preserves all the joins, i.e.

$$(1.1) \qquad h\left(\bigvee X\right) = \bigvee \{h(x) \mid x \in X\}, \quad \forall X \subseteq S.$$

We denote by **SL** the category whose objects are the sup-lattices and whose morphisms are the sup-lattice homomorphisms.

Example 2.1.1 Given sup-lattices M and N the set $\mathbf{Hom}(M, N)$ of sup-lattices homomorphisms has an order defined by $f \leq g$ iff $\forall x \in M, f(x) \leq g(x)$. This order turns $\mathbf{Hom}(M, N)$ into a sup-lattice.

Example 2.1.2 Given a set X, the power set $\wp(X)$ ordered by set inclusion is a sup-lattice, the joins are simply unions.

Example 2.1.3 If $(X, \Omega(X))$ is topological space then $\Omega(X)$ ordered by set inclusion is a sup-lattice. It is a sub-sup-lattice of 2^X, i.e. every family $(u_i)_{i\in I}$ of open sets has a join and this join is the same as the join computed in 2^X, because the union of opens sets is an open set.

In fact a sup-lattice L is a complete lattice[1] , i.e. every subset $X \in L$ has a join $\bigvee X$ and a meet (infimum or greatest lower bound) $\bigwedge X$, but on the category **SL**, the morphisms only have to preserve joins. As sup-lattice L is a complete lattice, its dual L° (the partially ordered set with the reversed order) is also a sup-lattice. Furthermore, each morphism of sup-lattices $f : M \to N$ has a (unique) right adjoint $f_* : N \to M$ that verifies:

$$f(m) \leq n \Leftrightarrow m \leq f_*(n), \forall m \in M, \forall n \in N$$

The function f_* can be explicitly defined by

$$f_*(n) = \bigvee \{m \in M \mid f(m) \leq n\}$$

It preserves arbitrary meets (infimum) so it defines is a sup-lattice homomorphism $f^\circ : N^\circ \to M^\circ$. Furthermore, one has $(M^\circ)^\circ = M$, $(f^\circ)^\circ = f$, $(fg)^\circ = g^\circ f^\circ$ and also $f \leq g$ iff $g_* \leq f_*$ iff $f^\circ \leq g^\circ$.

In the category **SL** the product $\prod_{i\in I} L_i$ is the product over I of the sets M_i, the product order is given by $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ iff $x_i \leq y_i, \forall i \in I$, with the usual projections defined by $p_i\left((x_j)_{j\in I}\right) = x_i$.

The equalizer of pair of morphisms $f, g : X \to Y$ is the set $\{x \in X \mid f(x) = f(y)\}$. Monomorphisms in **SL** are injective (or monomorphisms in the category that is "replacing" **Set** if we work over an arbitrary topos).

[1]The fact that a sup-lattice L is a complete lattice follows from the equality $\bigwedge X = \bigvee\{y \in X \mid y \leq x, \forall x \in X\}$. We also note that a sup-lattice L has a maximum element $1_L = \bigvee L$ which we call the *top* and a minimum $0_L = \bigvee \emptyset$ called the *bottom*.

This category admits coproducts. The coproduct $\coprod_{i\in I} L_i$ is the product $\prod_{i\in I}$ together with the adjoints of the projections p_i. Note that, the product order is computed "coordinatewise" so the projections preserve arbitrary meets and therefore their adjoints preserve arbitrary joins. See [9] for further details about this category.

2.2 Locales

In this section we introduce the category of locales, which are a generalization of topological spaces. Getting a locale from a topological space is just considering the lattice of opens sets, we will also see how to get a space from a locale and that we have an adjunction between the category **Top** and the category **Loc** of locales. This adjunction is an equivalence when restricted to the sub-category of sober topological spaces and the category of spacial locales.

As we saw in 2.1.3, the topology $\Omega(X)$ of a topological space X, ordered by set inclusion $\subseteq$, forms a sup-lattice. Furthermore it is also closed under binary intersections (meets), therefore it forms a sup-lattice where joins are unions and finite meets are finite intersections[2]. From basic set theory it follows that binary intersection (binary meet) distribute over arbitrary unions (joins). Furthermore, given a continuous map $f : X \to Y$ between topological spaces $(X, \Omega(X))$ and $(Y, \Omega(Y))$, the function $f^* : \Omega(Y) \to \Omega(X)$ defined by

$$f^*(U) = \{x \in X \mid f(x) \in U\}, \quad \forall U \in \Omega(Y)$$

preserves arbitrary unions (joins) and finite intersection (finite meets).

This motivates the following:

[2]Arbitrary meets also exist, they are the interior of the intersection, but, in general they are not preserved by preimages neither joins distribute over arbitrary meets.

Definition 2.2.1 By a *frame* L we mean a sup-lattice where the following distributive law holds:

$$a \wedge \left(\bigvee B \right) = \bigvee \{a \wedge b \mid b \in B\} \tag{2.2}$$

A *frame homomorphism* is a sup-lattice homomorphism $h : L \to L'$ between frames that also preserves binary meets, that is, a function that satisfies the equations:

$$h(x \wedge y) = h(x) \wedge h(y), \quad \forall x, y \in L \tag{2.3}$$

$$h\left(\bigvee X \right) = \bigvee \{h(x) \mid x \in X\}, \quad \forall X \subseteq L. \tag{2.4}$$

The category formed by frames together with frame homomorphisms will be denoted by **Frm** and is called the category of frames.

The category of locales, denoted by **Loc** is the dual category of **Frm**. We call *locale* to an object of **Loc**, and *map of locales* (or simply a *map*) to a morphism on this category. In other words, a locale is the same as a frame and a map, $f : L_1 \to L_2$ from the locale L_1 to the locale L_2, is a frame homomorphism $f : L_2 \to L_1$. As it is usual in the literature, given a map of locales $f : L_1 \to L_2$, we represent the correspondent frame homomorphism by $f^* : L_2 \to L_1$.

Example 2.2.2 Given topological spaces $(X, \Omega(X))$, $(Y, \Omega(Y))$ and a continuous map $f : X \to Y$, the preimage $f^* : \Omega(Y) \to \Omega(X)$ defined by $f^*(B) = \{a \in X \mid f(x) \in B\}$, $\forall B \in \Omega(Y)$, is a frame homomorphism.

Let $(X, \Omega(X))$, $(Y, \Omega(Y))$ and $(Y, \Omega(Z))$ be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. We have $(g \circ f)^* = f^* \circ g^*$. So we have a contravariant functor from **Top** to **Frm**, or equivalently, a (covariant) functor $\mathcal{O}$ from **Top** to **Loc**.

Now we will see how to obtain a topological space from a locale.

First we start with the points. Note that a point p of a topological space X can be identified with a map from a topological space with one element into X, namely the trivially continuous map that assigns the valour p to the only element of that topological space. Applying the functor $\mathcal{O}$ we get a frame homomorphism from $\Omega(X) \to \wp(\{0\})$, where $\wp(\{0\})$ is the power set of the set with one element[3] (which is order isomorphic to the topology of the space with one point). So, a point of topological space induces a locale map from $\wp(\{0\})$ into locale which is the topology of that space.

Definition 2.2.3 A *point* p of a locale L is a locale map $p : \wp(\{0\}) \to L$. We denote by $\Sigma(L)$ the set of points of the locale L.

A *completely prime filter* of a locale L is a filter F that satisfy the property $\bigvee X \in F$ implies $X \cap F \neq \emptyset$.

Proposition 2.2.4 *Given a locale L, there is an order isomorphism between the points and the completely prime filters of L.*

Proof. Given a point p, consider the corresponding frame homomorphism $p^* : L \to \wp(\{0\})$ and set $\bar{p} = \{x \in L \mid p^*(x) = \{0\}\}$, that is, $x \in \bar{p} \Leftrightarrow 0 \in p^*(x)$. It easy to check that $\bar{p}$ is a filter.

In order to show that $\bar{p}$ is completely prime, suppose S is a subset of L such that $\bigvee S \in \bar{p}$. Then $0 \in p^*\left(\bigvee S\right) = \bigcup\{p^*(s) \mid s \in S\}$, so, there is $s \in S$

[3] We use $\wp(\{0\})$ instead of the chain with two elements. Classically they are the same, but it may not be the same if our logic is not classic. The power set of a set may be always identified with the functions from that set into the set of logical values, but, in a intuitionistic logic, the set of logical values is not the boolean algebra with two elements, instead it is the initial object in the category of complete Heyting algebras, see [3].

such that $0 \in p^*(s)$. From this, we conclude that it exists $s \in S$ such that $s \in \bar{p}$ therefore, $S \cap \bar{p} \neq \emptyset$.

Now suppose that F is a completely prime filter of L. Define the function $p^* : L \to \wp(\{0\})$ by the condition $0 \in p^*(x) \Leftrightarrow x \in F$. We have

$$\begin{aligned} 0 \in p^*(x \wedge y) &\Leftrightarrow x \wedge y \in F \\ &\Leftrightarrow x \in F \text{ and } y \in F \\ &\Leftrightarrow 0 \in p^*(x) \text{ and } 0 \in p^*(y) \\ &\Leftrightarrow 0 \in p^*(x) \cap p^*(y) \end{aligned}$$

And therefore $p^*(x \wedge y) = p^*(x) \cap p^*(y)$.

Consider $S \subseteq L$,

$$\begin{aligned} 0 \in p^*\left(\bigvee S\right) &\Leftrightarrow \bigvee S \in F \\ &\Leftrightarrow S \cap F \neq \emptyset \\ &\Leftrightarrow \exists s \in S,\, s \in F \\ &\Leftrightarrow \exists s \in S,\, 0 \in p^*(s) \\ &\Leftrightarrow 0 \in \bigcup \{p^*(s) \mid s \in S\} \end{aligned}$$

So, $p^*\left(\bigvee S\right) = \bigcup\{p^*(s) \mid s \in S\}$ and p^* preserves arbitrary joins. Therefore $p : \wp(\{0\}) \to L$ is a point.

Given a completely prime filter F and letting p^* be the induced frame homomorphism, we have, for all $x \in L$, $x \in F \Leftrightarrow p^*(x) = 0 \Leftrightarrow x \in \bar{p}$, hence the correspondence $p \mapsto \bar{p}$ between points and completetely prime filters is surjective.

Given points p, q we have: $p \leq q \Leftrightarrow p^*(x) \subseteq q^*(x), \forall x \in L \Leftrightarrow \bar{p} \subseteq \bar{q}$. From lema 1.2.3 it follows that this correspondence is an order isomorphism, between the partial ordered set of points and the partially ordered set of completely prime filters. ∎

The function $\Phi : L \to \wp(\Sigma(L))$ defined by $\Phi(x) = \{p \in \Sigma(L) \mid p^*(x) = \{0\}\}$ satisfies

$$\begin{aligned} \Phi(x \wedge y) &= \Phi(x) \cap \Phi(y) \\ \Phi(\bigvee X) &= \bigcup\{\Phi(x) \mid x \in X\} \end{aligned}$$

So, Φ is a frame homomorphism and its image is a topology on $\Sigma(L)$. From now on, we consider $\Sigma(L)$ equipped with this topology. Since $\Sigma(L)$ is a topological space we denote its correspondent frame by $\mathcal{O}(\Sigma(L))$.

Given a map of locales $f : L \to L'$ (i.e. a frame homomorphism $f^* : L' \to L$) we define $\Sigma(f) : \Sigma(L) \to \Sigma(L')$ simply by $\Sigma(f)(p) = p \circ f^*$.

Given $x \in L'$ and a open set $U_x = \{p \in \Sigma(L') \mid p^*(x) = \{0\}\}$,

$$(\Sigma(f))^{-1}(U_x) = \{q \in \Sigma(L) \mid p(f^*(x)) = \{0\}\}$$

is therefore an open set of $\Sigma(L)$ and is determined by $f^*(x)$. Therefore $\Sigma(f)$ is a continuous function.

We can easily verify that $\Sigma(fg) = \Sigma(f) \circ \Sigma(g)$ and so Σ is a functor from the category **Loc** of locales into the category **Top** of topological spaces.

A *principal prime ideal* is an principal ideal of a lattice which is also prime. In a general lattice, a principal ideal $\downarrow a$ is prime iff a is a *prime element*, that is, $x \wedge y \leq a$ implies $x \leq a$ or $y \leq a$.

On following we will try to make when possible constructive proofs, when some result is only true if we accept the principle of the excluded third or the axiom of choice, we will say we are working in classical mathematics.

Classically a filter is completely prime if, and only if, its complement is a principal prime ideal.

So, classically, there are bijections between the sets formed by:

- Points of the locale L;

- Completely prime filters of L;
- Principal prime ideals of L;
- Prime elements of L.

If but one should be careful if we want that to produce arguments constructively valid. In the latter case, the argument is still valid in an arbitrary topos. See [3] or [8] for further details about how this can be done.

If X is a topological space and p is an element (topological point) in X, the completely prime filter of $\mathcal{O}(X)$ that corresponds to p is $\{U \in \mathcal{O}(X) \mid p \in U\}$. That is, the set of all opens neighbourhoods of p.

Also note that, classically, this latter set is a prime filter and bijectively corresponds to the greatest open set which do not contain the point p. This greatest open set is prime, that is, it can not be written has the intersection of two open sets. And so, its complement on the set X is an irreducible closed set[4], the intersection of all closed sets which contain the point (the closure of the singleton formed by the point).

Classically, a space X is *sober* if every closed irreducible subset is the closure of exactly one singleton (subset with one element) of X.

Some literature, for example [8], use a different definition. Accepting middle third excluded principle, both definitions are equivalent. from a intuitionist view point they are not, nevertheless, in both cases, the points of sober space correspond bijectively to the completely prime filters of its topology (considered as a locale). And, in both cases we have the following:

Proposition 2.2.5 *In a sober space X, the correspondence $X \to \Sigma(\mathcal{O}(X))$ that assigns to each $x \in X$ the maximal prime filter formed by the open*

[4]A closed subset F of a topological space is irreducible if, $F = F_1 \cap F_2$ implies $F = F_1$ or $F = F_2$, for all closed F_1, F_2 on X

neighbourhoods of x is a homeomorphism.

In terms of separation axioms, one has that any Hausdorff space (T_2) is sober, because in a Hausdorff space the only closed irreducible subsets are the subsets formed by one element. Any sober space is Kolmogorov (T_0). There is no relation between spaces that satisfy the T_1 condition and sober spaces.

Example 2.2.6 Consider a topological space X with the indiscrete topology. Then the only non empty closed irreducible subset is X. So X is not sober if it has more than one element.

Example 2.2.7 Consider the real plane with the Zarisky topology. Any straight line is a closed irreducible subset[5], so this topology is not sober.

This former example above is not sober because it has to many points, that is it has points that are not distinguished by the topology. In the latter one the situation is reversed. Note that we are considering as base set the real plane, if we consider the prime sprectrum of the ring of polynomials with two variables with its topology, then we will get a sober space, but we will have points that do not correspond to elements of the real plane.

The functor Σ from **Loc** to **Top** is not a category equivalence but it is a right adjoint to the functor $\mathcal{O}$ that associates to a topological space the locale formed by its open sets and to each continuous function the map of locales that correspond to the frame homomorphism given by the preimage function, See for example [8].

So we have a universal arrow η : **Top** $\to$ **Top** assigning to each topological space the spectrum of its topology.

[5]A straight line is the set of zeros of the prime ideal generated by the irreducible polynomial $aX + bY + c$ with a, b, c and $(a, b) \neq (0, 0)$.

If we restrict to sober spaces then we have an equivalence of categories between the category of sober spaces and a subcategory of **Loc**, the obejects of this category are called *spacial locales*. Since almost all topological spaces we use in mathematics are sober, we can see the category of locales has a generalization of the category of topological spaces.

Example 2.2.8 Consider S^1 and its universal cover H which is best seen geometrically as an helix parameterized by the real line by setting $t \mapsto (\cos(2\pi t), \sin(2\pi t), t)$, the projection $p : H \to S^1$ is defined by $p(\cos(2\pi t), \sin(2\pi t), t) = (\cos(2\pi t), \sin(2\pi t))$. This projection p is a local homeomorphism (i.e. H has an open cover $(U_i)_{i \in I}$ such that the restrictions $p\restriction_{U_i}: U_i \to p(U_i)$ are homeomorphisms). When considering the correspond locales formed by these topologies these restrictions became locale isomorphisms.

We recommend to the readers not familiar with the concepts in this work to keep this example in their minds.

2.3 Quantales and Modules

Definition 2.3.1 A *quantale* is a sup-lattice Q together with an associative binary operation $(a, b) \mapsto a \cdot b$ that distributes over arbitrary joins in both variables:

$$a \cdot \left(\bigvee_{i \in I} b_i \right) = \left(a \cdot \bigvee_{i \in I} b_i \right) \tag{3.5}$$

$$\left(\bigvee_{i \in I} b_i \right) \cdot a = \left(\bigvee_{i \in I} b_i \cdot a \right) \tag{3.6}$$

The binary operation is called *multiplication*.

We call a quantale Q *involutive* if in addition it also has a unary operation $*: Q \to Q$ satisfying:

$$\left(\bigvee_{i\in I} a_i\right)^* = \bigvee_{i\in I} a_i^* \tag{3.7}$$

$$(ab)^* = b^*a^* \tag{3.8}$$

Example 2.3.2 Consider a sup-lattice L, the set $\mathcal{Q}(S)$ of sup-lattices endomorphisms of L is quantale. The order is given by pointwise ordering and the multiplication given by $f \cdot g = g \circ f$.

Definition 2.3.3 Let Q be a quantale, a *(left) module* M over Q or simply a *(left) Q-module*, is a sup-lattice with an action $\_\cdot\_ : Q \times M \to M$, satisfying:

$$a \cdot \left(\bigvee X\right) = \bigvee\{a \cdot x \mid x \in X\} \tag{3.9}$$

$$\left(\bigvee S\right) \cdot x = \bigvee\{a \cdot x \mid a \in S\} \tag{3.10}$$

$$(a \cdot b) \cdot x = a \cdot (b \cdot x) \tag{3.11}$$

Note that we are using the same symbol, "$\cdot$", for the multiplication and for the action of the quantale on the module. But it always easy to see from the context which is the meaning of this symbol an we will frequently omit it and write simply ax instead of $a \cdot x$.

In a similar way we could define a right module over a quantale but, on this text we will consider modules over locales, which are commutative quantales, therefore it is irrelevant to consider left or right modules.

On a similar way as on the theory of modules over a ring (where a module M over the ring R is equivalent to a ring homomorphism from R into the ring of homomorphisms of an abelian group), giving a (left) module M over a quantale Q is equivalent to giving a quantale homomorphism $\varphi : Q \to \mathcal{Q}(S)$.

Given the module, we define φ by $\varphi(a)(x) = a \cdot x, \forall x \in M$, $\forall a \in Q$. From (3.9) it follows that $\varphi(a)$ is a sup-lattice endomorphism of M, so φ is a well defined application. The property (3.10) implies that φ preserves arbitrary joins. Finally, given $a, b \in Q$ and $x \in M$, $\varphi(a.b)(x) = (a \cdot b) \cdot x = a \cdot (b \cdot x) = \varphi(a)\,(\varphi(b)(x)) = (\varphi(a) \circ \varphi(b))\,(x)$.

Example 2.3.4 A quantale becomes left module over itself when considering the action given by the multiplication of the quantale.

Definition 2.3.5 Let Q be a quantale and M, N a (left) Q-modules. A Q-*homomorphism* or a *homomorphism of* Q*- modules* is a function $\varphi : M \to N$ such that

$$\varphi\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} \varphi(x_i) \tag{3.12}$$

$$\varphi(ax) = a\varphi(x) \tag{3.13}$$

for all $a \in Q$, $m, m_i \in M$, $i \in I$.

Note that a locale L is itself a commutative quantale where the multiplication is the binary meet. This quantale has multiplicative identity which is the top element 1_L.

2.4 Hilbert modules

Hilbert Q-modules are an analogue of Hilbert C*-modules where C*-algebras are replaced by quantales. They have been studied by Paseka mainly as a means of importing results and techniques from operator theory into the context of quantales (see, e.g., [14]), and also in connection with theoretical computer science [13]. We begin by recalling this notion in the special case that interests us in this paper, namely when the involutive quantale Q is the locale B.

Definition 2.4.1 By a *pre-Hilbert B-module* will be meant a B-module X equipped with a function

$$\langle -,-\rangle : X \times X \to B$$

called the *inner product*, that satisfies the following axioms, for all $x, y \in X$ and $b \in B$:

$$\langle bx, y\rangle = b \wedge \langle x, y\rangle \tag{4.2}$$

$$\left\langle \bigvee_\alpha x_\alpha, y \right\rangle = \bigvee_\alpha \langle x_\alpha, y\rangle \tag{4.3}$$

$$\langle x, y\rangle = \langle y, x\rangle\ . \tag{4.4}$$

(In short, a symmetric B-valued "bilinear" form.) A *Hilbert B-module* is a pre-Hilbert B-module whose inner product is non-degenerate,

$$\langle x, -\rangle = \langle y, -\rangle \Rightarrow x = y\ , \tag{4.5}$$

and it is said to be *strict* ("positive definite") if it satisfies

$$\langle x, x\rangle = 0 \Rightarrow x = 0\ .$$

A useful consequence of non-degeneracy is the following:

Lemma 2.4.6 *Let X be a Hilbert B-module. Then for all $b \in B$ and $x \in X$ we have*

$$bx = b1 \wedge x\ .$$

Hence, in particular, if X is a locale it is a B-locale.

Proof. The inequality $bx \le b1 \wedge x$ is immediate. For the other, it suffices to show that for all $y \in X$ we have $\langle b1 \wedge x, y\rangle \le \langle bx, y\rangle$:

$$\langle b1 \wedge x, y\rangle \le \langle b1, y\rangle \wedge \langle x, y\rangle = b \wedge \langle 1, y\rangle \wedge \langle x, y\rangle = \langle bx, y\rangle\ . \quad \blacksquare$$

Similarly to Hilbert C*-modules, the module homomorphisms which have "operator adjoints" play a special role:

Definition 2.4.7 Let X and Y be pre-Hilbert B-modules. A function

$$h : X \to Y$$

is *adjointable* if there is another function $h^\dagger : Y \to X$ such that for all $x \in X$ and $y \in Y$ we have

$$\langle h(x), y \rangle = \langle x, h^\dagger(y) \rangle \ .$$

[The usual notation for $h^\dagger$ is h^*, but we want to avoid confusion with the notation for inverse image homomorphisms of locale maps.]

Any adjointable function $h : X \to Y$ is necessarily a homomorphism of B-modules [13], and this fact is a consequence of the non-degeneracy of $\langle -, - \rangle_Y$ alone; that is, h satisfies $h\left(\bigvee a_\alpha x_\alpha\right) = \bigvee a_\alpha h(x_\alpha)$ because for all $y \in Y$ we have

$$\begin{aligned} \left\langle h\left(\bigvee a_\alpha x_\alpha\right), y \right\rangle &= \left\langle \bigvee a_\alpha x_\alpha, h^\dagger(y) \right\rangle = \bigvee a_\alpha \langle x_\alpha, h^\dagger(y) \rangle \\ &= \bigvee a_\alpha \langle h(x_\alpha), y \rangle = \left\langle \bigvee a_\alpha h(x_\alpha), y \right\rangle \ . \end{aligned}$$

Chapter 3

Background on sheaves

3.1 Sheaves and presheaves

We start by defining the notions of sheaf and presheaf on a locale. The main objective is to fix notation. For the reader who is not so familiar with this concept we recommend [1].

Any partially ordered set $(P, \leq)$ (in particular any locale) defines a category whose objects are the elements of P and, given objects $x, y \in P$, there is one morphism from x to y if $x \leq y$. If $x \nleq y$ then there are no arrows from x to y.

Definition 3.1.1 A *presheaf* over the locale L is a contravariant functor $F : L^{\mathrm{op}} \to \mathbf{Sets}$ from L to the category of **Sets**.

A presheaf over a topological space is a presheaf over the locale formed by its topology.

Given a presheaf $F : L^{\mathrm{op}} \to \mathbf{Sets}$ and $v \leq u$ elements of L, the functor F "transforms" a morphism $\rho : v \to u$ into a function $\rho_v^u : F(u) \to F(v)$.

Given $x \in F(u)$, we will abbreviate $\rho^u_v(x)$ by $x\restriction_v$, if there is no possible ambiguity.

Because F is a contravariant functor, ρ^u_u is the identity function on $F(U)$ and, if $w \leq v \leq u$, we have $\rho^v_w \circ \rho^u_v = \rho^u_w$. Or, using the notation above:

$$x\restriction_u = x, \quad \forall u \in L, \forall x \in F(u) \tag{1.1}$$

$$(x\restriction_v)\restriction_w = x\restriction_w, \quad \forall w \leq v \leq u \in L, \forall x \in F(U) \tag{1.2}$$

Because in many presheaves that appear on topological spaces the sets $F(u)$ are sets of functions defined on the open set u and the morphisms ρ^u_v are simply the restriction of these function to the open set v, on the presheaves over a locale, we will call also the morphisms ρ^u_v *restriction morphisms*. We will call the elements in $F(u)$ *sections on* u.

Definition 3.1.2 Consider a presheaf F over the locale L and a family $(u_i)_{i\in I}$ of elements of L. A family of elements $(x_i \in F(u_i))$ of the presheaf is *compatible* if

$$x_i\restriction_{u_i\wedge u_j} = x_j\restriction_{u_i\wedge u_j}, \forall i, j \in I.$$

Definition 3.1.3 A presheaf F over the locale L is *separated* if for every family $(u_i)_{i\in I}$ in L and $x, y \in F(\bigvee_{i\in I} u_i)$,

$$(\forall i \in I,\, x\restriction_{u_i} = y\restriction_{u_i}) \Rightarrow x = y$$

Definition 3.1.4 A *sheaf* F over a locale L is a presheaf F over L such that given $u = \bigvee_{i\in I} u_i$ in L and $(x_i \in F(u_i))$ a compatible family in F, there exist a unique $x \in F(u)$ such that $x\restriction_{u_i} = x_i$.

The element x above is called the *gluing* of the family $(x_i)_{i\in I}$.

Definition 3.1.5 The *morphisms of presheaves and of sheaves* are the natural transformations.

Example 3.1.6 Consider topological spaces $(X, \Omega(X))$ and $(Y, \Omega(Y))$. Recall that $\Omega(X)$ is a category whose objects are the open sets of $(X, \Omega(X))$ i.e. the elements of $\Omega(X)$ and whose morphisms are the set inclusion between their open sets. For each open set $U \in \Omega(X)$ let $F(U)$ be the set of continuous functions from U to Y. Given $V \subseteq U$ and $f \in F(U)$ define $\rho_V^U(f) = f\restriction_V$, i.e. $\rho_V^U(f)$ is the restriction of the function f to open set V.

It is easy to verify that F is a contravariant functor so it is a presheaf and if $U = \bigcup_{i\in I} U_i$ and if $(f_i : U_i \to Y)_i \in I$ is a compatible family of continuous functions then there is a unique extension f of the f_i to U such that $f\restriction_{u_i} = f_i$, so F is a sheaf.

There are also important presheaves which are not sheaves:

Example 3.1.7 Consider the set of real numbers with the usual topology. Consider the functor B that assigns to each open set U the set $B(U)$ of bounded functions defined on the open set U. This form a separated presheaf, but it is not a sheaf since the gluing of compatible sections may not be bounded.

3.2 Local homeomorphisms

Recall that a local homeomorphism $p : X \to Y$ from the topological space X onto the topological space Y is a continuous map such that for every $x \in X$, there is an open neighbourhood U of x such that $f(U)$ is open and the restriction $f\restriction_U; U \to f(U)$ is a homeomorphism. This is equivalent to saying that X has an open cover $(U_i)_{i\in I}$ such that $f(U_i)$ is open and the restrictions $f_i = f\restriction_{U_i} : U_i \to f(U_i)$ are homeomorphisms.

These restrictions f_i correspond to the frame homomorphism $f_i^* : {\downarrow} b_i \to {\downarrow} x_i$ defined by $f_i^*(b) = f^*(b) \wedge x_i$.

In the category of frames this corresponds to saying that, for each $i \in I$ we have a commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\quad f^* \quad} & L \\ {\scriptstyle (-)\wedge b_i}\downarrow & & \downarrow{\scriptstyle (-)\wedge x_i} \\ \downarrow b_i & \xrightarrow{\; f_i^*(-)=f^*(-)\wedge x_i \;} & \downarrow x_i \end{array}$$

where the f_i's are isomorphisms.

Definition 3.2.1 Let L be a locale, a *cover* of L is a family $(x_i)_{i\in I}$ of elements in L such that $\bigvee_{i\in I} x_i = 1_L$.

Definition 3.2.2 Let let $f : L \to B$ be a map of locales. We say that f is *local homeomorphism* is there is a cover $(x_i)_{i\in I}$ of L and a family $(b_i)_{i\in I}$ of elements from B such that:

- $(x_i)_{i\in I}$ is a cover of L;
- for each index $i \in I$, f restricts to an isomorphism $f_i : \downarrow x_i \to \downarrow b_i$ between the open sub-locales generated by x_i and b_i.

3.3 Local sets

Definition 3.3.1 Given two index sets I and J, a *B-valued matrix* of type $I \times J$ is a function $A : I \times J \to B$.

We denote the set of B valued matrices of type $I \times J$ by $M_{I\times J}(B)$.

Given $A \in M_{I\times J}(B)$ and $A' \in M_{J\times K}(B)$ the product $AA' \in M_{I\times K}(B)$ and the transpose $A^{\mathrm{T}} \in M_{J\times I}(B)$ are defined by:

$$\begin{aligned} (AA')_{ik} &= \bigvee_{j\in J} A_{ij} \wedge A'_{jk} \\ (A^{\mathrm{T}})_{ji} &= A_{ij} \ , \end{aligned}$$

It easy to verify that this product is associative and that $(AA')^{\mathrm{T}} = \mathcal{A}'^{\mathrm{T}}A^{\mathrm{T}}$.

The set $M_{I\times J}(B)$ is isomorphic to a direct product of "$I \times J$"copies of B, so it has the pointwise order that makes it a locale and, in particular, a sup-lattice.

Just as a curiosity we note that the set $M_{I\times I}(B)$ is an involutive quantale, the involution being given by transposition. We have mentioned that the product of matrices is associative. It is also immediate that transposition preserves arbitrary joins, since they are calculated pointwise, and reverses multiplication as in linear algebra.

We will just verify that the multiplication distributes over arbitrary joins. Let $A \in M_{I\times I}(B)$ and let $(M^p)_{p\in P}$ be a family of matrices in $M_{I\times I}(B)$. For all $i, k \in I$, one has

$$
\begin{aligned}
\left(A\left(\bigvee_{p\in P} M^p\right)\right)_{ik} &= \bigvee_{j\in I} A_{ij} \wedge \left(\bigvee_{p\in P} M^p\right)_{jk} \\
&= \bigvee_{j\in I} A_{ij} \wedge \left(\bigvee_{p\in P} M^p_{jk}\right) \\
&= \bigvee_{j\in I}\bigvee_{p\in P} A_{ij} \wedge M^p_{jk} \\
&= \bigvee_{p\in P}\bigvee_{j\in I} A_{ij} \wedge M^p_{jk} \\
&= \bigvee_{p\in P} (AM^p)_{ik} \\
&= \left(\bigvee_{p\in P} (AM^p)\right)_{ik}
\end{aligned}
$$

Therefore

$$A\left(\bigvee_{p\in P} M^p\right) = \bigvee_{p\in P} (AM^p)$$

And in an analogous way we can show that

$$\left(\bigvee_{p\in P} M^p\right) A = \bigvee_{p\in P} (M^p A)$$

We recall here the notion of locale-valued set of [3]:

Definition 3.3.2 By a *B-set* is meant a set $\mathit{\Gamma}$ equipped with an *equality relation* valued in B,

$$[\![- = -]\!] : \mathit{\Gamma} \times \mathit{\Gamma} \to B \ ,$$

which satisfies the following axioms, where Es, called the *extent* of s, is written as an abbreviation for $[\![s = s]\!]$:

$$[\![s = t]\!] \wedge [\![t = u]\!] \leq [\![s = u]\!] \tag{3.3}$$

$$[\![s = t]\!] = [\![t = s]\!] \ . \tag{3.4}$$

Mathematically, a B-set is just a matrix with entries in the locale B,

$$\mathcal{E} : \mathit{\Gamma} \times \mathit{\Gamma} \to B \ .$$

The properties that this the matrix has to verify in order to became a B-set are simply:

$$\mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \mathcal{E}_{su} \tag{3.5}$$

$$\mathcal{E}_{st} = \mathcal{E}_{ts} \ , \tag{3.6}$$

for all $s, t, u \in \mathit{\Gamma}$.

Proposition 3.3.7 *A matrix $\mathcal{E} : \mathit{\Gamma} \times \mathit{\Gamma} \to B$ is a B-set if, and only if, $\mathcal{E}$ is a projection matrix, that is:*

$$\mathcal{E}^2 = \mathcal{E} = \mathcal{E}^{\mathrm{T}}$$

Proof. The equality $\mathcal{E} = \mathcal{E}^{\mathrm{T}}$ is trivially equivalent to $\mathcal{E}_{st} = \mathcal{E}_{ts}$, for all $s, t \in \Gamma$.

Let us suppose that $\mathcal{E} : \Gamma \times \Gamma \to B$ is a B-set in order to prove that $\mathcal{E}$ is idempotent .

First note, since $\mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \mathcal{E}_{su}$ making $s = u$ we get $\mathcal{E}_{st} \leq \mathcal{E}_{ss}$.

Given $s, u \in \Gamma$,

$$\mathcal{E}^2_{su} = \bigvee_{t\in\Gamma} \mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \mathcal{E}_{su} = \mathcal{E}_{ss} \wedge \mathcal{E}_{su} \leq \bigvee_{t\in\Gamma} \mathcal{E}_{st} \wedge \mathcal{E}_{tu} = \mathcal{E}^2_{su}$$

therefore $\mathcal{E} = \mathcal{E}^2$.

For the converse, if $\mathcal{E}^2 = \mathcal{E}$ then $\mathcal{E}_{su} = \bigvee_{t\in\Gamma} \mathcal{E}_{st} \wedge \mathcal{E}_{tu}$, and therefore $\mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \mathcal{E}_{su}$. ∎

From the previous proof it follows the next corollary which will be useful.

Corollary 3.3.8 *On any B-set $\mathcal{E} : I \times I \to B$ we have*

$$\mathcal{E}_{ij} \leq \mathcal{E}_{ii}$$

for all $i, j \in I$.

We will think on B-sets as matrices and adopt the same notational conventions for matrices and their entries that are used in linear algebra.

Definition 3.3.9 Let $\mathcal{E} : I \times I \to B$ and $\mathcal{F} : J \times J \to B$ be B-sets. By a *relation* $T : \mathcal{E} \to \mathcal{F}$ from $\mathcal{E}$ to $\mathcal{F}$ is meant a matrix $T : J \times I \to B$ such that

$$T\mathcal{E} = T = \mathcal{F}T \tag{3.7}$$

It is immediate to show that, given B-sets $\mathcal{E}, \mathcal{F}, \mathcal{G}$ and relations $T : \mathcal{E} \to \mathcal{F}$ and $U : \mathcal{F} \to \mathcal{G}$, the product $UT : \mathcal{E} \to \mathcal{G}$ is a relation from $\mathcal{E}$ to $\mathcal{G}$. Since, by definition of relation $T : \mathcal{E} \to \mathcal{F}$, B-sets themselves are identities and product of matrices is associative so we have a category:

Definition 3.3.10 We denote by $Rel(B)$ the category that has as objects the B-sets and whose morphisms are the relations. Given relations $T : \mathcal{E} \to \mathcal{F}$ and $U : \mathcal{F} \to \mathcal{G}$, the composition is simply the matrix product UT.

Definition 3.3.11 A *map of B-sets* is a relation $T : \mathcal{E} \to \mathcal{F}$ that also satisfies the following inequalities:

$$TT^{\mathrm{T}} \leq \mathcal{F} \tag{3.8}$$

$$T^{\mathrm{T}}T \geq \mathcal{E} \tag{3.9}$$

Given maps of B-sets $T : \mathcal{E} \to \mathcal{F}$ and $S : \mathcal{F} \to \mathcal{G}$, we have $(ST)(ST)^{\mathrm{T}} = STT^{\mathrm{T}}S^{\mathrm{T}} \leq S\mathcal{F}S^{\mathrm{T}} = SS^{\mathrm{T}} \leq \mathcal{G}$. And $(ST)^{\mathrm{T}}(ST) = T^{\mathrm{T}}S^{\mathrm{T}}ST \geq T^{\mathrm{T}}\mathcal{F}T = T^{\mathrm{T}}T \geq \mathcal{E}$

Definition 3.3.12 We call $\mathbf{Set}(B)$ to the category whose objects are the B-sets and whose morphisms are the maps of B-sets.

The next proposition shows that this definition is equivalent to the one present on [8].

Proposition 3.3.13 *Given B-sets $\mathcal{E} : I \times I \to B$ and $\mathcal{F} : J \times J \to B$, matrix $T : J \times I \to B$ is a map of B-sets if and only if it satisfies the following conditions:*

$$T_{ji} \leq \mathcal{F}_{jj} \wedge \mathcal{E}_{ii} \tag{3.10}$$

$$T_{ji} \wedge \mathcal{F}_{jj'} \wedge \mathcal{E}_{ii'} \leq T_{j'i'} \tag{3.11}$$

$$T_{ji} \wedge T_{j'i} \leq \mathcal{F}_{jj'} \tag{3.12}$$

$$\mathcal{E}_{ii} \leq \bigvee_{j \in J} T_{ji} \tag{3.13}$$

for all $i, i' \in I$, $j, j' \in J$.

Proof. First note that the second condition can be written as $\bigvee_{j\in J}\bigvee_{i\in I}\mathcal{F}_{j'j}\wedge T_{jj'}\wedge\mathcal{E}_{ii'}\leq T_{j'i'}$, so this condition is equivalent to

$$\mathcal{F}T\mathcal{E}\leq T$$

Now we show that a map of B-sets satisfies these four conditions.

For the first one, note that $T\mathcal{E}=T$ implies $T_{ji}=\bigvee_{i'\in I}T_{ji'}\wedge\mathcal{E}_{i'i}\leq\bigvee_{i'\in I}T_{ji'}\wedge\mathcal{E}_{ii}\leq\mathcal{E}_{ii}$. Since $T^{\mathrm{T}}\mathcal{F}=T^{\mathrm{T}}$, we get $T_{ji}=T^{\mathrm{T}}_{ij}\leq\mathcal{F}_{jj}$, by the same argument.

For the second condition, on any relation we have $T=\mathcal{F}T=\mathcal{F}T\mathcal{E}$, so, in particular, the inequality holds.

The third condition is the same as $TT^{\mathrm{T}}\leq\mathcal{F}$, so it holds trivially.

The fourth follows from $\mathcal{E}\leq T^{\mathrm{T}}T$, because it implies $\mathcal{E}_{ii}\leq(T^{\mathrm{T}}T)_{ii}=\bigvee_{j\in J}T^{\mathrm{T}}_{ij}\wedge T_{ji}=\bigvee_{j\in J}T_{ij}$.

Let us assume now the four conditions above, in order to prove T is a map of B-sets.

Since $T_{ji}\leq\bigvee_{i'\in I}\bigvee_{j'\in J}T_{ji'}\wedge T_{j'i'}\wedge T_{j'i}$, it follows that $T\leq TT^{\mathrm{T}}T$, for every matrix $T:J\times I\to B$.

We have $\mathcal{F}T\mathcal{E}\leq T\leq TT^{\mathrm{T}}T\leq\mathcal{F}T$.

But

$$\begin{aligned}(\mathcal{F}T)_{ji} &= \bigvee_{j'\in J}\mathcal{F}_{jj'}\wedge T_{j'i}\\ &= \bigvee_{j'\in J}\mathcal{F}_{jj'}\wedge T_{j'i}\wedge\mathcal{E}_{ii}\\ &\leq \bigvee_{i'\in I}\bigvee_{j'\in J}\mathcal{F}_{jj'}\wedge T_{j'i'}\wedge\mathcal{E}_{i'i}\\ &\leq (\mathcal{F}T\mathcal{E})_{ji}.\end{aligned}$$

Therefore $\mathcal{F}T\mathcal{E}=T=TT^{\mathrm{T}}T=\mathcal{F}T$ and also $T\mathcal{E}=\mathcal{F}T\mathcal{E}=T$. So, T is a relation from $\mathcal{E}$ to $\mathcal{F}$.

We trivially have $TT^{\mathrm{T}} \leq \mathcal{F}$, so it remains to prove that $\mathcal{E} \leq T^{\mathrm{T}}T$.

We have

$$\begin{aligned} \mathcal{E}_{ii'} &= \mathcal{E}_{ii} \wedge \mathcal{E}_{ii'} \wedge \mathcal{E}_{i'i'} \\ &= \left(\bigvee_{j\in J} T_{ji}\right) \wedge \mathcal{E}_{ii'} \wedge \left(\bigvee_{j'\in J} T_{j'i'}\right) \\ &= \left(\bigvee_{j\in J} T_{ji} \wedge \mathcal{E}_{ii'}\right) \wedge \left(\bigvee_{j'\in J} T_{j'i'} \wedge \mathcal{E}_{ii'}\right) \end{aligned}$$

Since $T\mathcal{E} = T$, we have $T_{ji} \wedge \mathcal{E}_{ii'} \leq T_{ji'}$, so, $\bigvee_{j\in J} T_{ji} \wedge \mathcal{E}_{ii'} \leq \bigvee_{j\in J} T_{ji} \wedge T_{ji'} = (T^{\mathrm{T}}T)_{ii'}$.

Therefore it also holds $\bigvee_{j'\in J} T_{j'i'} \wedge \mathcal{E}_{ii'} \leq (T^{\mathrm{T}}T)_{i'i} = (T^{\mathrm{T}}T)_{ii'}$. From this we conclude $\mathcal{E} \leq T^{\mathrm{T}}T$. ∎

It is interesting to remark that the isomorphisms on **Set**(B) do not preserve the “size” of the B-sets. For instance, if two rows of a B-set $\mathcal{E} : \Gamma \times \Gamma \to B$ are the same then we may delete one of them (and then delete the corresponding column) and obtain an isomorphic B-set $\mathcal{F} : H \times H \to B$ with $H \subset \Gamma$, defined for all $s, t \in H$ by

$$\mathcal{F}_{st} = \mathcal{E}_{st} \ ,$$

and, similarly, an isomorphism $T : \mathcal{F} \to \mathcal{E}$ is defined by

$$T : \Gamma \times H \to B$$

$$T_{st} = \mathcal{E}_{st} \ .$$

In order to see this, notice that for all $s, t \in H$ we have

$$(T^{\mathrm{T}}T)_{st} = \bigvee_{u\in\Gamma} T_{us} \wedge T_{ut} = \bigvee_{u\in\Gamma} \mathcal{E}_{us} \wedge \mathcal{E}_{ut} = (\mathcal{E}^{\mathrm{T}}\mathcal{E})_{st} = \mathcal{E}_{st} = \mathcal{F}_{st} \ ,$$

and thus $T^{\mathrm{T}}T = \mathcal{F}$; and, similarly, if $s, t \in \Gamma$ then

$$(TT^{\mathrm{T}})_{st} = \bigvee_{u \in H} T_{su} \wedge T_{tu} = \bigvee_{u \in H} \mathcal{E}_{su} \wedge \mathcal{E}_{tu} \ ,$$

and this equals

$$\bigvee_{w \in \Gamma} \mathcal{E}_{sw} \wedge \mathcal{E}_{tw} = (\mathcal{E}^{\mathrm{T}}\mathcal{E})_{st} = \mathcal{E}_{st}$$

because, by the construction of $\mathcal{F}$, for all $w \in \Gamma$ there is $u \in H$ such that $\mathcal{E}_{su} = \mathcal{E}_{sw}$ and $\mathcal{E}_{tu} = \mathcal{E}_{tw}$. Hence, $TT^{\mathrm{T}} = \mathcal{E}$.

Consider the set of applications of A on L, L^A, that is, the L-module freely generated by A. Note that the action is given by $(xv)_a = x \wedge v_a$, for all $x \in L$, $v \in L^A$ and $a \in A$, and, one has that

$$x(y(v)) = (x \wedge y)v \tag{3.14}$$

$$x\left(\bigvee_{i \in I} v^i = \bigvee_{i \in I} xv^i\right) \tag{3.15}$$

$$\left(\bigvee_{j \in J} x^j\right) v = \bigvee_{j \in J} (x^j v) \tag{3.16}$$

with $x, y, x^j \in L$ and $v, v^i \in L^A$.

A matrix $\mathcal{E} \in M_L(A)$ induces an L-homomorphism (i.e.a homomorphism of L-modules) $f_{\mathcal{E}} : L^A \to L^A$. Given by $f_{\mathcal{E}}(v)_a = \bigvee_{a \in A} \mathcal{E}_{ab} \wedge v_b$

that is,

$$f_{\mathcal{E}}(v) = \mathcal{E}v$$

3.4 Complete local sets

Definition 3.4.1 Let B be a locale and $\mathcal{E} : \Gamma \times \Gamma \to B$ be a B-set. A *singleton* is a map $s : \Gamma \to B$ such that:

1. $s(a) \wedge \mathcal{E}_{ab} \leq s(b)$ and
2. $s(a) \wedge s(b) \leq \mathcal{E}_{ab}$,

for all $a, b \in \Gamma$.

We will use a similar notation as with B-sets and since a singleton s is a vector we will write s_a instead of $s(a)$. Using this notation, the first condition is equivalent to $\bigvee_{a \in \Gamma} s_a \wedge \mathcal{E}_{ab} \leq s_b$, for all $b \in \Gamma$. So these two conditions are equivalent to

$$s\mathcal{E} \leq s \tag{4.17}$$

$$s^{\top} s \leq \mathcal{E} \tag{4.18}$$

Given a vector $v : \Gamma \to B$ we have $(vv^{\mathrm{T}}v)_b = \left(\bigvee_{a \in A} v_a\right) \wedge v_b = v_b$, therefore $v = vv^{\mathrm{T}}v$, for all $v : \Gamma \to B$.

Let B be a locale, Γ a set of indexes and $\mathcal{E} : \Gamma \times \Gamma \to B$ a B-set, then, if s is a singleton, we have $s\mathcal{E} \leq s = ss^{\mathrm{T}}s \leq s\mathcal{E}$ and therefore $s = s\mathcal{E}$.

In a B-set, $\mathcal{E}$ each $s \in \Gamma$ determines a row vector, formally:

Definition 3.4.2 Given a B-set $\mathcal{E} : \Gamma \times \Gamma \to B$ and $s \in \Gamma$ we define the vector $\tilde{s} : \Gamma \to B$ by, $\tilde{s}(t) = \mathcal{E}_{st}$, for all $t \in \Gamma$.

Lemma 3.4.3 *Given a B-set $\mathcal{E} : \Gamma \times \Gamma \to B$ and $s \in \Gamma$, the vector $\tilde{s}$ is a singleton.*

Proof. We have $(\tilde{s}\mathcal{E})_u = \bigvee_{u \in \Gamma} \mathcal{E}_{st} \wedge \mathcal{E}_{tu} = \mathcal{E}_{su} = \tilde{s}_u$, for all $u \in \Gamma$, and $\left(\tilde{s}^{\mathrm{T}}\tilde{s}\right)_{tu} = \tilde{s}_u \wedge \tilde{s}_t = \mathcal{E}_{st} \wedge \mathcal{E}_{su} \leq \bigvee_{s \in \Gamma} \mathcal{E}_{ts} \wedge \mathcal{E}_{su} = \mathcal{E}_{tu}$. ∎

Fourman and Scott [3] define on a local set a binary relation called equivalence by

$$[\![s \equiv t]\!] = (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \to \mathcal{E}_{st}$$

By definition of the implication, $\rightarrow$, on the Heyting algebra B,

$$[\![s \equiv t]\!] = \bigvee \{b \in B \mid (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge b \leq \mathcal{E}_{st}\}$$

Definition 3.4.4 A B-set is separated iff $[\![s \equiv t]\!] = 1$ always implies $s = t$.

Lemma 3.4.5 *Let $\mathcal{E} : \Gamma \times \Gamma \rightarrow B$ be a B-set, and let $s, t \in \Gamma$. The following conditions are equivalent*

(a) $\bigvee\{U \in B \mid (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge U \leq \mathcal{E}_{st}\} = 1$;

(b) $\mathcal{E}_{sw} = \mathcal{E}_{tw}$, $\forall w \in \Gamma$;

(c) $\tilde{s} = \tilde{t}$.

Proof. It is immediate that conditions (b) and (c) are equivalent. If condition (b) holds then, $\mathcal{E}_{ss} = \mathcal{E}_{ts} = \mathcal{E}_{st} = \mathcal{E}_{tt}$ and therefore $(\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge 1 = \mathcal{E}_{st}$ and so $\bigvee\{U \in B \mid (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge U \leq \mathcal{E}_{st}\} = 1$.

Suppose now that condition (a) holds. By distributivity we get
$\bigvee \{U \wedge \mathcal{E}_{ss} \mid U \in B, (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge U \leq \mathcal{E}_{st}\} = \mathcal{E}_{ss}$.
But $\{U \wedge \mathcal{E}_{ss} \mid U \in B, (\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge U \leq \mathcal{E}_{st}\}$ is equal to
$\{U \in B \mid U \leq \mathcal{E}_{ss}$ and $(\mathcal{E}_{ss} \vee \mathcal{E}_{tt}) \wedge U \leq \mathcal{E}_{st}\}$ which is equal to
$\{U \in B \mid U \leq \mathcal{E}_{ss}$ and $U \leq \mathcal{E}_{st}\} = \downarrow(\mathcal{E}_{ss} \wedge \mathcal{E}_{st})$.
Therefore $\mathcal{E}_{ss} \wedge \mathcal{E}_{st} = \mathcal{E}_{ss}$ and so $\mathcal{E}_{ss} \leq \mathcal{E}_{st}$.

From this we conclude $\mathcal{E}_{ss} = \mathcal{E}_{st}$ and, since $\mathcal{E}$ is symmetric, also $\mathcal{E}_{st} = \mathcal{E}_{tt}$.

Let $w \in \Gamma$, $\mathcal{E}_{sw} = \mathcal{E}_{ss} \wedge \mathcal{E}_{sw} = \mathcal{E}_{ts} \wedge \mathcal{E}_{sw} \leq \bigvee_{v \in \Gamma} \mathcal{E}_{tv} \wedge \mathcal{E}_{vw} = \mathcal{E}_{tw}$.

Permuting the roles of s and t we get the other inequality $\mathcal{E}_{tw} \leq \mathcal{E}_{sw}$. ∎

From this lemma we immediately get

Proposition 3.4.6 *A B-set $\mathcal{E} : \Gamma \times \Gamma \rightarrow B$ is separated iff, for all $s, t \in \Gamma$, $\tilde{s} = \tilde{t}$ implies $s = t$.*

Definition 3.4.7 A B-set $\mathcal{E} : \mathit{\Gamma} \times \mathit{\Gamma} \to B$ is a *complete B-set* if every singleton v there exists one, and only one, $s \in \mathit{\Gamma}$ such that $v = \tilde{s}$

In other words, a B-set is complete if, and only if, all singletons came from a row of the matrix, and different rows form different singletons.

Theorem 3.4.8 *Every B-set $\mathcal{E} : \Gamma \times \Gamma \to B$ is isomorphic, in the category* **Set**(B)*, to a complete B-set.*

Proof. Let Σ be the set of all singletons of the B-set $\mathcal{E}$. Define $T : \Sigma \times \Gamma \to B$ by $T_{sa} = s(a), \forall a \in \Gamma, s \in \Sigma$. And set $\mathcal{F} = TT^{\mathrm{T}}$.

Given $s \in \Sigma$, and,$a \in \Gamma$,

$$(T\mathcal{E})_{sa} = \bigvee_{b\in\Gamma} s(b) \wedge \mathcal{E}_{ba} = (s\mathcal{E})_a = s(a) = T_{sa}$$

Given $a, b \in \Gamma$,

$$\left(T^{\mathrm{T}}T\right)_{ab} = \bigvee_{s\in\Sigma} (T_{sa} \wedge T_{sb}) = \bigvee_{s\in\Sigma} (s(a) \wedge s(b)) = \mathcal{E}_{ab}$$

This last equality holds because the rows of the matrix $\mathcal{E}$ are singletons, so $\bigvee_{s\in\Sigma} (s(a) \wedge s(b)) \geq \bigvee_{c\in\Gamma} (\tilde{c}(a) \wedge \tilde{c}(b)) = \bigvee_{c\in\Gamma} \mathcal{E}_{ac} \wedge \mathcal{E}_{cb} = \mathcal{E}_{ab}$.

Therefore, we have $T = T\mathcal{E} = TT^{\mathrm{T}}T = \mathcal{F}T$. Since, $T^{\mathrm{T}}T = \mathcal{E}$ and $TT^{\mathrm{T}} = \mathcal{F}$, the morphism $T : \mathcal{E} \to \mathcal{F}$ is an isomorphism whose inverse is T^{T}.

Now we will show that $\mathcal{F}$ is a complete B-set.

Let w be a singleton of $\mathcal{F}$, then we have $w\mathcal{F} = w$ and $w^{\mathrm{T}}w \leq \mathcal{F}$, it follows that

$$(wT)\,\mathcal{E} = wTT^{\mathrm{T}}T = w\mathcal{F}T = wT$$

and

$$(wT)^{\mathrm{T}}\,(wT) = T^{\mathrm{T}}w^{\mathrm{T}}wT \leq T^{\mathrm{T}}\mathcal{F}T = T^{\mathrm{T}}T = \mathcal{E}$$

and therefore wT is a singleton of the B-set $\mathcal{E}$.

So there is a unique $v \in \Sigma$ such that $wT = v$.

Given $s \in \Sigma$,

$$\begin{aligned}\tilde{v}_s = \mathcal{F}_{vs} = \left(TT^{\mathrm{T}}\right)_{vs} &= \bigvee_{a \in \Gamma} T_{va} \wedge T_{sa} \\ &= \bigvee_{a \in \Gamma} v(a) \wedge T_{sa} \\ &= \bigvee_{a \in \Gamma} (wT)_a \wedge T_{sa} \\ &= \left(wTT^{\mathrm{T}}\right)_s = (w\mathcal{F})_s = w_s\end{aligned}$$

Therefore $w = \tilde{v}$ and so, it exists $v \in \Sigma$ such that $w = \tilde{v}$.

Note that, in previous equalities we also proved that $\tilde{v} = vT^{\mathrm{T}}$. As a consequence, the correspondence $v \mapsto vT^{\mathrm{T}}$, for $v \in \Sigma$, is a bijection, between the singelons of $\mathcal{E}$ and the singletons of $\mathcal{F}$, whose inverse is $w \mapsto wT$ and for all singleton w of $\mathcal{F}$ there is one, and only one $v \in \Sigma$ such that $w = \tilde{v}$, furthermore $v = wT^{\mathrm{T}}$. ∎

3.5 Sheaves and presheaves II

In this section we introduce a different but equivalent definition of presheaf (which we will call FSsheaf) that is presented in [3].

Definition 3.5.1 A *FSpresheaf* over a locale B, is a triple $(A, \upharpoonright, E)$ where A is a set, $E : A \to B$ is a map from A into B called *extent* and $\upharpoonright : A \times B \to A$ is a map called *restriction* satisfying:

1. $s \upharpoonright_{E(s)} = s$;
2. $(s \upharpoonright_b) \upharpoonright_{b'} = s \upharpoonright_{b \wedge b'}$;
3. $E\left(s \upharpoonright_b\right) = E(s) \wedge b$,

For all $s \in A$ and all $b, b' \in B$.

Proposition 3.5.2 *Given a FSprsheaf* $(A, \upharpoonright, E)$ *over* B *the matrix* $\mathcal{E} : A \times A \to B$ *defined by*

$$\mathcal{E}_{st} = \bigvee\{b \leq E(s) \wedge E(t) \mid s\upharpoonright_b = t\upharpoonright_b\}$$

Is a B*-set and satisfies*

$$\mathcal{E}_{s\upharpoonright_b t} = \mathcal{E}_{st} \wedge b$$

Proof. It is immediate that $\mathcal{E}$ is a symmetric matrix, so, in order to show that $\mathcal{E}$ is a B-set, we just need to show that $\mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \mathcal{E}_{su}$. Using distributivity we have

$\mathcal{E}_{st} \wedge \mathcal{E}_{tu} = \bigvee\{b \wedge b' \mid b \leq E(s) \wedge E(t);\ b' \leq E(t) \wedge E(u);\ s\upharpoonright_b = t\upharpoonright_b;\ t\upharpoonright_{b'} = u\upharpoonright_{b'}\}$

And, therefore, $\mathcal{E}_{st} \wedge \mathcal{E}_{tu} \leq \bigvee\{c \leq E(s) \wedge E(u) \mid s\upharpoonright_c = u\upharpoonright_c\} = \mathcal{E}_{su}$.

In order to prove $\mathcal{E}_{s\upharpoonright_b t} = \mathcal{E}_{st} \wedge b$, note that

$$\mathcal{E}_{s\upharpoonright_b t} = \bigvee\{c \leq E(s) \wedge b \wedge E(t) \mid s\upharpoonright_c = t\upharpoonright_c\}$$

So, $\mathcal{E}_{s\upharpoonright_b t} \leq \mathcal{E}_{st} \wedge b$.

On the other hand, by distributivity,

$\mathcal{E}_{st} \wedge b = \bigvee\{c \wedge b \mid c \leq E(s) \wedge E(t); s\upharpoonright_c = t\upharpoonright_c\}$ and if $c \leq E(s) \wedge E(t)$ and $s\upharpoonright_c = t\upharpoonright_c$ then $c \wedge b \leq E(s) \wedge b \wedge E(t)$ and $s\upharpoonright_{c\wedge b} = s\upharpoonright_c\upharpoonright_{c\wedge b} = t\upharpoonright_c\upharpoonright_{c\wedge b} = t\upharpoonright_{c\wedge b}$. So $\{c \wedge b \mid c \leq E(s) \wedge E(t); s\upharpoonright_c = t\upharpoonright_c\} \subseteq \{c \leq E(s) \wedge b \wedge E(t) \mid s\upharpoonright_c = t\upharpoonright_c\}$ and therefore $\mathcal{E}_{st} \wedge c \leq \mathcal{E}_{s\upharpoonright_b t}$. ∎

Definition 3.5.3 Given elements x, y in a FSpresheaf X over B, we say x is a restriction of y, or y is an extension of x, and write $x \leq y$, iff $x = y\upharpoonright_{E(x)}$.

Lemma 3.5.4 *Let* X *be a FSpresheaf, then:*

1. $x \leq y$ *implies* $E(x) \leq E(y)$

2. *The relation $\leq$ is a partial order on X;*

Proof. If $x \leq y$, then $E(x) = E(y \restriction_{E(x)}) = E(y) \wedge E(x)$ so $E(x) \leq E(y)$.

Now we show that $\leq$ is a partial order on X.

Consider $x, y, z \in X$:

$x \restriction_{E(X)} = x$ implies $x \leq x$;

If $x \leq y$ and $y \leq x$ then, $E(x) \leq E(y)$ and $E(y) \leq E(x)$, hence $E(x) = E(y)$, and therefore, $x = y \restriction_{E(x)} = y \restriction_{E(y)} = y$.

If $x \leq y$ and $y \leq z$ then

$x = y \restriction_{E(x)} = \left(z \restriction_{E(y)}\right) \restriction_{E(x)} = z \restriction_{E(y) \wedge E(x)} = z \restriction_{E(x)}$, and consequently $x \leq z$.

Therefore $\leq$ is a partial order on X.

Definition 3.5.5 We say that $Y \subseteq X$ is a *compatible* if, for any two elements $y, y' \in Y$, $y \restriction_{E(y')} = y' \restriction_{E(Y)}$. A *join* for $B \subseteq X$ is a minimal upper bound for B under $\leq$[1]

A *FSsheaf* is a FSpresheaf such that every compatible family $Y \subseteq X$ has a unique join.

Proposition 3.5.6 *A separated presheaf is equivalent to separated a FSpresheaf*

Proof. Given a separated presheaf $F : L \to \mathbf{Sets}$ define extent $E : F(L) \to L$ by $E(F(x)) = x$ and restriction $\restriction : F(L) \times L \to F(L)$ by $F(x) \restriction_y = F(x \wedge y)$.

We have $F(x) \restriction_x = F(x)$ and $(F(x) \restriction_y) \restriction_z = F(x \wedge y \wedge z) = F(x) \restriction_{y \wedge z}$. And also $E(F(x) \restriction_y) = E(F(x \wedge y)) = x \wedge y = E(F(x)) \wedge y$. So every presheaf induces an FSpresheaf.

Given a FSpresheaf $(A, \restriction, E)$ we define a functor F from L to the category of sets, by $F(x) = \{a \in A \mid E(x) = a\}$. If $y \leq x$, (note that $y \leq x$ is the same

[1]Here we follow Fourman and Scot, but the reader should be careful because this is not the usual definition of join in a partial ordered set, usually the join is the least upper bound.

as saying that there is a morphism from y to x on the category correspondent to L) this morphism is transformed by F on the function between the sets $F(x)$ and $F(y)$ defined by $a \mapsto a\restriction_y$. If $y \leq x$ and. $E(x) = a$ then $E(a\restriction_y) = E(a) \wedge y = x \wedge y = y$ so $a\restriction_y \in F(Y)$ Furthermore, $(a\restriction_x)\restriction_y = a\restriction_{x\wedge y}$, and $a\restriction_{E(a)} = a$, so F is a contravariant functor from L to the category of sets.

$\blacksquare$

It is shown on [3] that a FSsheaf sheaf is equivalent to a complete B-set and, using proposition 3.3.13, it shown on [8] the category **Set**(B) is equivalent to the category of sheaves over B.

Chapter 4

Continuous maps as modules

4.1 General continuous maps

Let $p : X \to B$ be a map of locales. Then X is a B-module by "change of base ring" along the homomorphism $p^* : B \to X$: the action is given by, for all $x \in X$ and $b \in B$,

$$bx = p^*(b) \wedge x \; .$$

It follows that $b1 = p^*(b)$ and thus this module satisfies the condition

$$bx = b1 \wedge x \; , \tag{1.1}$$

which, as we shall see, completely characterizes the modules that arise in this way. (This condition has been called *stability* in [19], in the more general context of modules over unital quantales.) We remark that the action of such a module distributes over meets of non-empty sets $S \subset X$ in the right variable: $b\left(\bigwedge S\right) = b1 \wedge \bigwedge S = \bigwedge_{x\in S}(b1 \wedge x) = \bigwedge_{x \in S} bx$.

Let us define some terminology:

Definition 4.1.2 Let B be a locale. By a *B-locale* will be meant a locale X equipped with a structure of B-module satisfying (1.1). A *homomorphism* of

B-locales is a homomorphism of locales that is also a homomorphism of B-modules, and a *map* $f : X \to Y$ of B-locales is defined to be a homomorphism $f^* : Y \to X$ of B-locales. The *category of B-locales*, denoted by B-**Loc**, has as objects the B-locales and as morphisms the maps of B-locales. We shall denote the category $(B\text{-}\mathbf{Loc})^{\mathrm{op}}$ by B-**Frm** (the *category of B-frames*).

Theorem 4.1.3 *The category B-***Loc** *is isomorphic to* $\mathbf{Loc}/B$.

Proof. Each object $p : X \to B$ of $\mathbf{Loc}/B$ gives us a B-locale, as we have seen in the beginning of this section. Conversely, let X be a B-locale. Define a function $\phi : B \to X$ by

$$\phi(b) = b1 \ .$$

We have $\phi(1) = 11 = 1$, $\phi(b \wedge c) = (b \wedge c)1 = b(c1) = b1 \wedge c1 = \phi(b) \wedge \phi(c)$, and $\phi(\bigvee_\alpha b_\alpha) = (\bigvee_\alpha b_\alpha)1 = \bigvee_\alpha b_\alpha 1 = \bigvee_\alpha \phi(b_\alpha)$; that is, ϕ is a homomorphism of locales, and thus we have obtained a map $p : X \to B$ defined by $p^* = \phi$. This correspondence between objects of $\mathbf{Loc}/B$ and B-locales is clearly a bijection.

In order to see that the categories are isomorphic let $p : X \to B$ and $q : Y \to B$ be objects of $\mathbf{Loc}/B$, and let $f : X \to Y$ be a map of locales. We show that f is a morphism from p to q in $\mathbf{Loc}/B$ if and only if it is a map from X to Y in B-**Loc**. Let $b \in B$ and $y \in Y$. We have

$$f^*(by) = f^*(q^*(b) \wedge y) = (q \circ f)^* \wedge f^*(y)$$

and also

$$bf^*(y) = p^*(b) \wedge f^*(y) \ .$$

It follows that if $p = q \circ f$ then $f^*(by) = bf^*(y)$ for all $b \in B$ and $y \in Y$; that is, if f is in $\mathbf{Loc}/B$ then f^* is a homomorphism of B-modules and thus f is in B-**Loc**. Conversely, if f^* is a homomorphism of B-modules then letting $y = 1$ above we obtain $q \circ f = p$. ∎

From now on we shall freely identify B-modules with their associated locale maps, for instance calling B-module to a map $p : X \to B$, and for convenience we shall often refer to p as the *projection* of the B-module.

4.2 Open maps

Definition 4.2.1 A B-locale X is *open* if its projection p is an open map of locales; that is, p^* has a left adjoint $p_!$ which is a homomorphism of B-modules (but not in general a homomorphism of B-locales).

It is obvious that the direct image $p_!$ of the projection p of an open B-locale satisfies the property

$$p_!(x)x = x \ ,$$

for $p_!(x)x = p^*(p_!(x)) \wedge x$ and thus the equality $p_!(x)x = x$ is equivalent to the unit of the adjunction $p_! \dashv p^*$. This has a converse: if $\varsigma : X \to B$ is B-equivariant and monotone and it satisfies

$$\varsigma(x)1 \geq x$$

then ς is left adjoint to the map $(-)1 : B \to X$; the condition $\varsigma(x)1 \geq x$ is the unit of the adjunction and the counit $\varsigma(b1) \leq b$ is an immediate consequence of the equivariance, for $\varsigma(b1) = b \wedge \varsigma(1) \leq b$. This actually holds for any B-module, but for a B-locale X the condition $\varsigma(x)1 \geq x$ (equivalently, $\varsigma(x)x = x$ because $\varsigma(x)x = \varsigma(x)1 \wedge x$) implies that X is open. Summarizing, we have:

Theorem 4.2.2 *A B-locale X is open if and only if there is a monotone equivariant map*

$$\varsigma : X \to B$$

such that the following (necessarily equivalent) conditions are satisfied for all $x \in X$:

$$\varsigma(x)1 \geq x \tag{2.3}$$

$$\varsigma(x)x \geq x \tag{2.4}$$

$$\varsigma(x)x = x\ . \tag{2.5}$$

Furthermore there is at most one such map ς. If one exists it is necessarily a B-module homomorphism and it coincides with the direct image $p_!$ of the projection p of X.

Alternatively, a slightly different characterization of open B-locales is the following:

Theorem 4.2.6 *Let X be a locale which is also a B-module (but not necessarily a B-locale). Then X is an open B-locale if and only if there is a monotone equivariant map $\varsigma : X \to B$ such that (2.5) holds for all $x \in X$.*

Proof. By the previous theorem any open B-locale has such a map ς, so we only have to prove the converse. Assume that $\varsigma : X \to B$ is monotone, equivariant, and that it satisfies (2.5). Then we have

$$b1 \wedge x = \varsigma(b1 \wedge x)(b1 \wedge x) \leq \varsigma(b1)x = (b \wedge \varsigma(1))x \leq bx\ .$$

The converse inequality, $bx \leq b1 \wedge x$, is obvious, and thus (1.1) holds. This shows that X is an open B-locale. ∎

If X is an open B-locale with projection p then $p_!$ will be referred to as the *support* of X, and we shall usually write ς instead of $p_!$, following the analogous notation for supported quantales [17]. Similarly, we may refer to $p_!(x)$ as the *support* of x.

Example 4.2.7 Let S be any set. Then the free B-module generated by S, which is the function module B^S of maps $f : S \to B$, is an open B-locale whose support is defined by $\varsigma(f) = \bigvee_{s\in S} f(s)$. The projection of the B-locale is the obvious map $p : \coprod_{s\in S} B \to B$, where $\coprod_{s\in S} B$ is the coproduct in **Loc** of as many copies of B as there are elements in S; in other words, $p^* : B \to B^S$ is the diagonal homomorphism that to each $b \in B$ assigns the map $f : S \to B$ such that $f(s) = b$ for all $s \in S$.

Example 4.2.8 In a topos, any locale X has a unique Ω-locale structure determined by the continuous map $!_X : X \to \Omega$, and X is an open Ω-locale precisely if it is open in the usual sense [9]. The tensor product $B \otimes X$ (the product $B \times X$ in **Loc**) is a B-locale with action $a(b \otimes x) = (a \wedge b) \otimes x$ and projection $\pi_1^*(b) = b \otimes 1$, and it is open if X is open (because π_1 is the pullback of $!_X$ along $!_B$). Its support is computed from the Ω-action on B by $\varsigma(b \otimes x) = \varsigma(x)b$. In **Sets** every locale is open and the support of $B \otimes X$ is defined by the conditions $\varsigma(b \otimes 0) = 0$ and $\varsigma(b \otimes x) = b$ if $x \neq 0$.

4.3 Local homeomorphisms

Let $p : X \to B$ be a local homeomorphism, and let Γ be a cover of X (i.e., $\Gamma \subset X$ and $\bigvee \Gamma = 1$) such that, on each open sublocale determined by an element of Γ, p restricts to a homeomorphism onto its image; that is, for each $s \in \Gamma$ there is a commutative square

$$\begin{array}{ccc} B & \xrightarrow{p^*} & X \\ {\scriptstyle (-)\wedge\varsigma(s)}\downarrow & & \downarrow{\scriptstyle (-)\wedge s} \\ \downarrow\varsigma(s) & \xrightarrow{\theta_s} & \downarrow s \end{array} \qquad (3.1)$$

such that θ_s is an isomorphism. Then we have, for each $b \leq \varsigma(s)$,

$$\theta_s(b) = p^*(b) \wedge s = bs$$

and

$$\varsigma(\theta_s(b)) = \varsigma(bs) = b \wedge \varsigma(s) = b\ .$$

Hence, the restriction to $\downarrow s$ of ς splits θ_s, and thus it coincides with θ_s^{-1}. This motivates the following definition:

Definition 4.3.2 Let X be an open B-locale. A *local section* of X is an element $s \in X$ such that for all $x \leq s$ we have

$$\varsigma(x)s = x\ . \tag{3.3}$$

The set of local sections of X is denoted by Γ_X, and X is defined to be *étale* if $\bigvee \Gamma_X = 1$.

If s is a local section then

$$p^*(1) \wedge s = 1 \wedge s = s = p^*(\varsigma(s)) \wedge s\ ,$$

which means that the homomorphism $((-) \wedge s) \circ p^*$ factors as in (3.1). Moreover, the equivariance of ς gives us $\varsigma(bs) = b$ for all $b \leq \varsigma(s)$; this, together with (3.3), ensures that θ_s an isomorphism, and therefore we have the following characterization of local homeomorphisms:

Theorem 4.3.4 *An open B-locale is étale if and only if its projection is a local homeomorphism.*

Let us denote by **LH** the subcategory of **Loc** whose objects are the locales and whose morphisms are the local homeomorphisms. It follows from a basic property of local homeomorphisms that **LH**$/B$ is a full subcategory of **Loc**$/B$, and thus **LH**$/B$ is isomorphic to the following category, which provides our first example of a "category of sheaves as modules":

Definition 4.3.5 The *category of étale B-locales*, denoted by B-**LH**, is the full subcategory of B-**Loc** whose objects are the étale B-locales.

4.4 Sheaf homomorphisms

Although B-**LH** is meant to be a "category of sheaves as modules", it is not a category of modules; that is, $(B\text{-}\mathbf{LH})^{\mathrm{op}}$, rather than B-**LH**, is a subcategory of B-**Mod**. In order to remedy this let us first introduce the following terminology:

Definition 4.4.1 Let X and Y be étale B-locales. By a *sheaf homomorphism*

$$h : X \to Y$$

will be meant a homomorphism of B-modules satisfying the following two conditions:

1. $h(\mathit{\Gamma}_X) \subset \mathit{\Gamma}_Y$;
2. $\varsigma(h(s)) = \varsigma(s)$ for all $s \in \mathit{\Gamma}_X$ (equivalently, $\varsigma(h(x)) = \varsigma(x)$ for all $x \in X$).

The sheaf homomorphisms form a category that we shall denote by B-**Sh**.

The motivation for this terminology comes from the fact that, denoting by $\mathbf{Sh}(B)$ the category of sheaves on B in the usual sense (sheaves are separated and complete presheaves $B^{\mathrm{op}} \to \mathbf{Sets}$ and their morphisms are the natural transformations), we have a functor $G : B\text{-}\mathbf{Sh} \to \mathbf{Sh}(B)$ (which is part of an equivalence of categories — see the comments below) such that: (i) G assigns to each étale B-locale X the sheaf $G_X : B^{\mathrm{op}} \to \mathbf{Sets}$ defined for each $b \in B$ by $G_X(b) = \{s \in \mathit{\Gamma}_X \mid \varsigma(s) = b\}$, with the restriction map $G_X(b) \to G_X(a)$ for each pair $a \leq b$ in B being given by $s \mapsto as$; (ii) G assigns to each sheaf homomorphism $h : X \to Y$ the natural transformation $G_h : G_X \to G_Y$ whose components $(G_h)_b$ are all defined by $s \mapsto h(s)$.

Lemma 4.4.2 *If $f : X \to Y$ is a map of étale B-locales then $f_! : X \to Y$ is a sheaf homomorphism.*

Proof. Let $f : X \to Y$ be a map of B-locales with projections p and q, respectively. Then $f_!$ satisfies $\varsigma_Y(f_!(x)) = q_!(f_!(x)) = p_!(x) = \varsigma_X(x)$ for all $x \in X$. In addition, $f_!$ is Y-equivariant, and thus it is B-equivariant for the module structures of X and Y induced by $f^* \circ q^*$ and q^*, respectively. Finally, composing f with a local section of p yields a local section of q — a module theoretic proof of this is as follows: if $y \leq f_!(s)$ and $s \in \Gamma_X$ then

$$y = y \wedge f_!(s) = f_!(f^*(y) \wedge s) = f_!(\varsigma(f^*(y) \wedge s)s) = \varsigma(f^*(y) \wedge s)f_!(s) ,$$

and thus y is a restriction of $f_!(s)$. ∎

Hence, we have a functor $\mathcal{S} : B\text{-}\mathbf{LH} \to B\text{-}\mathbf{Sh}$ which is the identity on objects and to each map f assigns $f_!$. Using the (localic) correspondence between sheaves and local homeomorphisms (as in, e.g., [1, §2] or [8, pp. 502–513]) it is not hard to see that $\mathcal{S}$ is part of an adjoint equivalence of categories whose other functor is the composition

$$B\text{-}\mathbf{Sh} \xrightarrow{G} \mathbf{Sh}(B) \xrightarrow{\Lambda} \mathbf{LH}/B \xrightarrow{\cong} B\text{-}\mathbf{LH} ,$$

where, concretely, Λ can be the functor that to each sheaf assigns its locale of closed subobjects as in [1, §2.2]. But in fact one can prove something stronger:

Theorem 4.4.3 *The functor $\mathcal{S} : B\text{-}\mathbf{LH} \to B\text{-}\mathbf{Sh}$ is an isomorphism.*

A direct proof of this, using properties of Hilbert modules, will be postponed until §5.4.

Chapter 5

Hilbert modules with Hilbert bases

5.1 Supported modules and open B-locales

Let us see some relations between Hilbert B-modules and open B-locales.

We formulate where the definition of (pre)-Hilbert B-module present in [13] when the quantale is simply a locale B.

Definition 5.1.1 A (pre-)Hilbert B-module X is *supported* if it satisfies the condition

$$\langle x, x\rangle x = x$$

for all $x \in X$.

Theorem 5.1.2 *Any locale X which is also a supported Hilbert B-module is an open B-locale; its support function ς is defined by $\varsigma x = \langle x, x\rangle$.*

Proof. Assume that X is a locale equipped with a structure of Hilbert B-module. Then it is a B-locale due to 2.4.6. Besides, the function $\varsigma : X \to B$

defined by

$$\varsigma x = \langle x, x \rangle$$

is monotone and B-equivariant, and by hypothesis it satisfies $\varsigma(x)x = x$, whence by 4.2.2 X is open. ∎

There is a partial converse to this theorem:

Theorem 5.1.3 *Let X be an open B-locale. Then X is a supported pre-Hilbert B-module whose inner product is* weakly non-degenerate *in the sense that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in X$ then $\neg x = \neg y$ (where $\neg$ is the Heyting algebra pseudo-complement: $\neg x = x \to 0$).*

Proof. If X is an open B-locale we define $\langle x, y \rangle = \varsigma(x \wedge y)$. Being a B-locale implies that for all $x, y \in X$ and all $b \in B$ we have

$$bx \wedge y = (b1 \wedge x) \wedge y = b1 \wedge (x \wedge y) = b(x \wedge y) \ ,$$

and thus

$$\langle bx, y \rangle = \varsigma(bx \wedge y) = \varsigma(b(x \wedge y)) = b \wedge \varsigma(x \wedge y) = b \wedge \langle x, y \rangle \ .$$

If $y \in X$ and (x_α) is a family of elements in X we have

$$\left\langle \bigvee_\alpha x_\alpha, y \right\rangle = \varsigma \left(\bigvee_\alpha x_\alpha \wedge y \right) = \bigvee_\alpha \varsigma(x_\alpha \wedge y) = \bigvee_\alpha \langle x_\alpha, y \rangle \ .$$

Since $\langle -, - \rangle$ is of course symmetric, it follows that X is a pre-Hilbert B-locale. For the weak non-degeneracy let $x, y \in X$ be such that $\varsigma(x \wedge z) = \varsigma(y \wedge z)$ for all $z \in X$. Then, letting $z = \neg y$, we obtain

$$0 = \varsigma 0 = \varsigma(y \wedge \neg y) = \varsigma(x \wedge \neg y) \ ,$$

and thus $x \wedge \neg y = \varsigma(x \wedge \neg y)(x \wedge \neg y) = 0$. Hence, $\neg y \leq \neg x$. Similarly, letting $z = \neg x$ we conclude that $\neg x \leq \neg y$. ∎

The "weakly" in the theorem cannot be dropped. In order to see this, consider as an example of open map the first projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$. Let U be an open ball centered on $(0,0) \in \mathbb{R}^2$, and let $V = U \setminus \{(0,0)\}$. For all open sets $W \in \Omega(\mathbb{R}^2)$ we have $\pi_1(U \cap W) = \pi_1(V \cap W)$, but $U \neq V$ and thus the inner product associated to π_1 is degenerate.

5.2 Hilbert bases

Let us introduce a natural notion in the context of Hilbert B-modules, namely the analogue of a Hilbert basis of a Hilbert space. As we shall see, the existence of such a basis has strong consequences, notably modules equipped with a Hilbert basis are necessarily étale B-locales.

Definition 5.2.1 Let X be a pre-Hilbert B-module. By a *Hilbert basis* of X is meant a subset $\Gamma \subset X$ such that for all $x \in X$ we have

$$x = \bigvee_{s \in \Gamma} \langle x, s \rangle s \ .$$

(In particular, Γ is therefore a set of B-module generators for X.)

A Hilbert basis in this sense is not an actual basis as in linear algebra because there is no freeness (we only have projectivity — see 5.2.3 below). Therefore one might be better off calling it a Hilbert system of generators, but for the sake of simplicity we shall retain the shorter terminology.

Example 5.2.2 Let S be a set. The free B-module B^S (cf. Example 4.2.7) has a Hilbert basis Γ consisting of the "unit vectors" $f^{(s)} : S \to B$; for each $s \in S$ we define $f^{(s)} = \iota_s(1)$ where $\iota_s : B \to B^S \cong \bigoplus_{s \in S} B$ is the coproduct injection corresponding to the s-labeled copy of B. This definition of $f^{(s)}$ makes sense in any topos and it is equivalent, if S is decidable, to the following:

$$f^{(s)}(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \ . \end{cases}$$

The existence of a Hilbert basis has many useful consequences. In particular, any pre-Hilbert B-module with a Hilbert basis is necessarily supported, hence strict, and it is projective:

Lemma 5.2.3 *Let X be a pre-Hilbert B-module and let $\Gamma \subset X$. If Γ is a Hilbert basis then the following properties hold, for all $x, y \in X$.*

1. *X is a projective B-module.*
2. *$\bigvee \Gamma = 1$. (Γ is a* cover *of X.)*
3. *If $\langle x, s\rangle = \langle y, s\rangle$ for all $s \in \Gamma$ then $x = y$. (Hence, X is a Hilbert module.)*
4. *$\langle x, y\rangle = \bigvee_{s\in\Gamma} \langle x, s\rangle \wedge \langle s, y\rangle$.*
5. *$\langle x, x\rangle x = x$. (Hence, X is supported.)*
6. *$\langle x, y\rangle \leq \langle x, x\rangle$.*
7. *For all $s \in \Gamma$ the following conditions are equivalent:*
   - *(a) $x \leq s$;*
   - *(b) $x = \langle x, x\rangle s$;*
   - *(c) $x = \langle x, s\rangle s$.*
8. *The B-valued matrix $M : \Gamma \times \Gamma \to B$ defined by $m_{st} = \langle s, t\rangle$ (the "metric" of the inner product) is a projection matrix: $M^{\mathrm{T}} = M^2 = M$ (hence, M defines a B-set in the sense of [3]).*

Conversely, Γ is a Hilbert basis if $\langle -,-\rangle$ is non-degenerate and 4 holds.

Proof. Assume that Γ is a Hilbert basis. The first eight properties are proved as follows.

1. Since Γ is a set of B-module generators and B^Γ is a free module, there is a quotient of B-modules $\varphi : B^\Gamma \to X$ given by $\varphi(f) = \bigvee_{s\in\Gamma} f(s)s$, and in the opposite direction we define another homomorphism of B-modules $\psi : X \to B^\Gamma$ by $\psi(x)(s) = \langle x, s\rangle$. This splits φ, showing that X is a retract of a free module:
$$\varphi(\psi(x)) = \bigvee_{s\in\Gamma} \psi(x)(s)s = \bigvee_{s\in\Gamma} \langle x, s\rangle s = x\ .$$
2. $1 = \bigvee_{s\in\Gamma} \langle 1, s\rangle s \le \bigvee_{s\in\Gamma} 1s = \bigvee \Gamma$.
3. If $\langle x, s\rangle = \langle y, s\rangle$ for all $s \in \Gamma$ then $x = \bigvee_{s\in\Gamma} \langle x, s\rangle s = \bigvee_{s\in\Gamma} \langle y, s\rangle s = y$.
4. $\langle x, y\rangle = \big\langle \bigvee_{s\in\Gamma} \langle x, s\rangle s, y\big\rangle = \bigvee_{s\in\Gamma} \langle x, s\rangle \wedge \langle s, y\rangle$.
5. For all $x \in X$ and $s \in \Gamma$ we have $\langle \langle x, x\rangle x, s\rangle = \langle x, x\rangle \wedge \langle x, s\rangle = \bigvee_{t\in\Gamma} \langle x, t\rangle \wedge \langle t, x\rangle \wedge \langle x, s\rangle = \bigvee_{t\in\Gamma} \langle x, t\rangle \wedge \langle x, t\rangle \wedge \langle x, s\rangle = \langle x, s\rangle$, and thus by the non-degeneracy we conclude $\langle x, x\rangle x = x$.
6. Using 5 we have $\langle x, y\rangle = \langle \langle x, x\rangle x, y\rangle = \langle x, x\rangle \wedge \langle x, y\rangle$.
7. Either of the equations 7b or 7c implies 7a, of course, so let us assume that $x \le s$ in order to verify the converse implication. By 6 we have $\langle x, x\rangle = \langle x, s\rangle$ and thus 7b and 7c are equivalent; in addition, we have $\langle x, x\rangle s \ge \langle x, x\rangle x = x$, and, conversely, $\langle x, x\rangle s = \langle x, s\rangle s \le \bigvee_{t\in\Gamma} \langle x, t\rangle t = x$, whence $x = \langle x, x\rangle s = \langle x, s\rangle s$.
8. We have $M = M^{\mathrm{T}}$ by definition of the inner product, and $M = M^2$ follows from 4.

Now assume that $\langle -,-\rangle$ is non-degenerate and that 4 holds. Then for all $x, y \in X$ we have

$$\left\langle \bigvee_{s\in\Gamma} \langle x,s\rangle s, y \right\rangle = \bigvee_{s\in\Gamma} \langle x,s\rangle \wedge \langle s,y\rangle = \langle x,y\rangle \ ,$$

and by the non-degeneracy we obtain $\bigvee_{s\in\Gamma} \langle x,s\rangle s = x$. ∎

Proposition 5.2.3-8 has a converse, namely every projection matrix has an associated Hilbert module (in fact a locale — cf. 5.3.1) with a Hilbert basis (we shall write Mf for the product of the matrix M by the "column vector" $f : S \to B$ — that is, writing also f_s instead of $f(s)$ for such "vectors", we have $(Mf)_s = \bigvee_{t\in S} m_{st} \wedge f_t$):

Lemma 5.2.4 *Let S be a set and $M : S \times S \to B$ a B-valued projection matrix. Then the subset of B^S*

$$MB^S = \{Mf \mid f \in B^S\}$$

is a Hilbert B-module with the same inner product as B^S, it has a Hilbert basis Γ consisting of the functions $\tilde{s} : S \to B$ defined, for each $s \in S$, by $\tilde{s}_t = m_{ts}$ ($\tilde{s}$ is the "s^{th}-column" of M), and for all $s, t \in S$ we have

$$m_{st} = \langle \tilde{s}, \tilde{t}\rangle \ .$$

Proof. The assignment $j : f \mapsto Mf$ is a B-module endomorphism of B^S, and MB^S is its image, hence a submodule of B^S. Next note that Γ is a subset of MB^S because for each $t \in S$ we have $\tilde{t} = M\tilde{t} \in MB^S$:

$$\tilde{t}_s = m_{st} = (M^2)_{st} = \bigvee_{u\in S} m_{su} \wedge m_{ut} = \bigvee_{u\in S} m_{su} \wedge \tilde{t}_u = (M\tilde{t})_s \ .$$

For all $f \in MB^S$ we have $f = Mf$ and thus it follows that, for all $s \in \Gamma$,

$$\langle f, \tilde{s}\rangle = \bigvee_t f_t \wedge \tilde{s}_t = \bigvee_t f_t \wedge m_{ts} = \bigvee_t m_{st} \wedge f_t = (Mf)_s = f_s \ .$$

Hence, Γ is a Hilbert basis because for all $s \in \Gamma$ we have

$$\left(\bigvee_t \langle f, \tilde{t}\rangle \tilde{t}\right)_s = \left(\bigvee_t f_t \tilde{t}\right)_s = \bigvee_t f_t \wedge \tilde{t}_s = \bigvee_t m_{st} \wedge f_t = (Mf)_s = f_s \ . \ \blacksquare$$

This shows that for all $s \in S$, the singleton $\tilde{t}$ is a local section of MB^S. In general there could exist local sections that are not of this form. Now we will see that if the local set is complete then all local sections are of this form.

Proposition 5.2.5 *Let S be a set and $M : S \times S \to B$ a B-set. Then the set of local sections of MB^S coincides with the set of singletons of M.*

Proof. Suppose $u \in MB^S$ is a singleton, and let $v \in MB^S$ such that $v \leq u$. Then $v = \langle v, v\rangle v \leq \langle v, v\rangle u$.

Given $s \in S$,

$$\begin{aligned}
(\langle v, v\rangle u)_s &= \langle v, v\rangle \wedge u_s \\
&\leq \langle v, u\rangle \wedge u_s \\
&= \langle \bigvee_{t\in S} \langle v, \tilde{t}\rangle \tilde{t}, u\rangle \wedge u_s \\
&= \bigvee_{t\in S} \langle v, \tilde{t}\rangle \wedge \langle \tilde{t}, u\rangle \wedge u_s \\
&= \bigvee_{t\in S} v_t \wedge u_t \wedge u_s \\
&\leq \bigvee_{t\in S} v_t \wedge m_{ts} \\
&= v_s
\end{aligned}$$

So $\langle v, v\rangle u = v$, for all $v \in MB^S$ such that $v \leq u$ and therefore u is a local section.

Conversely, if $u : S \to B$ is a local section of MB^S, then we have $Mu = u$. So for $s \in S$, $u_s = \bigvee_{t\in S} m_{st} \wedge u_t \leq m_{ss}$.

Given $s, t \in S$, $u \wedge \tilde{s} \leq u$ therefore $\langle u \wedge \tilde{s}, u \wedge \tilde{s} \rangle u = u \wedge \tilde{s}$. So we have

$$\begin{aligned}
\bigvee_{w \in S} u_w \wedge m_{sw} \wedge u_t &= \bigvee_{w \in S} (u \wedge \tilde{s})_w \wedge u_t \\
&= (\langle u \wedge \tilde{s}, u \wedge \tilde{s} \rangle u)_t \\
&= (u \wedge \tilde{s})_t \\
&= u_t \wedge m_{st}
\end{aligned}$$

Making $w = s$ we get $u_s \wedge m_{ss} \wedge u_t \leq u_t \wedge m_{st}$.

But $u_s \wedge m_{ss} \wedge u_t = u_s \wedge u_t = (u^{\mathrm{T}} u)_{st}$ and $u_t \wedge m_{st} \leq m_{st}$. So $u^{\mathrm{T}} u \leq M$, and this proves u is a singleton. ∎

5.3 Étale B-locales

Now we establish an equivalence between local homeomorphisms, on one hand, and Hilbert B-modules equipped with Hilbert bases, on the other.

Lemma 5.3.1 *Any Hilbert B-module with a Hilbert basis is necessarily a B-locale and it arises, up to isomorphism, as in 5.2.4.*

Proof. Let X be a Hilbert B-module with a Hilbert basis Γ, let M be the matrix determined by $m_{st} = \langle s, t \rangle$ for all $s, t \in \Gamma$, and let $\varphi : B^{\Gamma} \to X$ be the B-module quotient defined by $\varphi(f) = \bigvee_{s \in \Gamma} f_s s$. Recalling that the inner product is non-degenerate we have, for all $f, g \in B^{\Gamma}$, the following series of equivalences:

$$\begin{aligned}
\varphi(f) = \varphi(g) \quad &\iff \quad \forall_{t\in\Gamma}\ \langle \varphi(f), t\rangle = \langle \varphi(g), t\rangle \\
&\iff \quad \forall_{t\in\Gamma}\ \left\langle \bigvee_{s\in\Gamma} f_s s, t\right\rangle = \left\langle \bigvee_{s\in\Gamma} g_s s, t\right\rangle \\
&\iff \quad \forall_{t\in\Gamma}\ \bigvee_{s\in\Gamma} f_s \wedge \langle s, t\rangle = \bigvee_{s\in\Gamma} g_s \wedge \langle s, t\rangle \\
&\iff \quad \forall_{t\in\Gamma}\ (Mf)_t = (Mg)_t \\
&\iff \quad Mf = Mg\ .
\end{aligned}$$

This shows that the B-module surjection φ factors uniquely through the quotient $f \mapsto Mf : B^\Gamma \to MB^\Gamma$ and an isomorphism of B-modules $X \overset{\cong}{\to} MB^\Gamma$. Finally, in order to conclude that X is a B-locale it suffices to show that it is a locale, due to 2.4.6, or, equivalently, that MB^Γ is a locale. Consider the B-module endomorphism $j : f \mapsto Mf$ of B^Γ, as in the proof of 5.2.4. It is easy to prove directly that MB^Γ is a locale (the restriction of j to $\downarrow(M1_{B^\Gamma})$ is a closure operator whose fixed points define a subframe — not a sublocale — of $\downarrow(M1_{B^\Gamma})$), but in fact this is already known, for MB^Γ coincides with the set of B-subsets of M as in [1, Def. 2.8.9 and Prop. 2.8.11]. ∎

Theorem 5.3.2 *Let X be a B-module. The following conditions are equivalent:*

1. *X can be equipped with a structure of pre-Hilbert B-module for which there is a Hilbert basis;*

2. *X can be equipped with a structure of Hilbert B-module for which there is a Hilbert basis;*

3. *X is an étale B-locale.*

Proof. The first two conditions are equivalent due to 5.2.3-3. Let us prove that 2 implies 3. Let X be a Hilbert B-module (and thus also a B-locale, by

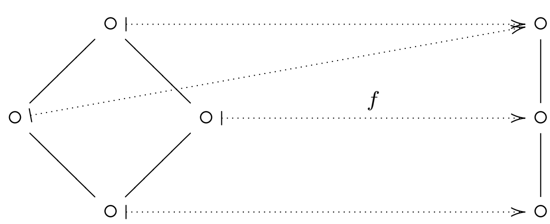
5.3.1). By 5.2.3-5 X is supported, and thus by 5.1.2 it is an open B-locale with ς defined by $\varsigma x = \langle x, x\rangle$. Now let $s \in \Gamma$, and let $x \leq s$. We have $\varsigma(x)s = x$, by 5.2.3-7, and thus s is a local section in the sense of 4.3.2. Hence, $\Gamma \subset \Gamma_X$. Since by 5.2.3-2 we know that $\bigvee \Gamma = 1$, we conclude that X is étale.

Now let us prove that 3 implies 1. If X is étale it is open and thus by 5.1.3 it is a supported pre-Hilbert B-module with the inner product defined by $\langle x, y\rangle = \varsigma(x \wedge y)$. For each $x \in X$ we have

$$x = 1 \wedge x = \left(\bigvee_{s \in \Gamma_X} s\right) \wedge x = \bigvee_{s \in \Gamma_X} s \wedge x\ ,$$

and, by the definition of local section, $s \wedge x = \varsigma(s \wedge x)s = \langle x, s\rangle s$, thus showing that Γ_X is a Hilbert basis. ∎

This provides us with an analogy between the view of sheaves on B as Hilbert B-modules, on one hand, and vector bundles on a compact space X as Hilbert $C(X)$-modules, on the other. The analogy extends to the fact that, just as vector bundles on X are projective $C(X)$-modules, sheaves on B are projective B-modules (due to 5.2.3-1). But, of course, the analogue of Swan's theorem does not hold because projective B-locales (i.e., the B-locales that are projective objects in B-**Mod**) are not necessarily étale B-locales. For instance, if $B = \Omega$ this is just the statement that not every locale which is projective as a sup-lattice is necessarily a free sup-lattice. The following diagram, where f is a retraction, illustrates this:



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5.4 Adjointable maps

The homomorphisms of Hilbert B-modules equipped with Hilbert bases are necessarily adjointable, and in order to prove this only the domain module need have a Hilbert basis:

Theorem 5.4.1 *Let X and Y be pre-Hilbert B-modules such that X has a Hilbert basis Γ (hence, X is Hilbert), and let $h : X \to Y$ be a homomorphism of B-modules. Then h is adjointable with a unique adjoint $h^\dagger$, which is given by*

$$h^\dagger(y) = \bigvee_{t\in\Gamma} \langle h(t), y\rangle t \ . \tag{4.2}$$

Proof. Let $x \in X$, $y \in Y$, and let us compute $\langle x, h^\dagger(y)\rangle$ using (4.2):

$$\begin{aligned}
\langle x, h^\dagger(y)\rangle &= \left\langle \bigvee_{s\in\Gamma} \langle x, s\rangle s, \bigvee_{t\in\Gamma} \langle h(t), y\rangle t \right\rangle \\
&= \bigvee_{s,t\in\Gamma} \langle x, s\rangle \wedge \langle s, t\rangle \wedge \langle h(t), y\rangle \\
&= \bigvee_{t\in\Gamma} \langle x, t\rangle \wedge \langle h(t), y\rangle = \left\langle \bigvee_{t\in\Gamma} \langle x, t\rangle h(t), y \right\rangle \\
&= \left\langle h\left(\bigvee_{t\in\Gamma} \langle x, t\rangle t\right), y \right\rangle = \langle h(x), y\rangle \ .
\end{aligned}$$

This shows that $h^\dagger$ is adjoint to h, and the uniqueness is a consequence of the non-degeneracy of the inner product of X. ∎

Corollary 5.4.3 *If X and Y are Hilbert B-modules and X has a Hilbert basis then any function $h : X \to Y$ is adjointable if and only if it is a homomorphism of B-modules.*

Definition 5.4.4 The *category of Hilbert B-modules with Hilbert bases*, denoted by B-**HMB**, is the category whose objects are those Hilbert B-modules for which there exist Hilbert bases and whose morphisms are the homomorphisms of B-modules (equivalently, the adjointable maps).

Corollary 5.4.5 *The assignment from homomorphisms h to their adjoints $h^\dagger$ is a strong self-duality $(-)^\dagger : (B\text{-}\mathbf{HMB})^{op} \to B\text{-}\mathbf{HMB}$.*

The matrix representations of Hilbert modules with Hilbert bases (cf. 5.2.3-8 and 5.3.1) can be extended to homomorphisms in a natural way. In particular the adjoint $h^\dagger$ of a homomorphism h corresponds to the transpose of the matrix of h.

Theorem 5.4.6 *The categories B-$\mathbf{HMB}$ and $\mathbf{Rel}(B)$ are equivalent.*

Proof. The assignment from modules X to matrices $\langle -,- \rangle_X : \Gamma_X \times \Gamma_X \to B$ extends to the functor $\mathcal{M} : B\text{-}\mathbf{HMB} \to \mathbf{Rel}(B)$ that to each homomorphism $h : Y \to X$ assigns the matrix $\mathcal{M}(h) : \Gamma_X \times \Gamma_Y \to B$ defined by

$$(\mathcal{M}(h))_{st} = \langle h(t), s \rangle \ .$$

In the converse direction, the construction of a module MB^S from each matrix $M : S \times S \to B$ extends to a functor $\mathcal{X} : \mathbf{Rel}(B) \to B\text{-}\mathbf{HMB}$: given projection matrices $M : S \times S \to B$ and $N : T \times T \to B$, and an arrow $F : M \to N$, we define $\mathcal{X}(F) : MB^S \to NB^T$ by $\mathcal{X}(F)(f) = Ff$. There is a natural isomorphism $\mathcal{X} \circ \mathcal{M} \cong \mathrm{id}$, due to 5.3.1, and a natural isomorphism $\mathcal{M} \circ \mathcal{X} \cong \mathrm{id}$ follows from the equivalence of [3] between the category of B-sets and that of complete B-sets (see also [1, §2.9] or [8, pp. 502–513]). Hence, the functors $\mathcal{M}$ and $\mathcal{X}$ form an adjoint equivalence of categories. ∎

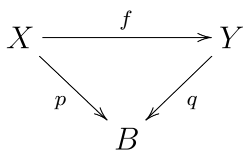
We remark that the maps $\mathbf{Set}(B)$ of [3] are arrows of $\mathbf{Rel}(B)$, and thus the category $\mathbf{Set}(B)$ is a subcategory of $\mathbf{Rel}(B)$.

5.5 Sheaf homomorphisms

In this section we exhibit an identification of "operator adjoints" with categorical adjoints, which as a consequence shows that the duality between homomorphisms of étale B-locales and sheaf homomorphisms is a restriction of the strong self-duality of B-**HMB**. In what follows we shall always consider an étale B-locale X to be a Hilbert B-module with respect to the Hilbert basis of local sections Γ_X.

Theorem 5.5.1 *Let X and Y be étale B-locales, and let $f : X \to Y$ be a map of B-locales. Then $f_! = (f^*)^\dagger$ (equivalently, $f^* = (f_!)^\dagger$).*

Proof. Let the B-locales, their projections, and f be as follows:



Since f commutes with the projections, which are local homeomorphisms, it is itself a local homeomorphism and thus it satisfies the Frobenius reciprocity condition $f_!(x \wedge f^*(y)) = f_!(x) \wedge y$ (i.e., $f_!$ is Y-equivariant). Hence, we have, for all $x \in X$ and $y \in Y$:

$$\begin{aligned} \langle x, f^*(y) \rangle_X &= p_!(x \wedge f^*(y)) = q_!(f_!(x \wedge f^*(y))) \\ &= p_!(f_!(x) \wedge y) = \langle f_!(x), y \rangle_Y \ . \quad \blacksquare \end{aligned}$$

Now we shall look at a converse to the above theorem, whose proof depends on the following lemmas:

Lemma 5.5.2 *Let X be an étale B-locale. If $s \in \Gamma_X$ and (b_α) is a non-empty family of elements of B, we have $(\bigwedge_\alpha b_\alpha)\, s = \bigwedge_\alpha (b_\alpha s)$.*

Proof. Let (b_α) be a non-empty family of elements of B, and let $s \in \Gamma_X$. Then $(\bigwedge_\alpha b_\alpha)\, s$ is a lower bound of the set $\{b_\alpha \sigma\}$. Let t be another lower bound. Then $\varsigma(t) \leq b_\alpha$ for all α and, since (b_α) is non-empty, we have $t \leq b_\alpha s$ for some α and thus $t \leq s$. Hence,

$$\varsigma(t)\left(\left(\bigwedge_\alpha b_\alpha\right) s\right) = \left(\varsigma(t) \wedge \bigwedge_\alpha b_\alpha\right) s = \varsigma(t)s = t\ ,$$

and it follows that $t \leq (\bigwedge_\alpha b_\alpha)\, s$. This shows that $(\bigwedge_\alpha b_\alpha)\, s = \bigwedge_\alpha (b_\alpha s)$. ∎

Lemma 5.5.3 *Let X be an étale B-locale, and let $S \subset \Gamma_X$ be a non-empty set such that $\bigvee S \in \Gamma_X$. Then $\varsigma\left(\bigwedge S\right) = \bigwedge_{t\in S} \varsigma(t)$.*

Proof. Let $s = \bigvee S$. We have $\varsigma\left(\bigwedge S\right) = \varsigma\left(\bigwedge_{t\in S} \varsigma(t)s\right)$ and, by 5.5.2, this equals

$$\varsigma\left(\left(\bigwedge_{t\in S} \varsigma(t)\right) s\right) = \left(\bigwedge_{t\in S} \varsigma(t)\right) \wedge \varsigma\left(s\right)\ .$$

Since S is non-empty the latter equals $\bigwedge_{t\in S} \varsigma(t)$. ∎

Theorem 5.5.4 *Let $h : X \to Y$ be a sheaf homomorphism of étale B-locales. Then its adjoint $h^\dagger$ preserves arbitrary meets.*

Proof. Let $S \subset Y$. We shall show that $h^\dagger\left(\bigwedge S\right) = \bigwedge h^\dagger(S)$ by using the non-degeneracy of the inner product of X; that is, we shall prove, for all $s \in \Gamma_X$, that $\left\langle s, h^\dagger\left(\bigwedge S\right)\right\rangle = \left\langle s, \bigwedge h^\dagger(S)\right\rangle$. Let then $s \in \Gamma_X$. We have

$$\begin{aligned} \left\langle s, h^\dagger\left(\bigwedge S\right)\right\rangle &= \left\langle h(s), \bigwedge S\right\rangle = \varsigma\left(h(s) \wedge \bigwedge S\right) \\ &= \varsigma\left(h(s) \wedge \bigwedge_{y\in S} (h(s) \wedge y)\right) = \varsigma\left(\bigwedge S'\right)\ , \end{aligned}$$

where the set $S' = \{h(s)\} \cup \{h(s) \wedge y \mid y \in S\}$ is non-empty, it is contained in Γ_Y because $h(s) \in \Gamma_Y$, and $\bigvee S' \in \Gamma_Y$ because S' is upper bounded by

$h(s)$. Hence, by 5.5.3, we have $\varsigma\left(\bigwedge S'\right) = \bigwedge \varsigma(S')$. Moreover, $\varsigma(h(s)) = \varsigma(s)$ and thus

$$\begin{aligned} \bigwedge \varsigma(S') &= \varsigma(h(s)) \wedge \bigwedge_{y\in S} \varsigma(h(s) \wedge y) = \varsigma(s) \wedge \bigwedge_{y\in S} \langle h(s), y\rangle \\ &= \varsigma(s) \wedge \bigwedge_{y\in S} \langle s, h^\dagger(y)\rangle = \varsigma(s) \wedge \bigwedge_{y\in S} \varsigma(s \wedge h^\dagger(y)) = \bigwedge \varsigma(S'')\ , \end{aligned}$$

where the set $S'' = \{s\} \cup \{s \wedge h^\dagger(y) \mid y \in S\}$ is non-empty, it is contained in Γ_X, and $\bigvee S'' \in \Gamma_X$ because S'' is upper bounded by s. Hence, again by 5.5.3, we have

$$\begin{aligned} \bigwedge \varsigma(S'') &= \varsigma\left(\bigwedge S''\right) = \varsigma\left(s \wedge \bigwedge_{y\in S} (s \wedge h^\dagger(y))\right) \\ &= \varsigma\left(s \wedge \bigwedge h^\dagger(S)\right) = \left\langle s, \bigwedge h^\dagger(S)\right\rangle\ , \end{aligned}$$

which concludes the proof. ∎

Theorem 4.4.3, whose proof has been postponed until now, is a simple corollary of these results. Instead of having proved directly that the right adjoints of sheaf homomorphisms are module homomorphisms, as we might have attempted in §4, we have instead shown that the "operator adjoints" of sheaf homomorphisms, which are module homomorphisms, are also homomorphisms of locales:

Corollary 5.5.5 (cf. Theorem 4.4.3) *The categories B-**Sh** and B-**LH** are isomorphic.*

Proof. By 5.5.4, the adjoint $h^\dagger$ of a sheaf homomorphism h is a homomorphism of B-locales. This defines a map of B-locales f such that $f^* = h^\dagger$. By 5.5.1, $f_! = (f^*)^\dagger = h$, and thus the faithful functor $\mathcal{S}$ of 4.4.3 is full. ∎

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