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Selective Revision

Abstract. We introduce a constructive model of selective belief revision in which it is possible to accept only a part of the input information. A selective revision operator $\circ$ is defined by the equality $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$, where $*$ is an AGM revision operator and f a function, typically with the property $\vdash \alpha \to f(\alpha)$. Axiomatic characterizations are provided for three variants of selective revision.

Key words: belief change, belief revision, success postulate, non-prioritized revision, choice function.

1. Introduction

In standard accounts of belief revision [1, 9], the new information is always accepted. This is an unrealistic feature, since actual epistemic agents, when confronted with information that contradicts previous beliefs, often reject it altogether or accept only parts of it. Recently, several models of belief revision have been developed that allow for two options: either the new information is fully accepted or it is completely rejected [4, 6, 7, 10, 11]. In this paper we introduce a model that also allows a third possibility: to accept parts of the new information and reject the rest of it.

The following example illustrates the practical relevance of this third option: One day when you return back from work, your son tells you, as soon as you see him: "A dinosaur has broken grandma's vase in the living-room". You probably accept one part of the information, namely that the vase has been broken, while rejecting the part of it that refers to a dinosaur.

The AGM model is briefly introduced in Section 2, together with some formal preliminaries. In Section 3, postulates for selective revision are proposed, and in Section 4 a constructive model for selective revision is introduced. The construction is very simple: Selective revision ($\circ$) is constructed out of AGM revision ($*$) through the relationship $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$, where f is a function that, intuitively speaking, selects the credible part out of every sentence α. Three versions of this model are axiomatically characterized. All proofs can be found in the Appendix.

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2. Background

Just as in the AGM model we will assume that the beliefs of a rational agent are represented by a belief set $\boldsymbol{K}$, a set of sentences in a language $\mathcal{L}$. $\boldsymbol{K}$ is closed under logical consequence Cn, where Cn satisfies: $A \subseteq Cn(A)$, $Cn(Cn(A)) \subseteq Cn(A)$, and $Cn(A) \subseteq Cn(B)$ if $A \subseteq B$. We assume that Cn includes classical logical consequence, satisfies the rule of introduction of disjunction into premises and is compact. $\vdash \alpha$ is an alternative notation for $\alpha \in Cn(\emptyset)$, $\perp$ is the falsity constant and $\boldsymbol{K}_\perp$ the inconsistent belief set. $\boldsymbol{K}+\alpha$ denotes the expansion of $\boldsymbol{K}$ by α and is defined by $\boldsymbol{K} + \alpha = Cn(\boldsymbol{K} \cup \{\alpha\})$.

Partial meet revision, the revision operator of AGM theory, is defined as follows:

Definition 2.1. ([2]) Let A be a set of sentences and α a sentence. The set $A \perp \alpha$ is the set of sets such that $B \in A \perp \alpha$ if and only if:

(i) $B \subseteq A$

(ii) $\alpha \notin Cn(B)$

(iii) There is no set B' such that $B \subset B' \subseteq A$ and $\alpha \notin Cn(B')$.

Definition 2.2. ([1]) Let A be a set of sentences. A *selection function* for A is a function γ such that for all sentences α:

(i) If $A \perp \alpha$ is non-empty, then $\gamma(A \perp \alpha)$ is a non-empty subset of $A \perp \alpha$.

(ii) If $A \perp \alpha$ is empty, then $\gamma(A \perp \alpha) = \{A\}$.

The selection function γ is *relational* if and only if there is a relation $\sqsubseteq$ such that for all sentences α, if $A \perp \alpha$ is non-empty, then

$$\gamma(A \perp \alpha) = \{\, B \in A \perp \alpha \mid C \sqsubseteq B \text{ for all } C \in A \perp \alpha \,\}.$$

The selection function γ is *transitively relational* if and only if this hold for some transitive relation $\sqsubseteq$.

Definition 2.3. ([1]) Let A be a set of sentences and γ a selection function for A. The *partial meet contraction* on A that is generated by γ is the operation $\sim_\gamma$ such that for all sentence α:

$$A \sim_\gamma \alpha = \bigcap \gamma(A \perp \alpha)$$

An operation $-$ on A is a partial meet contraction if and only if there is a selection function γ for A such that for all sentences $\alpha : A - \alpha = A \sim_\gamma \alpha$.

Furthermore, $-$ is (transitively) relational if and only if it can be generated from a (transitively) relational selection function.

DEFINITION 2.4. ([1]) The operator $*$ on a belief set $\boldsymbol{K}$ is an operator of *partial meet revision* if and only if there is some operator $-$ of partial meet contraction on $\boldsymbol{K}$ such that for all sentences α

$$\boldsymbol{K} * \alpha = (\boldsymbol{K} - \neg\alpha) + \alpha \qquad \text{(the Levi identity)}$$

Furthermore, $*$ is (transitively) relational if and only if this hold for some $-$ that is (transitively) relational.

One of the major achievements of AGM theory is the characterization of partial meet revision, and its transitively relational variant, in terms of a set of intuitively reasonable postulates [1]. The six basic AGM postulates for revision are:

$(\boldsymbol{K}*1)$	$\boldsymbol{K} * \alpha$ is a belief set.	(Closure)
$(\boldsymbol{K}*2)$	$\boldsymbol{K} * \alpha \vdash \alpha$	(Success)
$(\boldsymbol{K}*3)$	$\boldsymbol{K} * \alpha \subseteq \boldsymbol{K} + \alpha$	(Inclusion)
$(\boldsymbol{K}*4)$	If $\boldsymbol{K} \nvdash \neg\alpha$, then $\boldsymbol{K} + \alpha \subseteq \boldsymbol{K} * \alpha$	(Vacuity)
$(\boldsymbol{K}*5)$	If $\nvdash \neg\alpha$ then $\boldsymbol{K} * \alpha \neq \boldsymbol{K}_{\perp}$	(Consistency)
$(\boldsymbol{K}*6)$	If $\vdash \alpha \leftrightarrow \beta$, then $\boldsymbol{K} * \alpha = \boldsymbol{K} * \beta$	(Extensionality)

The supplementary AGM postulates are as follows:

$(\boldsymbol{K}*7)$	$\boldsymbol{K} * (\alpha \wedge \beta) \subseteq (\boldsymbol{K} * \alpha) + \beta$	(Superexpansion)
$(\boldsymbol{K}*8)$	If $\boldsymbol{K} * \alpha \nvdash \neg\beta$, then $(\boldsymbol{K} * \alpha) + \beta \subseteq \boldsymbol{K} * (\alpha \wedge \beta)$	(Subexpansion)

OBSERVATION 2.5. ([1]) *Let $\boldsymbol{K}$ be a belief set. An operator $*$ on $\boldsymbol{K}$ is a* partial meet revision function *if and only if $*$ satisfies* closure, success, inclusion, vacuity, consistency, *and* extensionality.

It is a transitively relational partial meet revision function *if and only if also satisfies* superexpansion *and* subexpansion.

The following postulates will be useful in the following sections.

$(\boldsymbol{K} * \alpha) \cap (\boldsymbol{K} * \beta) \subseteq \boldsymbol{K} * (\alpha \vee \beta)$	(Disjunctive overlap)
If $\boldsymbol{K} * (\alpha \vee \beta) \nvdash \neg\alpha$, then $\boldsymbol{K} * (\alpha \vee \beta) \subseteq \boldsymbol{K} * \alpha$	(Disjunctive inclusion)
Either $\boldsymbol{K}*(\alpha\vee\beta) = \boldsymbol{K}*\alpha$, or $\boldsymbol{K}*(\alpha\vee\beta) = \boldsymbol{K}*\beta$, or $\boldsymbol{K} * (\alpha \vee \beta) = (\boldsymbol{K} * \alpha) \cap (\boldsymbol{K} * \beta)$	(Disjunctive factoring)

OBSERVATION 2.6. ([5]) *Let $\boldsymbol{K}$ be a belief set and $*$ be an operator for $\boldsymbol{K}$ that satisfies* closure, success, inclusion, vacuity, consistency, *and* extensionality. *Then:*

1. $*$ *satisfies* disjuntive overlap *if and only if it satisfies* superexpansion.
2. $*$ *satisfies* disjuntive inclusion *if and only if it satisfies* subexpansion.
3. $*$ *satisfies both* disjuntive overlap *and* disjunctive inclusion, *if and only if it satisfies* disjunctive factoring.

3. Postulates for selective revision

Four of the six AGM postulates are equally plausible for selective revision as for standard revision. These are *closure*, *inclusion*, *consistency*, and *extensionality*. The *vacuity* postulate is more debatable. It has been questioned even in a non-selective framework [9], and the reasons to do so are stronger in a framework for selective revision. It may be argued that even if the input sentence α does not logically contradict $\boldsymbol{K}$, there may be non-logical reasons for not accepting it completely, so that *vacuity* should not hold.

The *success* postulate should clearly not hold for selective revision, but it is of interest to investigate weakened versions of it. The following postulate, introduced in [7], ensures that an input is accepted if it is consistent with the original belief set:

If $\boldsymbol{K} \nvdash \neg\alpha$, then $\boldsymbol{K} \circ \alpha \vdash \alpha$ (Weak success)

Weak success follows logically from *vacuity.*

Another way to weaken *success* is to require that revision should take the form of accepting and fully incorporating some part of the input information. That part then acts as a proxy for the input:

There is a sentence β, such that $\boldsymbol{K} \circ \alpha \vdash \beta$, $\vdash \alpha \rightarrow \beta$, and $\boldsymbol{K} \circ \alpha = \boldsymbol{K} \circ \beta$ (Proxy success)

There is an obvious way to weaken this postulate:

There is a sentence β, such that $\boldsymbol{K} \circ \alpha \vdash \beta$ and $\boldsymbol{K} \circ \alpha = \boldsymbol{K} \circ \beta$ (Weak proxy success)

Proxy success and its weak variant are unusual among belief change postulates due to their existential nature. (On the use of existential conditions, see [12, p. 16])

The following postulate captures the intuition that previous beliefs are given up only if this is required to avoid inconsistency.

If $\boldsymbol{K} \nsubseteq \boldsymbol{K} \circ \alpha$ then $\boldsymbol{K} \cup (\boldsymbol{K} \circ \alpha) \vdash \bot$ (Consistent expansion)

This postulate is a direct consequence of *success* and *vacuity*:

OBSERVATION 3.1. *If* $\circ$ *satisfies* vacuity *and* success *then it satisfies* consistent expansion.

OBSERVATION 3.2. Consistent expansion *does not follow from* closure, inclusion, vacuity, consistency, *and* extensionality.

As was pointed to us by an anonymous referee for this journal, *consistent expansion* is also a weakening of:

If $\beta \in \boldsymbol{K} \setminus (\boldsymbol{K} \circ \alpha)$ then $\boldsymbol{K} \circ \alpha \vdash \neg\beta$ (Tenacity)

that has been used as a characteristic postulate of maxichoice revision in AGM theory [3] [5, p. 58–59] [8].

4. Constructing selective revision

In this section we provide a constructive model for selective revision that makes use of the power of the AGM apparatus. We also provide the corresponding representation theorems.

DEFINITION 4.1. Let $\boldsymbol{K}$ be a belief set, $*$ a *partial meet revision* for $\boldsymbol{K}$ and f a function from $\mathcal{L}$ to $\mathcal{L}$. The *selective revision* $\circ$, based on $*$ and f, is the operation such that for all sentences α:

$$\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$$

f is the *transformation function* on which $\circ$ is based.

The following is a list of properties that the transformation function may satisfy:

$\vdash \alpha \rightarrow f(\alpha)$ (implication)

If $\boldsymbol{K} \nvdash \neg\alpha$, then $\vdash \alpha \rightarrow f(\alpha)$ (weak implication)

$\vdash f(f(\alpha)) \leftrightarrow f(\alpha)$ (idempotence)

$\vdash \neg f(\alpha) \rightarrow f(\neg\alpha)$ internalized negation)

$\vdash f(\neg\alpha) \rightarrow \neg f(\alpha)$ (externalized negation)

If $\vdash \alpha \rightarrow \beta$ then $\vdash f(\alpha) \rightarrow f(\beta)$ (monotony)

If $\vdash \alpha \leftrightarrow \beta$ then $\vdash f(\alpha) \leftrightarrow f(\beta)$ (extensionality)

If $\nvdash \neg\alpha$, then $\nvdash \neg f(\alpha)$ (consistency preservation)

$\nvdash \neg f(\alpha)$ (consistency)

$\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha) \vee f(\beta)$ (disjunctive distribution)

Either $\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha)$ or $\vdash f(\alpha \vee \beta) \leftrightarrow f(\beta)$ or $\vdash f(\alpha \vee \beta) \leftrightarrow f(\alpha) \vee f(\beta)$ (disjunctive factoring)

$\vdash f(\alpha \wedge \beta) \leftrightarrow f(\alpha) \wedge f(\beta)$ (conjunctive distribution)

$\vdash f(\alpha) \leftrightarrow \alpha$ (maximality)

If $\boldsymbol{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$ (weak maximality)

Either $\vdash f(\alpha) \leftrightarrow \alpha$ or $\vdash f(\neg\alpha) \leftrightarrow \neg\alpha$ (disjunctive maximality)

Some interrelations among properties of the transformation function are listed in the following observation:

OBSERVATION 4.2. (1) *If f satisfies implication, then it satisfies internalized negation.*

(2) *If f satisfies implication and externalized negation then it satisfies maximality.*

(3) *If f satisfies extensionality and disjunctive distribution then it satisfies monotony.*

(4) *If f satisfies weak maximality with respect to $\boldsymbol{K}$ and $\boldsymbol{K} \neq \boldsymbol{K}_\perp$, then f satisfies disjunctive maximality.*

(5) *If $\boldsymbol{K} \nvdash \perp$ and $\boldsymbol{K} \notin \mathcal{L}\perp\perp$ (i.e., $\boldsymbol{K}$ is consistent but not a maximal consistent subset of the language) then f cannot satisfy simultaneously monotony, consistency, and weak maximality with respect to $\boldsymbol{K}$.*

The following two observations show how these properties of the transformation function give rise to properties of the selective revision function.

OBSERVATION 4.3. *Let $\boldsymbol{K}$ be a belief set in a language $\mathcal{L}$, $*$ a revision operator for $\boldsymbol{K}$ that satisfies the six basic* AGM *postulates, and f a transformation function. Let $\circ$ be the selective revision function on $\boldsymbol{K}$ generated from $*$ and f. Then:*

1. *$\circ$ satisfies* closure *and* consistent expansion.
2. *If f satisfies extensionality then $\circ$ satisfies* extensionality.
3. *If f satisfies weak implication then $\circ$ satisfies* inclusion.
4. *If f satisfies weak maximality then $\circ$ satisfies* inclusion *and* vacuity.
5. *If f satisfies consistency preservation then $\circ$ satisfies* consistency.
6. *If f satisfies maximality then $\circ$ satisfies* success.
7. *If f satisfies implication then $\circ$ satisfies* consistency.
8. *If f satisfies idempotence, then $\circ$ satisfies* weak proxy success.
9. *If f satisfies idempotence and implication, then $\circ$ satisfies* proxy success.

In the limiting case when f satisfies maximality, $\circ$ is a partial meet revision function.

It should be noted that *weak implication* and *weak maximality* differ from the rest of the listed properties of f by referring to a belief set $\boldsymbol{K}$. It is no surprise that these properties can be used to obtain the postulates of *inclusion* and *vacuity*, the only basic AGM revision postulates that refer to the belief set $\boldsymbol{K}$.

OBSERVATION 4.4. *Let $\boldsymbol{K}$ be a belief set in a language $\mathcal{L}$, $*$ a revision operator for $\boldsymbol{K}$ that satisfies the eight basic and supplementary* AGM *postulates, and f a transformation function. Let $\circ$ be the selective revision function on $\boldsymbol{K}$ generated from $*$ and f. Then:*

1. *If f satisfies implication and conjunctive distribution, then $\circ$ satisfies* superexpansion.
2. *If f satisfies disjunctive distribution, then $\circ$ satisfies* disjunctive overlap.
3. *If f satisfies disjunctive factoring, then $\circ$ satisfies* disjunctive factoring.
4. *If f satisfies implication and disjunctive distribution, then $\circ$ satisfies* disjunctive inclusion.

The following representation theorems have been obtained for three classes of selective revision functions.

THEOREM 4.5. *Let $\mathcal{L}$ be a finite language, $\boldsymbol{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\boldsymbol{K}$. Then the following conditions are equivalent:*

1. $\circ$ *satisfies* closure, inclusion, vacuity, consistency, extensionality, *and* consistent expansion.
2. *There exists a revision function $*$ for $\boldsymbol{K}$ that satisfies the six basic* AGM *postulates, and a transformation function f that satisfies extensionality, consistency preservation, and weak maximality, such that $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$ for all α.*

THEOREM 4.6. *Let $\mathcal{L}$ be a finite language, $\boldsymbol{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\boldsymbol{K}$. Then the following conditions are equivalent:*

1. $\circ$ *satisfies* closure, inclusion, vacuity, consistency, extensionality, consistent expansion, *and* weak proxy success.
2. *There exists a revision function $*$ for $\boldsymbol{K}$ that satisfies the six basic* AGM *postulates, and a transformation function f that satisfies extensionality, consistency preservation, weak maximality, and idempotence, such that $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$ for all α.*

THEOREM 4.7. *Let $\mathcal{L}$ be a finite language, $\boldsymbol{K}$ a belief set in $\mathcal{L}$ and $\circ$ an operator on $\boldsymbol{K}$. Then the following two conditions are equivalent:*

1. $\circ$ *satisfies* closure, inclusion, vacuity, consistency, extensionality, consistent expansion, *and* proxy success.
2. *There exists a revision function $*$ for $\boldsymbol{K}$ that satisfies the six basic AGM postulates, and a transformation function f that satisfies extensionality, consistency preservation, weak maximality, idempotence, and implication, such that $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$ for all α.*

The (1) to (2) direction of Theorem 4.5 can be proved with a construction such that, in the principal case when α is inconsistent with $\boldsymbol{K}$, $f(\alpha)$ is taken to be equivalent with the whole of $\boldsymbol{K} \circ \alpha$ and $\boldsymbol{K} * \alpha$ is taken to be equivalent with α itself. Refinements of this construction can be used for Theorems 4.6 and 4.7. For details, the reader is referred to the Appendix.

The postulates for selective revision referred to in Theorems 4.5–4.7 are, in addition to five of the six AGM postulates: *consistent expansion* that follows form the AGM postulates *vacuity* and *success*; and *weak proxy success* and *proxy success* that both follow from *success*. Hence, these operations of selective revision are weakened variants of AGM revision.

The representation theorems indicate that these constructions provide a fairly faithful extension of the AGM framework to allow for less than total acceptance of new information.

Appendix: Proofs

PROOF OF OBSERVATION 3.1. Let $\boldsymbol{K} \nsubseteq \boldsymbol{K} \circ \alpha$. It follows from *vacuity* that $\boldsymbol{K} \vdash \neg\alpha$ and from *success* that $\boldsymbol{K} \circ \alpha \vdash \alpha$. Hence, $\boldsymbol{K} \cup (\boldsymbol{K} * \alpha) \vdash \bot$. ■

PROOF OF OBSERVATION 3.2. (The idea for this proof was provided by David Makinson.) Let

$$\boldsymbol{K} * \alpha = \begin{cases} \boldsymbol{K} + \alpha & \text{if } \boldsymbol{K} \nvdash \neg\alpha \\ Cn(\emptyset) & \text{otherwise} \end{cases}$$

It is trivial to prove that $*$ satisfies *closure*, *inclusion*, *vacuity*, *consistency*, and *extensionality*. However it does not satisfies *consistent expansion*: Let α be such that $\nvdash \alpha$ and $\nvdash \neg\alpha$ and let $\boldsymbol{K} = Cn(\{\neg\alpha\})$. Then $\boldsymbol{K} * \alpha = Cn(\emptyset)$, so $\boldsymbol{K} \nsubseteq \boldsymbol{K} * \alpha$ but $\boldsymbol{K} \cup (\boldsymbol{K} * \alpha) = Cn(\{\neg\alpha\}) \cup Cn(\emptyset) \nvdash \bot$. ■

PROOF OF OBSERVATION 4.2. *Part 1*: It follows from implication that $\vdash \alpha \to f(\alpha)$ and $\vdash \neg\alpha \to f(\neg\alpha)$. From this it follows truth-functionally that $\vdash \neg f(\alpha) \to f(\neg\alpha)$.

Part 2: From implication we obtain $\vdash \neg\alpha \to f(\neg\alpha)$ and from externalized negation $\vdash f(\neg\alpha) \to \neg f(\alpha)$. Hence $\vdash \neg\alpha \to \neg f(\alpha)$ or equivalently $\vdash f(\alpha) \to \alpha$. Since implication also yields $\vdash \alpha \to f(\alpha)$ we can conclude that $\vdash f(\alpha) \leftrightarrow \alpha$.

Part 3: Let $\vdash \alpha \to \beta$. Then $\vdash \beta \leftrightarrow \alpha \vee \beta$, and it follows from extensionality that $\vdash f(\alpha \vee \beta) \leftrightarrow f(\beta)$. We can combine this with $\vdash f(\alpha) \to f(\alpha \vee \beta)$, that follows from disjunctive distribution, and obtain $\vdash f(\alpha) \to f(\beta)$.

Part 4: It follows from $\boldsymbol{K} \neq \boldsymbol{K}_\bot$ that $\boldsymbol{K} \nvdash \alpha$ or $\boldsymbol{K} \nvdash \neg\alpha$, so by weak maximality $\vdash f(\alpha) \leftrightarrow \alpha$ or $\vdash f(\neg\alpha) \leftrightarrow \neg\alpha$.

Part 5: Since $\boldsymbol{K} \in \mathcal{L}\bot\bot$, there is some β such that $\boldsymbol{K} \nvdash \beta$ and $\boldsymbol{K} \nvdash \neg\beta$. It follows from monotony that $\vdash f(\beta \wedge \neg\beta) \to f(\beta)$ and $\vdash f(\beta \wedge \neg\beta) \to f(\neg\beta)$, and from weak maximality that $\vdash f(\beta) \leftrightarrow \beta$ and $\vdash f(\neg\beta) \leftrightarrow \neg\beta$. Hence $\vdash f(\beta \wedge \neg\beta) \to (\beta \wedge \neg\beta)$, which contradicts consistency. ■

PROOF OF OBSERVATION 4.3. *Part 1*: Trivial, since by Observations 2.5 and 3.1 $*$ satisfies *closure* and *consistent expansion.*

Part 2: Let $\vdash \alpha \leftrightarrow \beta$. Then, by f-extensionality $\vdash f(\alpha) \leftrightarrow f(\beta)$, and by $*$-*extensionality* $\boldsymbol{K} * f(\alpha) = \boldsymbol{K} * f(\beta)$, or equivalently $\boldsymbol{K} \circ \alpha = \boldsymbol{K} \circ \beta$.

Part 3: We prove by cases: (a) $\boldsymbol{K} \vdash \neg\alpha$ then $\boldsymbol{K} + \alpha = \boldsymbol{K}_\perp$, so that $\boldsymbol{K} \circ \alpha \subseteq \boldsymbol{K} + \alpha$. (b) $\boldsymbol{K} \nvdash \neg\alpha$, then $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha)$, $\boldsymbol{K} * f(\alpha) \subseteq \boldsymbol{K} + f(\alpha)$ ($*$-*inclusion*), $\boldsymbol{K} + f(\alpha) \subseteq \boldsymbol{K} + \alpha$ (weak implication), hence $\boldsymbol{K} \circ \alpha \subseteq \boldsymbol{K} + \alpha$.

Part 4: *Inclusion* follows from part 3 since weak maximallity implies weak implication. For *vacuity*, suppose that $\boldsymbol{K} \nvdash \neg\alpha$. Then by weak maximality, $\vdash \alpha \leftrightarrow f(\alpha)$ so that $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * \alpha$, and by $*$-*vacuity* $\boldsymbol{K} + \alpha \subseteq \boldsymbol{K} \circ \alpha$.

Part 5: Suppose that $\nvdash \neg\alpha$. Then by consistency preservation $\nvdash \neg f(\alpha)$, hence by $*$-*consistency* $\boldsymbol{K} * f(\alpha) \nvdash \perp$, hence $\boldsymbol{K} \circ \alpha \nvdash \perp$.

Part 6: Trivial, since by definition $*$ satisfies *success* and by maximality $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * \alpha$.

Part 7: From part 5, since implication implies consistency preservation.

Part 8: By Definition 4.1 and idempotence $\boldsymbol{K} \circ \alpha = \boldsymbol{K} * f(\alpha) = \boldsymbol{K} * f(f(\alpha)) = \boldsymbol{K} \circ f(\alpha)$. Since $\boldsymbol{K} * f(\alpha) \vdash f(\alpha)$ we therefore have $\boldsymbol{K} \circ \alpha = \boldsymbol{K} \circ f(\alpha)$ and $\boldsymbol{K} \circ \alpha \vdash f(\alpha)$, which is sufficient to prove that $\circ$ satisfies *proxy success.*

Part 9: This follows from the proof of part 8 since f satisfies implication. ∎

PROOF OF OBSERVATION 4.4. *Part 1*: $\boldsymbol{K} \circ (\alpha \wedge \beta) = \boldsymbol{K} * f(\alpha \wedge \beta)$ (definition of $\circ$) $= \boldsymbol{K} * (f(\alpha) \wedge f(\beta))$ (conjunctive distribution and $*$-*extensionality*) $\subseteq \boldsymbol{K} * f(\alpha)) + f(\beta)$ (since $*$ satisfies *superexpansion*) $\subseteq (\boldsymbol{K} * f(\alpha)) + \beta$ (implication) $= (\boldsymbol{K} \circ \alpha) + \beta$.

Part 2: $(\boldsymbol{K} \circ \alpha) \cap (\boldsymbol{K} \circ \beta) = (\boldsymbol{K} * f(\alpha)) \cap (\boldsymbol{K} * f(\beta)) \subseteq \boldsymbol{K} * (f(\alpha) \vee f(\beta))$ (*disjuntive overlap* for $*$) $= \boldsymbol{K} * f(\alpha \vee \beta)$ (disjunctive distribution and $*$-*extensionality*) $= \boldsymbol{K} \circ (\alpha \vee \beta)$.

Part 3: By f-disjunctive factoring and $*$-*extensionality*

$$\boldsymbol{K} * f(\alpha \vee \beta) = \begin{cases} \boldsymbol{K} * f(\alpha), \text{ or} \\ \boldsymbol{K} * f(\beta), \text{ or} \\ \boldsymbol{K} * (f(\alpha) \vee f(\beta)) \end{cases}$$

By observation 2.6, $*$ satisfies *disjunctive factoring*, thus

$$\boldsymbol{K} * (f(\alpha) \vee f(\beta)) = \begin{cases} \boldsymbol{K} * f(\alpha), \text{ or} \\ \boldsymbol{K} * f(\beta), \text{ or} \\ \boldsymbol{K} * f(\alpha) \cap \boldsymbol{K} * f(\beta) \end{cases}$$

Part 4: Let $\boldsymbol{K} \circ (\alpha \vee \beta) \nvdash \neg\alpha$. Then: $\boldsymbol{K} \circ (\alpha \vee \beta) \nvdash \neg f(\alpha)$ (implication), then $\boldsymbol{K} * (f(\alpha) \vee f(\beta)) \nvdash \neg f(\alpha)$ (disjunctive distribution, $*$-*extensionality*),

$\boldsymbol{K}*(f(\alpha)\vee f(\beta)) \subseteq \boldsymbol{K}*f(\alpha)$ (*$*$-disjunctive inclusion*), $\boldsymbol{K}*f(\alpha\vee\beta) \subseteq \boldsymbol{K}*f(\alpha)$ (disjunctive distribution), $\boldsymbol{K}\circ(\alpha\vee\beta) \subseteq \boldsymbol{K}\circ\alpha$ (definition of $\circ$). ■

PROOF OF THEOREM 4.5. (1) *implies* (2): We first define f and $*$: Let e be any function such that for any two sentences α and β if $\vdash \alpha \leftrightarrow \beta$ then $e(\alpha) = e(\beta)$ and $\vdash \alpha \leftrightarrow e(\alpha)$.

$$f(\alpha) = \begin{cases} e(\alpha) & \text{if } \boldsymbol{K} \nvdash \neg\alpha \\ e(\&(\boldsymbol{K}\circ\alpha)) & \text{otherwise} \end{cases}$$

$$\boldsymbol{K}*\beta = \begin{cases} \boldsymbol{K}+\beta & \text{if } \boldsymbol{K} \nvdash \neg\beta \\ Cn(\{\beta\}) & \text{otherwise} \end{cases}$$

We need to show (a) that f satisfies the properties, (b) that $*$ is a partial meet revision and (c) that $\boldsymbol{K}\circ\alpha = \boldsymbol{K}*f(\alpha)$ for all α.

(a) It follows directly that f satisfies extensionality (since $\circ$ satisfies *extensionality*) and weak maximality. To show that it satisfies consistency preservation we need to consider the two clauses of the definition of f. First, if $\boldsymbol{K} \nvdash \neg\alpha$, then α is consistent. Secondly, since $\circ$ satisfies *consistency*, $\&(\boldsymbol{K}\circ\alpha)$ is consistent if $\nvdash \neg\alpha$.

(b) To show that $*$ is a partial meet revision, we need to prove that it satisfies the six AGM postulates. It follows directly from the definition that *closure*, *success*, *inclusion*, *vacuity*, and *extensionality* are satisfied. To show that it satisfies *consistency*, let $\nvdash \neg\beta$. If $\boldsymbol{K} \nvdash \neg\beta$ then it follows from our definition of $*$ that $\boldsymbol{K}*\beta = \boldsymbol{K}+\beta$, and hence $\boldsymbol{K}*\beta$ is consistent. If $\boldsymbol{K} \vdash \neg\beta$, then $\boldsymbol{K}*\beta = Cn(\{\beta\})$, and since $\nvdash \neg\beta$, $\boldsymbol{K}*\beta$ is consistent.

(c) Finally, we need to prove that $\boldsymbol{K}\circ\alpha = \boldsymbol{K}*f(\alpha)$. There are two major cases, according to whether or not $\boldsymbol{K}$ implies $\neg\alpha$:

(c1) If $\boldsymbol{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$ so that $\boldsymbol{K} \nvdash \neg f(\alpha)$. Hence $\boldsymbol{K}*f(\alpha) = \boldsymbol{K}+f(\alpha) = \boldsymbol{K}+\alpha$ and since $\circ$ satisfies vacuity $\boldsymbol{K}\circ\alpha = \boldsymbol{K}+\alpha$.

(c2) If $\boldsymbol{K} \vdash \neg\alpha$, then $f(\alpha) = e(\&(\boldsymbol{K}\circ\alpha))$. We have two subcases. First, if $\boldsymbol{K} \vdash \neg f(\alpha)$ then $\boldsymbol{K}*f(\alpha) = Cn(\{f(\alpha)\}) = \boldsymbol{K}\circ\alpha$. Seccondly if $\boldsymbol{K} \nvdash \neg f(\alpha)$ or equivalently $\boldsymbol{K} \nvdash \neg e(\&(\boldsymbol{K}\circ\alpha))$, then we can use *consistent expansion* to obtain $\boldsymbol{K} \subseteq \boldsymbol{K}\circ\alpha$, hence $\boldsymbol{K}*f(\alpha) = \boldsymbol{K}+f(\alpha) = \boldsymbol{K}+e(\&(\boldsymbol{K}\circ\alpha)) = \boldsymbol{K}\circ\alpha$.

(2) *implies* (1): This direction of the proof follows from Observation 4.3. ■

PROOF OF THEOREM 4.6. (1) *implies* (2): We first define f and $*$:

$$f(\alpha) = \begin{cases} \alpha & \text{if } \boldsymbol{K} \nvdash \neg\alpha; \\ r(\alpha) & \begin{array}{l}\text{otherwise, where } r \text{ is a function such that} \\ \text{for all } \alpha \text{ and } \alpha',\ r(\alpha) = r(\alpha'),\ \boldsymbol{K}\circ\alpha \vdash r(\alpha), \\ \text{and } \boldsymbol{K}\circ\alpha = \boldsymbol{K}\circ r(\alpha).\end{array} \end{cases}$$

This definition is possible since $\circ$ satisfies *weak proxy success*.

$$K * \beta = \begin{cases} K \circ \beta & \text{if } K \circ \beta \vdash \beta; \\ K *' \beta & \text{otherwise, where } *' \text{ is any operation that satisfies the six basic AGM postulates.} \end{cases}$$

We need to show (a) that f satisfies the properties, (b) that $*$ is a partial meet revision and (c) that $K \circ \alpha = K * f(\alpha)$ for all α.

(a) That f satisfies weak maximality follows directly from the definition of f. To show that f satisfies consistency preservation let $\nvdash \neg\alpha$: If $K \nvdash \neg\alpha$, then $f(\alpha) = \alpha$ is consistent. If $K \vdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow r(\alpha)$ and $K \circ \alpha \vdash r(\alpha)$. Since α is consistent so is $K \circ \alpha$, thus $r(\alpha)$ is consistent, hence $f(\alpha)$ is consistent. To show that f satisfies extensionality, let $\vdash \alpha \leftrightarrow \gamma$: If $K \nvdash \neg\alpha$, then $K \nvdash \neg\gamma$, and we have $f(\alpha) = \alpha$ and $f(\gamma) = \gamma$, hence $\vdash f(\alpha) \leftrightarrow f(\gamma)$. If $K \vdash \neg\alpha$, then $f(\alpha) = r(\alpha)$. Since $K \vdash \neg\gamma$ we also have $f(\gamma) = r(\gamma)$. By $\circ$-*extensionality* $K \circ \alpha = K \circ \gamma$, from which follows that $r(\alpha) = r(\gamma)$, hence $f(\alpha) = f(\gamma)$. Finally we show that f satisfies idempotence. If $K \nvdash \neg\alpha$ then $f(f(\alpha)) = f(\alpha)$ follows directly. Let $K \vdash \neg\alpha$ then $f(\alpha) = r(\alpha)$. If $K \nvdash \neg r(\alpha)$, then $f(f(\alpha)) = r(\alpha)$. If $K \nvdash \neg r(\alpha)$, then $f(f(\alpha)) = r(r(\alpha))$. By definition $K \circ r(r(\alpha)) = K \circ r(a)$, from which it follows that $r(\alpha) = r(r(\alpha))$. Hence $f(f(\alpha)) = r(\alpha) = f(\alpha)$.

(b) That $*$ satisfies *closure*, *inclusion*, *vacuity*, *extensionality* and *consistency* is trivial, since $\circ$ and $*'$ both satisfy these five postulates. That $*$ satisfies *success* also follows directly from the definition.

(c) We need to prove that $K \circ \alpha = K * f(\alpha)$. If $K \nvdash \neg\alpha$, then $f(\alpha) = \alpha$ and $K \circ f(\alpha) = K \circ \alpha$ follows directly. By $\circ$-*vacuity* $K \circ f(\alpha) \vdash f(\alpha)$. By the definition of $*$, $K * f(\alpha) = K \circ f(\alpha)$. Hence $K * f(\alpha) = K \circ \alpha$.

If $K \vdash \neg\alpha$, then it follows from the definitions of f and r, and $\circ$-*extensionality* that $K \circ \alpha \vdash f(\alpha)$ and $K \circ f(\alpha) = K \circ \alpha$.

Hence $K * f(\alpha) = K \circ \alpha$.

(2) *implies* (1): This part of the proof follows from Observation 4.3. ■

PROOF OF THEOREM 4.7. This proof is quite similar to that of Theorem 4.6. To show (1) *implies* (2), we define f to be a function such that for all α, $K \circ \alpha \vdash f(\alpha)$, $\vdash \alpha \rightarrow f(\alpha)$, and $K \circ \alpha = K \circ f(\alpha)$ and that if $\vdash \alpha \leftrightarrow \alpha'$, then $f(\alpha) = f(\alpha')$. The existence of such function follows from *proxy success*. The proof that f satisfies extensionality, consistency preservation, weak maximality, and idempotence are essentially the same, and the implication property follows trivially. To show that (2) *implies* (1) we only have to add a proof of proxy success. This follows from Observation 4.3. ■

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References

[1] ALCHOURRÓN, C., P. GÄRDENFORS, and D. MAKINSON, 'On the logic of theory change: Partial meet contraction and revision functions', *Journal of Symbolic Logic* 50 (1985), 510–530.

[2] ALCHOURRÓN, C., and D. MAKINSON, 'Hierarchies of regulations and their logic', in R. Hilpinen (ed.), *New Studies in Deontic Logic: Norms, Actions, and the Foundations of Ethics*, pages 125–148, 1981.

[3] ALCHOURRÓN, C., and D. MAKINSON, 'On the logic of theory change: Contraction functions and their associated revision functions', *Theoria* 48 (1982), 14–37.

[4] CANTWELL, J., 'On the logic of small change in hypertheories' *Theoria* 63 (1997), 54–89.

[5] GÄRDENFORS, P., *Knowledge in Flux: Modeling the Dynamics of Epistemic States*, The MIT Press, Cambridge, 1988.

[6] HANSSON, S. O., *Belief Base Dynamics*, PhD thesis, Uppsala University, 1991.

[7] HANSSON, S. O., 'Semi-revision', *Journal of Applied Non-Classical Logic* 7 (1-2) (1997), 151–175.

[8] HANSSON, S. O., *A Textbook of Belief Dynamics. Theory Change and Database Updating.* (Vol. 11 in the Applied Logic Series), Kluwer Academic Publishers, Dordrecht, 1999.

[9] KATSUNO, H., and A. MENDELZON, 'On the difference between updating a knowledge base and revising it', in P. Gärdenfors (ed.), *Belief Revision*, number 29 in Cambridge Tracts in Theoretical Computer Science, pages 183–203. Cambridge University Press, 1992.

[10] MAKINSON, D., 'Screened revision', *Theoria* 63 (1997), 54–89.

[11] OLSSON, E., *Coherence*, PhD thesis, Uppsala University, 1997.

[12] VAN BENTHEM, J., *The Logic of Time*, Second Edition, Kluwer Academic Publisher, 1991.

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